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A NONSTANDARD PROOF OF THE JORDAN CURVE THEOREM

Abstract

We give a nonstandard variant of Jordan's proof of the Jordan curve theorem which is free of the defects his contemporaries criticized and avoids the epsilontic burden of the classical proof. The proof is self-contained, except that the Jordan theorem for polygons is taken for granted.

Introduction

The *Jordan curve theorem* [5] (often abbreviated as JCT in the literature) was one of the starting points in the modern development of *topology* (originally called *Analysis Situs*). This result is considered difficult to prove, at least compared to its intuitive evidence.

C. Jordan [5] considered the assertion to be evident for simple polygons and reduced the case of a simple closed continuous curve to that of a polygon by approximating the curve by a sequence of suitable simple polygons.

Although the idea appears natural to an analyst it is not so easy to carry through. Jordan's proof did not satisfy mathematicians of his time. On the one hand it was felt that the case of polygons also needed a proof based on clearly stated geometrical principles, on the other hand his proof was considered incomplete. (See the criticisms formulated in [12] and in [9].)

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If one is willing to assume slightly more than mere continuity of the curve, then much simpler proofs (including the case of polygons) are available (see Ames [1] and Bliss [3] under restrictive hypotheses).

O. Veblen [12] is considered the first to have given a rigorous proof which, in fact, makes no use of metrical properties, or, in the words of Veblen: *We accordingly assume nothing about analytic geometry, the parallel axiom, congruence relations, nor the existence of points outside a plane.* His proof is based on the incidence and order axioms for the plane and the natural topology defined by the basis consisting of nondegenerate triangles. He also defines simple curves intrinsically as specific sets without parametrizations by intervals of the real line. He finally discusses how the introduction of one additional axiom, existence of a point outside the plane, allows him to reduce his result to the context Jordan was working in. Veblen also gave a specific proof for polygons based on the incidence and order axioms exclusively (see [11]) which was later criticized as inconclusive by H. Hahn [4] who published his own version of a proof based on Veblen's incidence and order axioms of the plane (which, by the way, are equivalent to the incidence and order axioms of Hilbert's system).

Jordan's proof in his *Cours d' analyse* of 1893 is elementary as to the tools employed. Nevertheless the proof extends over nine pages and, as mentioned above, cannot be considered complete. We are interested here in this proof. It depends on some facts for polygons and an approximation argument. It is, therefore, a natural idea to use nonstandard arguments to eliminate the epsilontic burden of the approximation. There is an article by L. Narens [7] in which this point of view is adopted. Unfortunately, some part of this proof has been criticized recently as inconclusive and, in any case, the reasoning is not essentially shorter than, or as elementary as, Jordan's proof.

It is certainly true that not all classical arguments can be replaced in some useful or reasonable way by simpler nonstandard arguments. But as we shall show it is possible to simplify the approximation argument specific to Jordan's proof. We shall follow the proof quite closely but take a somewhat different approach when proving path-connectedness. That nonstandard analysis can even give some additional insight into the geometric problem is manifest from the proof by N. Bertoglio and R. Chuaqui [2] which avoids polygons and approximations entirely by looking at a nonstandard discretization of the plane and reducing the problem to a combinatorial version of the JCT proved by L. N. Stout [10]. This reduction of the problem to a (formally) discrete one is interesting and leads to a proof which establishes a link to a context totally different from Jordan's.

As a curiosity we note in passing that Jordan speaks of *infinitesimals* in his proof but it is only a figure of speech for a number which may be chosen as small as one wishes or for a function which tends to zero.

For reference we state the JCT.

The Jordan Curve Theorem. *A simple closed continuous curve* $\mathcal{K}$ *in the plane separates its complement into two open sets of which it is the common boundary; one of them is called the* outer (*or* exterior) region $\mathcal{K}_{\text{ext}}$ *which is an open, unbounded, path-connected set and another set called the* inner (*or* interior) region $\mathcal{K}_{\text{int}}$ *which is an open, simply path-connected, bounded set.*

Notation

By *simple* (polygon, curve) we shall always mean one having no self-intersections. A *broken line* will be a curve consisting of finitely (or hyperfinitely – in a nonstandard domain) many nonzero segments.

The reader is assumed to have a basic knowledge of nonstandard analysis. [1] In what follows we shall always identify reals and points from the standard domain with their "asterisk" images in the nonstandard domain (although the standard curve $\mathcal{K}$ will be distinguished from ${}^*\mathcal{K}$). We shall understand the words *point, polygon, real, curve etc.* as meaning internal objects in the nonstandard domain unless otherwise specified, for instance by the adjective "standard". Hopefully this way of exposition will be equally understandable by both **IST** followers and those who prefer the model–theoretic version of nonstandard analysis (although the latter should understand as hyperreals, hyperpoints *etc.* what we will call reals, points *etc.*).

Plan of the Proof

Starting the proof of the Jordan theorem, we consider a standard simple closed curve $\mathcal{K} = \{K(t) : 0 \le t < 1\}$ where $K : \mathbb{R} \to \mathbb{R}^2$ is a (standard) continuous 1-periodic function which is injective modulo 1 (*i. e.* $K(t) = K(t')$ implies $t - t' \equiv 0 \bmod 1$). From $K(t) \approx K(t')$ it follows then that $t \approx t' \bmod 1$.

Section 1. Working in a fixed nonstandard domain, we infinitesimally approximate $\mathcal{K}$ by a simple (nonstandard) polygon Π, using a construction, essentially due to Jordan, of consecutively cutting off loops from an originally self-intersecting approximation.

Section 2. We define the interior region $\mathcal{K}_{\text{int}}$ as the open set of all standard points which belong to Π_{int} but do not belong to the monad of Π. (The Jordan

[1] We refer to Lindstrøm [6] and Nelson [8].

theorem for polygons is taken for granted; this attaches definite meaning to $\Pi_{\rm int}$ and $\Pi_{\rm ext}$ in the nonstandard domain.) $\mathcal{K}_{\rm ext}$ is defined accordingly.

Section 3. We prove that any point of $\mathcal{K}$ is a limit point for both $\mathcal{K}_{\rm int}$ and $\mathcal{K}_{\rm ext}$; this also implies the non-emptiness of the regions.

Section 4. To prove that $\mathcal{K}_{\rm int}$ is path-connected we define a simple nonstandard polygon Π' which lies entirely within $\Pi_{\rm int}$, does not intersect $\mathcal{K}$, and contains all (standard) points of $\mathcal{K}_{\rm int}$. This easily implies the path-connectedness.

1 Approximation by a Simple Polygon

We say that a polygon $\Pi = P_1P_2\dots P_nP_1$ (n may be infinitely large) *approximates* $\mathcal{K}$ if there is an internal sequence of (perhaps nonstandard) reals $0 \le t_1 < t_2 < \dots < t_n < 1$ such that

(†) $P_i = K(t_i)$ for $1 \le i < n$, and

(‡) $t_n - t_1 \ge \frac{1}{2}$ and $t_{i+1} - t_i \le \frac{1}{2}$ for all $1 \le i < n$.

We say that Π *approximates* $\mathcal{K}$ *infinitesimally* if in addition $\Delta(\Pi) \approx 0$, where $\Delta(\Pi) = \max_{1\le k\le n} |P_kP_{k+1}|$. (It is understood that $P_{n+1} = P_1$.)

Lemma 1. *Let $\Pi = P_1\dots P_nP_1$ approximate $\mathcal{K}$ infinitesimally. Then*

(i) *n is infinitely large, $t_{i+1} \approx t_i$ for all $1 \le i < n$, $t_1 \approx 0$, and $t_n \approx 1$,*

(ii) *there is an infinitesimal $\varepsilon > 0$ such that ${}^*\mathcal{K}$ is in the ε-neighborhood of Π and Π is in the ε-neighborhood of ${}^*\mathcal{K}$ and*

(iii) *if $P \approx Q$ are on Π, then precisely one of the two arcs Π is decomposed into by these points must be included in the monad of P.*

Proof.(i) The requirement (‡) does not allow the hyperreals t_k to collapse into a sort of infinitesimal "cluster" or into a pair of them around 0 and 1, which are compatible with $\Delta(\Pi) \approx 0$ alone. (Note that the injectivity modulo 1 of ${}^*\mathcal{K}$ is used in the proof that $t_1 \approx 0$ and $t_n \approx 1$.)

(ii) $\delta_i = \max_{t_i\le t\le t_{i+1}} |K(t) - K(t_i)|$ is infinitesimal for each $1 \le i \le n$ and therefore $\varepsilon = 2\max_{1\le i\le n}\delta_i$ is infinitesimal and proves the assertion.

(iii) Since all edges of Π are infinitesimal by (i), we may assume that P and Q are vertices, say $P = P_i$ and $Q = P_j$. Then either $t_i \approx t_j$ or $t_i \approx 0$ while $t_j \approx 1$. (Indeed otherwise ${}^*\mathcal{K}$ would have a self-intersection.) Consider the first case. The arc determined by $t_i \le t \le t_j$ is clearly within the monad of P. To see that the other arc is not included in the monad consider any t_k which is $\not\approx$ any of t_i, 0, 1. Then $P_k \not\approx P$ as otherwise $\mathcal{K}$ would have a self-intersection. □

Lemma 2. *There is a simple polygon which infinitesimally approximates* $\mathcal{K}$.

Proof. Taking $t_i = \frac{i}{n}$ for some infinitely large n results in a polygon which infinitesimally approximates $\mathcal{K}$. But it may have self-intersections.

Assume two non-adjacent sides intersect, *i. e.* P_iP_{i+1} intersects P_jP_{j+1} for some $1 \leq i < j-1 < n$. By the triangle inequality the shorter of the segments P_iP_j and $P_{i+1}P_{j+1}$ is not longer than the longer of the segments P_iP_{i+1} and P_jP_{j+1} which is bounded in length by $\Delta(\Pi)$.

Let us assume that $|P_iP_j| \leq |P_{i+1}P_{j+1}|$. We now replace in $P_1 \dots P_nP_1$ the arc $P_i \dots P_j$ by a new side P_iP_j if $t_j - t_i \leq \frac{1}{2}$ which ensures that (‡) is satisfied for the new polygon. If $t_j - t_i > \frac{1}{2}$ we replace the complementary arc $P_j \dots P_nP_1 \dots P_i$ by a new side P_jP_i such that again (‡) is satisfied. The case $|P_iP_j| > |P_{i+1}P_{j+1}|$ is treated in the same way. (The dots $\dots$ indicate that all indices in between are involved.)

In all the cases the resulting polygon Π_{new} still infinitesimally approximates $\mathcal{K}$ because (†) and (‡) are satisfied (for the accordingly reduced system of parameter values t_i) and $\Delta(\Pi_{\text{new}}) \leq \Delta(\Pi)$.

This (internal) procedure does not necessarily reduce the number of self-intersections because for the one which is removed there may be others appearing on the newly introduced side of the reduced polygon Π_{new}. But the number of vertices of Π_{new} is strictly less than that of Π. Therefore the internal sequence of polygons arising from Π by iterated applications of this reduction procedure eventually ends with a simple polygon Π' which approximates $\mathcal{K}$ infinitesimally. □

2 Definition of the Interior and Exterior Region

Let us fix for the remainder a simple (nonstandard) polygon $\Pi = P_1P_2 \dots P_nP_1$ which approximates $\mathcal{K}$ infinitesimally.

Let $\mathcal{K}_{\text{int}}$ be the open standard set of all standard points $A \in \Pi_{\text{int}}$ which have a non-infinitesimal distance from Π. We call this the *interior region* of the curve $\mathcal{K}$. In the same way we define the open standard set $\mathcal{K}_{\text{ext}}$ of all standard points from Π_{ext} which have non-infinitesimal distance from Π and call this the *exterior region* of the curve $\mathcal{K}$.

Omitting rather elementary proofs that $\mathcal{K}_{\text{int}}$ is bounded, $\mathcal{K}_{\text{ext}}$ is unbounded, and the complement of the union of both sets equals the curve $\mathcal{K}$, let us prove that for $A \in \mathcal{K}_{\text{int}}$ and $B \in \mathcal{K}_{\text{ext}}$ any standard continuous arc

α from A to B intersects $\mathcal{K}$. Indeed $^*\alpha$ must intersect Π in some point P because it starts in Π_{int} and ends in Π_{ext}. (The JCT, transferred to the nonstandard domain, is applied.) By Lemma 1, and the fact that $\mathcal{K}$ is compact, there is a (standard) point $P' \in \mathcal{K}$ infinitesimally close to $P \in \Pi$. As $\mathcal{K}$ and the arc are standard and closed, P' is in $\mathcal{K} \cap \alpha$.

3 The Curve is the common Boundary

We prove that each (standard) point of $\mathcal{K}$ is a limit point for both the interior region $\mathcal{K}_{\text{int}}$ and the exterior region $\mathcal{K}_{\text{ext}}$; this clearly implies that the interior region is not empty (that the exterior region is not empty is trivial).

By the choice of Π and the definition of $\mathcal{K}_{\text{int}}$ and $\mathcal{K}_{\text{ext}}$, it suffices to prove the following: *given a vertex A on Π, , then for any square S with center in A and non-infinitesimal (possibly nonstandard) size the domain S_{int} contains points in both Π_{int} and Π_{ext} which have non-infinitesimal distance from Π.* We prove this assertion for Π_{int} only; the proof for the exterior region is similar.

Let B be another vertex of Π chosen such that the distance $|AB|$ is non-infinitesimal. We can assume that B lies in S_{ext} and has non-infinitesimal distance from S, and in addition S itself does not contain any vertex of Π. Let α and β be the simple broken lines – connecting A with B – into which Π is partitioned by the vertices A and B.

The interior region Π_{int} is decomposed by S into a number of polygonal domains. Let Π' be the polygon which bounds that domain among them the boundary of which contains A. Then Π' consists of parts of the broken lines α and β and connected parts of S. Since A is the only common point of α and β except for B (which is far away from S), going around Π' we find a connected "interval" C_1C_2 of S (which may occasionally contain one or more of the four vertices of S) such that the points C_1 and C_2 belong to different curves among α, β. Since C_1C_2 is also a part of Π', any inner point E of C_1C_2 belongs to Π_{int}.

Consider a point E in C_1C_2 which has equal distance $d = \mathrm{d}(E,\alpha) - \mathrm{d}(E,\beta)$ from both α and β. Note that d is not infinitesimal. Indeed otherwise there are points $A' \in \alpha$ and $B' \in \beta$ such that $A' \approx E \approx B'$, which is impossible by Lemma 1(iii) as S has non-infinitesimal distance from both A and B. Thus $E \in \Pi_{\text{int}}$ has a non-infinitesimal distance from Π, as required.

4 Path-Connectedness

Let A and B be two (standard) points in $\mathcal{K}_{\text{int}}$. We have to prove that there is a (standard) broken line joining A with B and not intersecting $\mathcal{K}$. This is based on the following lemma.

Lemma 3. *There exists a simple polygon* Π' *lying entirely within* Π_{int}, *containing no point of* ${}^*\mathcal{K}$ *in* Π'_{int}, *and containing every standard point of* $\mathcal{K}_{\text{int}}$ *in* Π'_{int}.

The lemma clearly implies the result: indeed, by the JCT for polygons, A can be connected to B by a broken line which lies within Π'_{int} therefore does not intersect ${}^*\mathcal{K}$. By Transfer we get a standard broken line which connects A and B and does not intersect $\mathcal{K}$, as required.

Moreover, the lemma implies the *simple* path-connectedness of $\mathcal{K}_{\text{int}}$. Indeed we have to prove that every standard simple closed curve $\mathcal{K}_1$ lying entirely within $\mathcal{K}_{\text{int}}$ can be appropriately contracted into a point. To see this note that ${}^*\mathcal{K}_1$ is evidently situated within Π'_{int}, which is the interior of a simple polygon, so that ${}^*\mathcal{K}_1$ has the required property in the nonstandard domain by the JCT for polygons. It remains to apply Transfer.

As for the path-connectedness of the *exterior* region $\mathcal{K}_{\text{ext}}$, we choose a point in $\mathcal{K}_{\text{int}}$ and apply an inversion with center in this point. The interior region becomes a neighborhood of ∞ and the exterior region becomes the interior region of the image of the curve. To this we apply the result above.

Proof of Lemma 3. Let an infinitesimal $\varepsilon > 0$ be defined as in Lemma 1(ii), so that $\mathcal{K}$ and ${}^*\mathcal{K}$ are included in the ε-neighborhood of Π.

Note that each side of Π is infinitesimal by definition. For any side PQ of Π we draw a rectangle of the size $(|PQ| + 4\varepsilon) \times (4\varepsilon)$ so that the side PQ lies within the rectangle at equal distance 2ε from each of the four sides of the rectangle.

Let us say that a point E is the *inner intersection* of two straight segments σ and σ' iff E is an inner point of both σ and σ', and $\sigma \cap \sigma' = \{E\}$. For any point $C \in \Pi_{\text{int}}$ which is either a vertex of some of the rectangles above, or an inner intersection of sides of two different rectangles in this family let CC' be a shortest straight segment which connects C with a point C' on Π; obviously each CC' is infinitesimal.

Let us fix a standard point A in $\mathcal{K}_{\text{int}}$. The parts of the rectangles lying within Π and the segments CC' decompose the interior region Π_{int} into a (possibly hyperfinite) number of polygonal domains. Let the polygon Π' be the boundary of the domain containing A. (Note that all the lines involved lie in the monad of Π, hence none of them contains A.) It remains to prove that Π'_{int} also contains any other standard point B of $\mathcal{K}_{\text{int}}$.

Note that each side of Π' is a part of either a side of one of the rectangles covering Π or of a segment of the form CC' — therefore it is infinitesimal.

Let $\Pi' = C_1C_2\ldots C_n$. We observe that by construction, for any $k = 1,...,n$, there is a shortest segment $\sigma_k = C_kC'_k$, connecting C_k with a point C'_k in Π which does not intersect Π'_{int}. Moreover, by the triangle equality, the segments σ_k have no inner intersections. Therefore any two of them intersect each other only in such a manner that either the only intersection point is the common endpoint $C'_k = C'_l$ or one of them is an end-part the other one. Then the segments σ_k decompose the ring-like polygonal region $\mathcal{R}$ between Π and Π' into n open domains $\mathcal{D}_k$ $(k = 1,...,n)$ defined as follows.

If σ_k and σ_{k+1} are disjoint (σ_{n+1} equals σ_1), then the border of $\mathcal{D}_k$ consists of σ_k, σ_{k+1}, the side C_kC_{k+1} of Π', and that arc $\widehat{C'_kC'_{k+1}}$ of Π which does not contain any of the points C'_l as an inner point.

If σ_k and σ_{k+1} have the common endpoint $C_k = C_{k+1}$ and no more common points, then the border shrinks to σ_k, σ_{k+1}, and C_kC_{k+1}. If, finally, one of the segments is included in the other, then $\mathcal{D}_k$ is empty.

If now $B \in \Pi'_{\text{ext}}$, then B belongs to one of the domains $\mathcal{D}_k$. If this is a domain of the first type, then the infinitesimal simple arc $C'_kC_kC_{k+1}C'_{k+1}$ separates A from B within Π, which easily implies, by Lemma 1(iii), that either A or B belongs to the monad of Π, which contradicts the choice of the points. If $\mathcal{D}_k$ is a domain of second type, then the barrier accordingly shrinks, leading to the same contradiction. $\square$

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