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YET A SHORTER PROOF OF AN INEQUALITY OF CUTLER AND OLSEN

Abstract

A very short proof of an inequality due to Cutler and Olsen is presented.

For $E \subseteq \mathbb{R}^d$, let $\dim E$ denote the Hausdorff dimension of E. Let $\mathcal{P}(E)$ denote the family of Borel probability measures on E. For $\mu \in \mathcal{P}(E)$ and $\delta > 0$ write

$$h_\delta(\mu) = \inf \Big\{ -\sum_{i\in\mathbb{N}} \mu E_i \log \mu E_i : \{E_i\} \text{ is a disjoint } \delta\text{–cover of } E \Big\} .$$

The lower Rényi dimension of μ is defined by $\underline{R}(\mu) = \underline{\lim}_{\delta\to 0} \frac{h_\delta(\mu)}{|\log \delta|}$. Cutler and Olsen [1] proved and Olsen [3] reproved (with a shorter proof) the following.

Theorem. *If $E \subseteq \mathbb{R}^d$ is a Borel set, then* $\dim E \leqslant \sup_{\mu\in P(E)} \underline{R}(\mu)$.

We present a remarkably shorter proof utilizing the full strength of the following well–known Frostman's Lemma as it appears in [2, Theorem 5.6]. The point is that we use a version better than that in [3] and thus we do not have to state Lemma 1 of [3] and we can avoid the use of potentials and energies thus skipping completely the proof on page *659* of [3] and reducing the entire proof to less than five lines.

For $x \in \mathbb{R}^d$ and $r > 0$, $B(x, r)$ denotes the closed ball of radius r centered at x.

Frostman Lemma. *Let $E \subseteq \mathbb{R}^d$ be a Borel set. If $0 < s < \dim E$, then there is a measure $\mu \in \mathcal{P}(E)$ and a constant b such that $\mu B(x, r) \leqslant br^s$ for each $r > 0$ and $x \in E$.*

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PROOF. [Proof of the Theorem] Let $0 < s < \dim E$ and let μ be the measure of the Frostman Lemma. Let $\delta > 0$ and let $\{E_i\}$ be a disjoint δ–cover of E. Each E_i is contained in a ball of radius δ and thus $\mu E_i \leqslant b\delta^s$. It follows that $-\sum_{i\in N} \mu E_i \log \mu E_i \geqslant -\log(b\delta^s) \sum_{i\in N} \mu E_i = -\log(b\delta^s)$. Therefore $h_\delta(\mu) \geqslant -\log(b\delta^s)$. Taking the limits yields $\underline{R}(\mu) \geqslant \underline{\lim}_{\delta\to 0} \frac{\log(b\delta^s)}{\log\delta} = s$. □

References

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