Real Analysis Exchange
Vol. 24(2), 1998/9, pp. 815–820

Valentin Skvortsov, Department of Mathematics, Moscow State University, Moscow, 119899, Russia; e-mail: vaskvor@nw.math.msu.su or: Institute of Mathematics, WSP, Plac Weyssenhoffa 11, Bydgoszcz, 85-072, Poland; e-mail: skworcow@wsp.bydgoszcz.pl

A MARTINGALE CLOSURE THEOREM FOR A-INTEGRABLE MARTINGALE SEQUENCES

Abstract

A generalized conditional expectation and the corresponding martingale is defined in terms of the Kolmogorov A-integral. It is proved that the uniform A-integrability of a martingale sequence is a sufficient condition for the sequence to be closed on the right by the A-integrable last element.

A well known theorem in martingale theory states that a martingale sequence $\{X_n, \mathcal{F}_n,\ n = 1, 2, \dots\}$ is closed on the right by the last element $X_\infty(\omega) = \lim_{n\to\infty} X_n(\omega)$ iff $\{X_n, \mathcal{F}_n\}$ is a uniformly integrable sequence (see [1, p.300], [3, p.239] or [6, p.60]). The conditional expectation in this theory is defined in terms of the Lebesgue integral.

Meanwhile there are some other versions of a notion of the mathematical expectation which involve integration more general than the Lebesgue integration. One of such generalization was introduced by Kolmogorov in [4] who defined generalized mathematical expectation as a non-absolutely convergent integral which later became known as the Kolmogorov A-integral (see [2], [5]). In this note we are extending this definition to the case of the conditional expectation and applying this extension to the investigation of A-integrable martingales.

We recall some definitions.

Definition 1. A random variable (r. v.) X defined on a probability space $(\Omega, \mathcal{B}, P)$ is said to be *A-integrable* over a set $B \in \mathcal{B}$ if

$$P\big\{\omega \in B \colon |X(\omega)| > C\big\} = \bar{o}(1/C) \text{ as } C \to \infty \tag{1}$$

Key Words: A-integral, Uniform A-integrability, Martingale sequence
Mathematical Reviews subject classification: 26A39, 60G46
Received by the editors April 20, 1998

and if there exists a finite limit

$$\lim_{C\to\infty} \int\limits_{\{\omega\in B:\ |X(\omega)|\le C\}} X\,dP = I.$$

Then I is called *the A-integral* of X over B and is denoted by $(A)\int\limits_B X\,dP$.

Note that if a r. v. X is A-integrable over some set $B \in \mathcal{B}$ and $X(\omega) \ge 0$ a. s. on B then X is also L-integrable and $(A)\int\limits_B X\,dP = (L)\int\limits_B X\,dP$. This fact implies that if an $\mathcal{F}$-measurable r. v. X ($\mathcal{F}$ being a sub-σ-field of $\mathcal{B}$) is A-integrable over any $\mathcal{F}$-measurable subset of some $\mathcal{F}$-measurable set B then X is L-integrable over B. Indeed, put $B_+ = \{\omega \in B\colon X(\omega) \ge 0\}$ and $B_- = \{\omega \in B\colon X(\omega) < 0\}$. Then, being the r. v. X A-integrable over B_+ and over B_-, it must be L-integrable over each of these sets and consequently over B. We use this observation in the following definition.

Definition 2. Let a r. v. X be A-integrable over any set $B \in \mathcal{F}$ where $\mathcal{F}$ is a sub-σ-field of $\mathcal{B}$. *The conditional A-expectation of X with respect to $\mathcal{F}$* is defined as an $\mathcal{F}$-measurable r. v. $AE(X|F)$ such that for every $B \in \mathcal{F}$ we have

$$\int\limits_B AE(X|\mathcal{F})\,dP = (A)\int\limits_B X\,dP. \tag{2}$$

Definition 3. An A-integrable r. v. X_∞ is said to be *the last element* of a martingale sequence $\{X_n, \mathcal{F}_n, n = 1, 2, \ldots\}$ if $X_n = AE(X_\infty|\mathcal{F}_n)$ for each $n = 1, 2, \ldots$. We also say in this case that X_∞ *closes the martingale sequence from the right.*

We have omitted the "A"-sign in front of the left hand side of (2) because of the above observation, meaning that the Lebesgue integral can be used here.

It follows from the same observation that the use of the A-integral in the definition of a martingale sequence gives an essentially more general notion only for the last element and only in the case where each σ-field $\mathcal{F}_n$ is a proper subset of the σ-field $\mathcal{F}_\infty$generated by $\cup_n \mathcal{F}_n$.

Here we are going to give a sufficient condition for a martingale sequence to be closed on the right by the A-integrable last element. This condition is formulated in terms of uniform A-integrability which is a non-absolute analogue of the uniform Lebesgue integrability.

Definition 4. A family of r. v. $\{X_\gamma\}_{\gamma\in\Gamma}$, defined on $(\Omega, \mathcal{B}, P)$ (Γ is some index set) is said to be *uniformly A-integrable on* $B \in \mathcal{B}$ iff the sets $D_\gamma(C) =$

$\{\omega \in B\colon |X_\gamma(\omega)| > C\}$ satisfy the conditions: $P(D_\gamma(C)) = \overline{o}(\frac{1}{C})$ uniformly in γ as $C \to \infty$ and

$$\sup_{\gamma\in\Gamma} \left| (A) \int\limits_{D_\gamma(C)} X_\gamma \, dP \right| \longrightarrow 0 \quad \text{as } C \to \infty.$$

Now let $\{\mathcal{F}_n\}$ be an increasing sequence of sub-σ-fields of $\mathcal{B}$ and let $\{X_n, \mathcal{F}_n, n = 1, 2, \dots\}$ be a martingale. For any $B \in \cup_n \mathcal{F}_n$ we define a set function Φ by putting

$$\Phi(B) = \int\limits_B X_n \, dP \quad \text{if } B \in \mathcal{F}_n. \tag{3}$$

We call Φ *the associated set function* for $\{X_n, \mathcal{F}_n\}$.

Note that Φ is well defined. Indeed, if $m \geq n$ then for the same B by (3) with n substituted by m we get

$$\Phi(B) = \int\limits_B X_m \, dP \quad \text{if } B \in \mathcal{F}_n \subseteq \mathcal{F}_m. \tag{4}$$

Now by the definition of the martingale and by the definition of the conditional expectation we get

$$\int\limits_B X_m \, dP = \int\limits_B E(X_m|\mathcal{F}_n) \, dP = \int\limits_B X_n \, dP$$

and this proves that the values of Φ on B given by (3) and (4) coincide.

Φ is of course additive on $\cup_n \mathcal{F}_n$ but we do not assume that Φ can be extended to the σ-field $\mathcal{F}_\infty$ generated by $\cup_n \mathcal{F}_n$.

Lemma 1. *A r. v. X_∞ is the last element of a martingale $\{X_n, \mathcal{F}_n, n = 1, 2, \dots, \infty\}$ in the sense of the A-integral iff for the associated set function Φ of the martingale sequence $\{X_n, \mathcal{F}_n, n = 1, 2, \dots\}$ we have $(A)\int_B X_\infty \, dP = \Phi(B)$ for any $B \in \cup_n \mathcal{F}_n$.*

Proof. This follows directly from the definition of Φ and from the definition of the last element. $\square$

Note that this Lemma is true for any other integral which can be used in the above equality.

Theorem 1. *Let $\{X_n, \mathcal{F}_n, n = 1, 2, \ldots\}$ be a martingale sequence convergent a. s. to a r. v. X_∞. Let*

$$P\big\{\omega \in \Omega\colon |X_n(\omega)| > C\big\} = \bar{o}(1/C) \text{ uniformly in } n \text{ as } C \to \infty \tag{5}$$

and $\{X_n\}$ be uniformly A-integrable on any $B \in \cup_n \mathcal{F}_n$ in the sense of Definition 3. Then X_∞ is A-integrable on each $B \in \cup_n \mathcal{F}_n$ and closes the martingale sequence $\{X_n, \mathcal{F}_n\}$ on the right, i. e. $\{X_n, \mathcal{F}_n, n = 1, 2, \ldots, \infty\}$ is an A-integrable martingale with

$$X_n = AE(X_\infty | \mathcal{F}_n)$$

X_∞ being its last element.

PROOF. We show first that (5) implies (1) with $X = X_\infty$.

It is enough to prove (1) with $B = \Omega$. Denote $D(C) = \big\{\omega \in \Omega\colon |X_\infty(\omega)| > C\big\}$, $D_n(C) = \big\{\omega \in \Omega\colon |X_n(\omega)| > C\big\}$, $G_n(C) = \bigcap\limits_{m=n}^{\infty} D_m(C)$. Then obviously $P(D(C)) \le P(\bigcup\limits_{n=1}^{\infty} G_n(C))$, $G_n(C) \subseteq G_{n+1}(C)$,

$$P\big(D(C)\big) \le \lim_{n\to\infty} P\big(G_n(C)\big), \tag{6}$$

$$G_n(C) \subseteq D_n(C). \tag{7}$$

Fix any $\varepsilon > 0$. By (5) there exists $C_\varepsilon > 0$ such that $P\big(D_n(C)\big) \le \varepsilon/C$ for all $n = 1, 2 \ldots$ and for any $C \ge C_\varepsilon$. Fix such C. Then by (6) and (7) $P\big(D(C)\big) \le \varepsilon/C$. As $\varepsilon > 0$ is arbitrary and C is such that $C \ge C_\varepsilon$, then (1) with $X = X_\infty$ is proved for $B = \Omega$ and therefore for any $B \in \mathcal{B}$.

For any r. v. X and $C > 0$ define

$$X^C(\omega) = \begin{cases} X(\omega), & \text{if } |X(\omega)| \le C, \\ C \operatorname{sign} X(\omega), & \text{otherwise.} \end{cases}$$

Notice that

$$\lim_{n\to\infty} X_n^C(\omega) = X_\infty^C(\omega) \quad \text{a. s. on } \Omega. \tag{8}$$

Now fix $B \in \cup_n \mathcal{F}_n$. Then $B \in \mathcal{F}_n$ for some n and hence for the associated function Φ the equality (4) holds for any $m \ge n$.

Fix $\varepsilon > 0$. Since the sequence $\{X_n, \mathcal{F}_n, n = 1, 2, \ldots\}$ is uniformly A-integrable on B and (5) holds, we can find C_0 such that for all $C \ge C_0$ and for all m

$$\left| \int_B \big(X_m - X_m^C\big)\, dP \right| = \left| \int_{\{\omega \in B\colon\ |X_m(\omega)| > C\}} X_m \, dP \right.$$

$$-CP\big\{\omega\in B\colon\ X_m(\omega)>C\big\}+CP\big\{\omega\in B\colon\ X_m(\omega)<-C\big\}\Bigg|\le$$

$$\le\Bigg|\int\limits_{\{\omega\in B\colon\ |X_m(\omega)|>C\}} X_m\,dP\Bigg|+CP\big\{|X_m|>C\big\}<\varepsilon/2 \tag{9}$$

Let C be also fixed for the moment. Then (8) and the Lebesgue dominated convergence theorem imply that for some $m=m_{\varepsilon,C}\ge n$

$$\int\limits_B\big|X_m^C-X_\infty^C\big|\,dP<\varepsilon/2 \tag{10}$$

Now combining (4), (9) and (10) we get for the chosen m

$$\Bigg|\Phi(B)-\int\limits_B X_\infty^C\,dP\Bigg|=\Bigg|\int\limits_B X_m\,dP-\int\limits_B X_\infty^C\,dP\Bigg|\le$$

$$\le\Bigg|\int\limits_B\big(X_m-X_m^C\big)\,dP\Bigg|+\int\limits_B\big|X_m^C-X_\infty^C\big|\,dP<\varepsilon$$

This together with (1) proved already for $X=X_\infty$ implies that X_∞ is A-integrable on B to $\Phi(B)$ and by Lemma, the r. v. X_∞ is the last element, in the sense of the A-integral, of the considered martingale sequence. This completes the proof. □

Note that unlike in the case of the uniform Lebesgue integrability, the above condition in terms of the uniform A-integrability is not necessary for existence of the last element. This can be shown by constructing an example of a Haar series such that it is the A-Fourier series of an A-integrable function and its partial sums are not uniformly A-integrable. (See [7] for details.)

References

[1] R. B. Ash, *Real Analysis and Probability*, Academic Press, New York, 1972.

[2] N. K. Bary,*A Treatise on Trigonometric Series*, Macmillan, New York, 1964.

[3] Y. S. Chow, H. Teicher, *Probability Theory*, 2nd edition, Springer-Verlag, 1988.

[4] A. N. Kolmogorov, *Foundations of the Theory of Probability*, Chelsea Publishing Company, New York, 1950.

[5] L. P. Lee, *Lanzhou Lectures on Henstock Integration*, World Scientific, Singapore, New Jersey, London, 1989.

[6] M. Loeve, *Probability Theory II*, 4th edition, Springer-Verlag, New York, 1978.

[7] V. A. Skvortsov, V. V. Kostin, *Martingale Sequences in the Theory of Orthogonal Series*, Vestnik of Moscow University, in press.