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THE HAUSDORFF DIMENSION OF HILBERT'S COORDINATE FUNCTIONS

Abstract

We characterize the coordinate functions of Hilbert's space-filling curve using a directed-graph iterated function system and use this to analyze their fractal properties. In particular, we show that both coordinate functions have graphs of Hausdorff dimension $\frac{3}{2}$ and level sets of dimension $\frac{1}{2}$.

1 Introduction

Let $I = [0,1]$ denote the unit interval and let I^2 denote the unit square. Hilbert's space filling curve is a continuous, surjective function $h : I \to I^2$. The coordinate functions x and y are given by $h(t) = (x(t), y(t))$. An excellent general reference for h is [7], where one may find arithmetical expressions for x and y. More importantly, for this paper, are functional equations given in [7]. We use these to characterize the coordinate graphs using a directed-graph iterated functions system, henceforth referred to as a DiGraph IFS. This, in turn, allows us to show that the graphs of x and y have positive, finite $\frac{3}{2}$-dimensional Hausdorff measure. This is similar to the result in [5] that the coordinate functions of Peano's space filling curve are each self-affine, and also have positive, finite $\frac{3}{2}$-dimensional Hausdorff measure. Our techniques, however, more closely model those applied to Kiesswetter's curve in [1].

Let X denote the graph of x and let Y denote the graph of y. We will see that X may be decomposed into 4 parts: 2 affine images of itself and 2 affine images of Y. Similarly, Y may be decomposed into 2 affine images of itself and 2 affine images of X. This is exactly the type of situation which may be described by a DiGraph IFS. Our treatment follows that of [6] and [2].

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A DiGraph IFS consists of a directed multi-graph, G, together with a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ associated with each edge of G. We will assume that all of these functions are contractions, although this condition may be relaxed somewhat. The directed multi-graph, G, consists of a finite set, V, of vertices and a finite set, E, of directed edges. Given two vertices, u and v, denote the set of all edges from u to v by E_{uv}. We denote the set of all paths of length n with initial vertex u by E_u^n. G is called strongly connected if for every $u, v \in E$, there is a path from u to v. Theorem 4.3.5 of [2] states that given any DiGraph IFS, there is a unique set of compact sets K_v, one for every $v \in V$, such that for every $u \in V$

$$K_u = \bigcup_{v \in V,\ e \in E_{uv}} f_e(K_v).$$

Such a set is called the invariant list of the DiGraph IFS. Note that if $e \in E_{uv}$, then f_e maps K_v into K_u. More generally, if $\alpha \in E_u^n$ has terminal vertex v, then we may form $f_\alpha : K_v \to K_u$ by composing the functions f_e over $e \in \alpha$ taken in reverse order along the path α. If K is any compact set, then the sets $\bigcup_{\alpha \in E_u^n} f_\alpha(K)$ converge to K_u in the Hausdorff metric as $n \to \infty$.

The sets X and Y will be characterized using affine functions. Define A and B to be the following matrices.

$$A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \qquad B = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

Let $\vec{w} \in \mathbb{R}^2$ represent a column vector and define affine functions using matrix multiplication as follows.

$$\begin{aligned}
a_{xx}(\vec{w}) &= A\vec{w} + \begin{pmatrix} 1/4 \\ 0 \end{pmatrix} & a_{yy}(\vec{w}) &= A\vec{w} + \begin{pmatrix} 1/4 \\ 1/2 \end{pmatrix} \\
b_{xx}(\vec{w}) &= A\vec{w} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} & b_{yy}(\vec{w}) &= A\vec{w} + \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \\
c_{xy}(\vec{w}) &= A\vec{w} & c_{yx}(\vec{w}) &= A\vec{w} \\
d_{xy}(\vec{w}) &= B\vec{w} + \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} & d_{yx}(\vec{w}) &= B\vec{w} + \begin{pmatrix} 3/4 \\ 1/2 \end{pmatrix}
\end{aligned}$$

We may associate these functions to the edges in a DiGraph simply by labeling the edges. We may, also, label the vertices to indicate which one corresponds to X and which one corresponds to Y. The labeled DiGraph for X and Y is shown in figure 1.

Lemma 1.1. *X and Y form the invariant list of the DiGraph IFS shown in figure 1.*

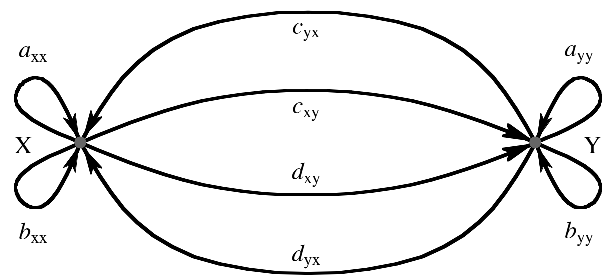

Figure 1: The DiGraph for X and Y

PROOF. This follows immediately from the fact that x and y satisfy the following list of functional equations ([7], page 30, ex. 13). We have labeled the functional equations to clarify the correspondence.

$$\begin{array}{ll} a_{xx} : x((1+t)/4) = x(t)/2 & a_{yy} : y((1+t)/4) = 1/2 + y(t)/2 \\ b_{xx} : x((2+t)/4) = 1/2 + x(t)/2 & b_{yy} : y((2+t)/4) = 1/2 + y(t)/2 \\ c_{xy} : x(t/4) = y(t)/2 & c_{yx} : y(t/4) = x(t)/2 \\ d_{xy} : x((3+t)/4) = 1 - y(t)/2 & d_{yx} : y((3+t)/4) = 1/2 - x(t)/2 \end{array}$$

□

2 Hausdorff Measure and Dimension

In this section, we recall the definitions of Hausdorff measure and dimension, show that X and Y have dimension $\leq 3/2$, and state some useful lemmas. Our notation has been influenced by [2] and [4], where one may find proofs of the basic facts.

Let $s \geq 0$. We will define the s-dimensional Hausdorff measure, $\mathcal{H}^s$, on Euclidean space, $\mathbb{R}^n$. Let $F \subset \mathbb{R}^n$. The diameter of F will be denoted by $\mathrm{diam}(F)$. Let $\varepsilon > 0$. An ε-cover, $\mathcal{C}$, of F is a countable collection of sets such that $F \subset \cup_{U\in\mathcal{C}} U$ and $\mathrm{diam}(U) \leq \varepsilon$ for every $U \in \mathcal{C}$. Now define

$$\mathcal{H}^s_\varepsilon(F) = \inf\Big\{\sum_{U\in\mathcal{C}} \mathrm{diam}(U)^s : \mathcal{C} \text{ is an } \varepsilon\text{-cover of } F\Big\}$$

and

$$\mathcal{H}^s(F) = \lim_{\varepsilon\to 0^+} \mathcal{H}^s_\varepsilon(F).$$

Note that this limit is well defined since $\mathcal{H}^s_\varepsilon(F)$ increases as ε decreases. It may be shown that $\mathcal{H}^s$ is a Borel outer measure on $\mathbb{R}^n$. We denote its restriction to the $\mathcal{H}^s$-measurable sets by $\mathcal{H}^s$, also, and call this the s-dimensional Hausdorff measure.

The Hausdorff dimension of F, $\dim(F)$, is defined by

$$\dim(F) = \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\}.$$

If F is an infinite set, this is equivalent to

$$\dim(F) = \sup\{s \geq 0 : \mathcal{H}^s(F) = \infty\}.$$

We will show that $0 < \mathcal{H}^{3/2}(X) < \infty$, so that $\dim(X) = 3/2$, and similarly for Y. We prove the upper bound here and the lower bound in the next section.

Theorem 2.1. *$\mathcal{H}^{3/2}(X) \leq 2^{3/4}$, so $\dim(X) \leq 3/2$. A similar statement holds for Y.*

PROOF. We write the proof for X. The proof for Y is identical. Note that each of the affine functions in the DiGraph IFS defining X and Y maps I^2 into I^2. Thus $X_n = \bigcup_{\alpha \in E_X^n} f_\alpha(I^2)$ forms a nested sequence of sets containing the invariant set X. Furthermore, each set $f_\alpha(I^2)$ is a rectangle with width 4^{-n} and height 2^{-n}, due to the affine nature of the functions. There are 4^n of these sets since there are 4^n paths of length n leaving any vertex in the DiGraph.

Now, each of the rectangles $f_\alpha(I^2)$ may be decomposed into 2^n squares of side length 4^{-n}. Thus, we may cover X by $2^n 4^n$ squares of side length 4^{-n}. Therefore,

$$\mathcal{H}^{3/2}_{\sqrt{2}\,4^{-n}}(X) \leq 2^n 4^n (\sqrt{2}\,4^{-n})^{3/2} = 2^{3/4}$$

and $\mathcal{H}^{3/2}(X) \leq 2^{3/4}$, as n is arbitrary. □

Lower bounds for Hausdorff measure are, typically, more difficult. Our strategy will be to show that $\mathcal{H}^{1/2}(x^{-1}(z)) > 0$ for all $z \in [0,1]$. The lower bound for X will then follow from a result of Besicovitch. We will obtain the lower bound for level sets by using following measure comparison lemma ([4], page 55).

Lemma 2.1. *Let μ be a Borel measure on the Borel set F and suppose that for some $s > 0$, there are numbers $c, \delta > 0$ such that $\mu(U) \leq c$ diam$(U)^s$ for all open sets U with diam$(U) \leq \delta$. Then, $\mathcal{H}^s(F) \geq \mu(F)/c$.*

We will, also, need the following scaling property of Hausdorff measure ([4], page 27).

Lemma 2.2. *If $F \subset \mathbb{R}^n$, $\lambda > 0$, and $\lambda F = \{\lambda x : x \in F\}$, then $\mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$.*

Finally, we will need Mauldin and Williams' computation of the Hausdorff measure of DiGraph self-similar sets. We associate a similarity dimension with any DiGraph IFS with similarities as follows. Suppose that for each $e \in E$, f_e is a similarity with ratio r_e. Construct a matrix $M(s)$ whose rows and columns are indexed by the vertex set V. The element in row u and column v is $\displaystyle\sum_{e \in E_{uv}} r_e^s$. The similarity dimension of the DiGraph IFS is the unique value of s such that $M(s)$ has spectral radius 1. This coincides with the Hausdorff dimension of the corresponding DiGraph self-similar sets, provided an open set condition is satisfied. The open set condition states that there should be open sets U_v, one for each $v \in V$, so that $U_u \supset \displaystyle\bigcup_{e \in E_{uv}} f_e(U_v)$ with this union disjoint. The following is the main result of [6]. See, also, [2] theorem 6.4.8.

Lemma 2.3. *Let $\{K_v\}_{v \in V}$ be the invariant list of a self-similar DiGraph IFS arising from a strongly connected directed multi-graph and with similarity dimension s. Then $\mathcal{H}^s(K_v) < \infty$ for all $v \in V$. If, in addition, the open set condition is satisfied, then $\mathcal{H}^s(K_v) > 0$ for all $v \in V$.*

Note that self-similarity is the special case of DiGraph self-similarity where the DiGraph has one vertex. In this case, lemma 2.3 reduces to the standard formula for similarity dimension.

3 The Structure of Level Sets

We now turn our attention to the structure of level sets. We first consider the sets $x^{-1}(0)$, $y^{-1}(0)$, $x^{-1}(1)$, and, $y^{-1}(1)$ and show they all have Hausdorff dimension 1/2. Then, we will use the DiGraph structure of X and Y to extend these results to other sets.

Consider the functions a_{xx}, c_{xy}, c_{yx}, and d_{yx}. These are the four affine transformations from figure 1 which leave the x-axis invariant. They are all similarities of ratio $\frac{1}{4}$ when restricted to $\mathbb{R}$. Thus, the sets $x^{-1}(0)$ and $y^{-1}(0)$ form the invariant list of the corresponding self-similar DiGraph IFS. The open set condition is satisfied using the open unit interval. Lemma 2.3 shows that

$$0 < \mathcal{H}^{1/2}(x^{-1}(0)) < \infty \quad \text{and} \quad 0 < \mathcal{H}^{1/2}(y^{-1}(0)) < \infty.$$

In fact, note that $\mathcal{H}^{1/2}(x^{-1}(0)) \leq 1$ since $x^{-1}(0)$ may be covered by 2^n intervals of length 4^{-n} for any n. A similar statement holds for y.

We need to highlight a certain regularity in these sets in order to extend results to other sets.

Lemma 3.1. *$\mathcal{H}^{1/2}(x^{-1}(0) \cap U) \leq 4$ diam$(U)^{1/2}$ for all Borel sets $U \subset I$. A similar statement holds for y.*

PROOF. First, consider the case where U is a closed interval of the form

$$[i/4^n, (i+1)/4^n] \text{ where } i, n \in \mathbb{N} \text{ and} 0 \leq i < 4^n.$$

Then, either $\mathcal{H}^{1/2}(x^{-1}(0) \cap U) = 0$ or $x^{-1}(0) \cap U$ is a set similar to $x^{-1}(0)$ or $y^{-1}(0)$ scaled by a factor 4^{-n}. In either case,

$$\mathcal{H}^{1/2}(x^{-1}(0) \cap U) \leq (4^{-n})^{1/2}\, \mathcal{H}^{1/2}(x^{-1}(0)) = \text{diam}(U)^{1/2}.$$

Now, suppose that U satisfies $4^{-(n+1)} \leq \text{diam}(U) < 4^{-n}$. Then, U may be covered by at most 2 intervals of the form $[i/4^n, (i+1)/4^n]$. Thus,

$$\mathcal{H}^{1/2}(x^{-1}(0) \cap U) \leq 2(4^{-n})^{1/2} = 4(4^{-(n+1)})^{1/2} \leq 4 \text{ diam}(U)^{1/2}.$$

□

Next, consider the sets $x^{-1}(1)$ and $y^{-1}(1)$. We see that $x^{-1}(1)$ is isometric to $x^{-1}(0)$ by using another of Sagan's functional equations: $x(t)+x(1-t) = 1$ ([7], ex. 13, page 30). In particular, if t satisfies $x(t) = 0$, then $1 - t$ satisfies $x(1-t) = 1$. Therefore, $0 < \mathcal{H}^{1/2}(x^{-1}(1)) \leq 1$ and $x^{-1}(1)$ satisfies the conclusions of lemma 3.1.

Finally, $y^{-1}(1)$ is a self-similar set for the similarities a_{yy} and b_{yy} restricted to the horizontal line $y = 1$. Again, $0 < \mathcal{H}^{1/2}(y^{-1}(1)) \leq 1$ and $y^{-1}(1)$ satisfies the conclusions of lemma 3.1.

We now consider the extension to other level sets.

Lemma 3.2. *Suppose that $n \in \mathbb{N}$ and that j is an odd integer satisfying $1 \leq j < 2^n$. Then, $x^{-1}(j/2^n)$ and $y^{-1}(j/2^n)$ both consist of 2^{n+1} sets which are similar to one of the basic sets $x^{-1}(0)$, $y^{-1}(0)$, $x^{-1}(1)$, or, $y^{-1}(1)$, scaled by a factor 4^{-n}.*

PROOF. First, note that the result is true for $n = 1$, as $x^{-1}(1/2)$ consists of a copy of $y^{-1}(1)$ over $[0, 1/4]$, a copy of $x^{-1}(1)$ over $[1/4, 1/2]$, a copy of $x^{-1}(0)$ over $[1/2, 3/4]$, and a copy of $y^{-1}(1)$ over $[3/4, 1]$. This may be seen from the action of the DiGraph IFS. Similarly, $y^{-1}(1/2)$ consists of a copy of $x^{-1}(1)$, 2 copies of $y^{-1}(0)$, and a copy of $x^{-1}(0)$.

Proceeding by induction, suppose the result is true for $n \in \mathbb{N}$. Let j be an odd integer satisfying $1 \leq j < 2^{n+1}$.

Case 1: $j < 2^n$. Then, $x^{-1}(j/2^{n+1})$ consists of a copy of $y^{-1}(j/2^n)$ over $[0, 1/4]$ and a copy of $x^{-1}(j/2^n)$ over $[1/4, 1/2]$, each scaled by a factor 1/4.

Case 2: $j > 2^n$. Then, $x^{-1}(j/2^{n+1})$ consists of a copy of $x^{-1}(1 - j/2^n)$ over $[1/2, 3/4]$ and a copy of $y^{-1}(j/2^n)$ over $[3/4, 1]$, each scaled by a factor 1/4.

In both cases, the induction hypotheses shows that we have a total of 2^{n+2} copies of the basic sets scaled by a factor $4^{-(n+1)}$. A similar argument applies to y. □

Now, let

$$m = \min\{\mathcal{H}^{1/2}(x^{-1}(0)), \mathcal{H}^{1/2}(y^{-1}(0)), \mathcal{H}^{1/2}(x^{-1}(1)), \mathcal{H}^{1/2}(y^{-1}(1))\}.$$

Corollary 3.1. *If z is a dyadic rational and U is a Borel set, then*

$$m \le \mathcal{H}^{1/2}(x^{-1}(z)) \le 2$$

and

$$\mathcal{H}^{1/2}(x^{-1}(z) \cap U) \le 4 \; diam(U)^{1/2}.$$

PROOF. This follows immediately from the scaling lemma 2.2 and lemma 3.2. □

The fact that $\mathcal{H}^{1/2}(x^{-1}(z)) > 0$ for all $z \in I$ now follows from the following lemma, which generalizes a technique applied to Kiesswetter's curve by Edgar.

Lemma 3.3. *Let $s > 0$ and let f be a continuous, real valued function defined on some closed interval J. Suppose there are numbers a and b such that*

$$0 < a \le \mathcal{H}^s(f^{-1}(z)) \le b$$

for all z in some dense subset $D \subset range(f)$. Suppose further that there is a $c > 0$ such that for all $z \in D$ and for all open sets U we have

$$\mathcal{H}^s(f^{-1}(z) \cap U) \le c \; diam(U)^s.$$

Then, $\mathcal{H}^s(f^{-1}(z)) \ge a/c$ for all $z \in range(f)$.

PROOF. Fix $z \in \text{range}(f)$ and choose a sequence (z_n) from D such that $z_n \ne z$ for any n, and $z_n \to z$ as $n \to \infty$. For each $n \in \mathbb{N}$, define the Borel measure μ_n on J to be $\mu_n = \mathcal{H}^s|_{f^{-1}(z_n)}$, the restriction of $\mathcal{H}^s$ to $f^{-1}(z_n)$. Since $a \le \mu_n(J) \le b$ for every n, this sequence has some weak-$*$ cluster point, say μ, satisfying $a \le \mu(J) \le b$.

We claim that μ is supported on $f^{-1}(z)$. Suppose U is an open set containing $f^{-1}(z)$. Then there is an open set V such that $f^{-1}(z) \subset V \subset \overline{V} \subset U$. By the continuity of f, we have $f^{-1}(z_n) \subset V$ for large enough n. Thus,

$$\mu(J \setminus U) \le \mu(J \setminus \overline{V}) \le \liminf_{n\to\infty} \mu_n(J \setminus \overline{V}) = 0$$

and $\mu(J \setminus f^{-1}(z)) = 0$. Furthermore, if $U \subset J$ is any open set, then

$$\mu(U) \leq \liminf_{n\to\infty} \mathcal{H}^s(f^{-1}(z_n) \cap U) \leq c\,\mathrm{diam}(U)^s.$$

Thus, $\mathcal{H}^s(f^{-1}(z)) \geq a/c$ by lemma 2.1. □

Corollary 3.2. *$\mathcal{H}^{1/2}(x^{-1}(z)) \geq m/4 > 0$ and $\mathcal{H}^{1/2}(y^{-1}(z)) \geq m/4 > 0$ for all $z \in I$.*

PROOF. Simply combine corollary 3.1 and lemma 3.3. □

The following lemma is Theorem 5.8 of [3], but is originally due to Besicovitch. We will need it to transfer results to X and Y. If $F \subset \mathbb{R}^2$, where $\mathbb{R}^2$ is the xz plane, then $F_z = \{x \in \mathbb{R} : (x, z) \in F\}$ represents a level set.

Lemma 3.4. *Let F be a subset of the xz plane and let A be any subset of the z-axis. Suppose that if $z \in A$, then $\mathcal{H}^t(F_z) > c$, for some constant c. Then*

$$\mathcal{H}^{s+t}(F) \geq b\,c\,\mathcal{H}^s(A),$$

where b depends only on s and t.

Corollary 3.3. *$\mathcal{H}^{3/2}(X) > 0$ and $\mathcal{H}^{3/2}(Y) > 0$.*

PROOF. This follows immediately from corollary 3.2 and lemma 3.4 by taking A to be $[0, 1]$, $t = 1/2$, and $s = 1$. □

Comments

We have proved that $0 < \mathcal{H}^{3/2}(X) < \infty$ and similarly for Y. More that this, we have obtained the stronger fact that $0 < \mathcal{H}^{1/2}(x^{-1}(z))$ for all $z \in [0, 1]$. Not only does this imply that $0 < \mathcal{H}^{3/2}(X)$, but the reverse implication is not true in general. Indeed, all of the vertical cross-sections of X are singletons and, therefore, zero dimensional.

The fact that $\mathcal{H}^{3/2}(X) < \infty$, implies $\mathcal{H}^{1/2}(x^{-1}(z)) < \infty$ for almost all $z \in [0, 1]$. This statement is easily improved. Consider the rectangular covers of X used in the proof of theorem 2.1. One may prove by induction that any horizontal line intersects at most $2 \cdot 2^n$ of the rectangles of width 4^{-n}. Thus, for any $z \in [0, 1]$,

$$\mathcal{H}^{1/2}_{4^{-n}}(x^{-1}(z)) \leq 2 \cdot 2^n (4^{-n})^{1/2} = 2$$

and $\mathcal{H}^{1/2}(x^{-1}(z)) \leq 2$ since n is arbitrary.

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