Real Analysis Exchange
Vol. 24(2), 1998/9, pp. 781–790

T. H. Steele, Department of Mathematics, Weber State University, Ogden, UT 84408-1702; e-mail: thsteele@weber.edu

A NOTE ON PERIODIC POINTS AND COMMUTING FUNCTIONS

Abstract

Alikhani-Koopaei has recently conjectured that two commuting continuous functions typically share no periodic points. After discussing the history behind Alikhani-Koopaei's conjecture, we use the Baire Category Theorem to investigate the likelihood that two commuting continuous functions have disjoint sets of periodic points, as well as the structure of the set $\mathcal{F} = \{g \in C(I, I) : gf = fg\}$. We then turn our attention to the iterative properties possessed by commuting pairs of continuous functions.

1 Introduction

Throughout the better part of this century, the subject of commuting functions defined on an interval has received considerable attention. In the 1920's, J. F. Ritt published several papers in which he investigated the algebraic properties of function composition as a binary operation on the set of rational complex functions. In particular, he was able to show that commuting polynomials always have a common fixed point. In 1954, Eldon Dyer asked whether two commuting continuous functions must always share a fixed point; A. J. Shields posed this question in 1955, as did Lester Dubins in 1956. As a partial answer, A. J. Schwartz was able to show that if one of the two commuting continuous functions is also continuously differentiable, then it would necessarily follow that the functions share a periodic point. He published these results in 1965 [6]. It was not until 1967, however, that Dyer's fixed point problem was finally solved. In his doctoral dissertation at Tulane University, William M. Boyce constructed a pair of continuous functions f and g defined on the unit interval for which $fg(x) = gf(x)$ for all $x \in [0, 1]$, yet have no fixed point in common [3]. Six months later, John Huneke submitted two methods of constructing commuting functions that also solve the fixed point problem [5].

Key Words: commuting functions, periodic point, ω-limit set
Mathematical Reviews subject classification: Primary 54C50, Secondary 26A21
Received by the editors January 12, 1998

More recently, Aliasghar Alikhani-Koopaei has turned his attention to commuting continuous functions and the question of whether two such functions must always share a periodic point [2] [1]. Recently, Alikhani-Koopaei has conjectured that two commuting continuous functions will typically have disjoint sets of periodic points. In this paper, we investigate some aspects of Alikhani-Koopaei's conjecture as well as develop generalizations of, and alternate proofs for, some earlier results.

We proceed through four sections. In section 2 we present the notation and definitions we will use throughout the text. We also recall a few important, previously known results. In section 3 we use the Baire Category Theorem in an effort to better understand Alikhani-Koopaei's conjecture, as well as discuss a possible solution. Our fourth section is dedicated to the iterative properties of commuting pairs of continuous functions; it is here that we are able to develop some new proofs to old results, and extend them a bit. We conclude with a few observations in our final section.

2 Preliminaries

We will be concerned with the class $C(I,I)$ of continuous functions mapping the unit interval $I=[0,1]$ into itself, and the properties that commuting members of this class possess. For $f \in C(I,I)$, and any integer $n \geq 1, f^n$ denotes the n^{th} iterate of f. Let $F_n(f)=\{x \in I : f^n(x)=x\}, F(f)=F_1(x), P_n(f)=\{x \in I : f^n(x)=x, f^m(x) \neq x \text{ whenever } m \mid n\}$, and $P(f)=\cup_{n=1}^{\infty} P_n(f)$. Then $F_n(f)$ is the set of points in I left fixed by $f^n, P_n(f)$ represents those points which are periodic under f with period n, and $P(f)$ is the set of f's periodic points. Suppose $x \in P_n(f)$. If $f^n(x)-x$ is not unisigned in any deleted neighborhood of x, we call x a stable periodic point of f, and write $x \in S(f)$. For all $x \in I$, let $O(x,f)=\{x, f(x), f^2(x), ...\}$ be the orbit of x under f, and take $\omega(x,f)$ — the ω-limit set of f generated by x — to be the set of all subsequential limits of $O(x,f)$. We let $\Omega(f)=\{\omega(x,f) : x \in I\}$ represent the set of a function's ω-limit sets. A point x is said to be a recurrent point of f if $x \in \omega(x,f)$. We let $R(f)$ represent the set of recurrent points of f.

We now recall a couple results that will be of use to us in the sequel. The first is Sarkovskii's well known result concerning the ordering of periodic orbits; the second is from [4].

Theorem 2.1. *Let the positive integers be totally ordered in the following way:* $1<2<2^2<2^3<...<2^2\circ 5<2^2\circ 3<...<2\circ 5<2\circ 3<...<9<7<5<3$. *If $f \in C(I,I)$ has a periodic orbit of period n, and if $n>m$, then f also has a periodic orbit of period m.*

Theorem 2.2. *If $f : I \longrightarrow I$ is continuous, then $\overline{P(f)}=\overline{R(f)}$.*

With our definitions for $F_n(f), P_n(f)$ and $P(f)$ in mind, we record the following elementary results. These will be used extensively in the ensuing sections.

Lemma 2.1. *Let $f : I \longrightarrow I$ be continuous.*

- *For any n, $F_n(f)$ is closed.*
- *For any n, $F_n(f)$ is either nowhere dense, or $F_n(f)$ contains an interval.*
- *For any n, $F_n(f) = \cup_{m|n} P_m(f)$.*

Let us now turn our attention to the Baire Category Theorem. Let (X, ρ) be a metric space. A set is of the first category in (X, ρ) if it can be written as a countable union of nowhere dense sets; otherwise, the set is of the second category. A set is residual if it is the complement of a first category set; an element of a residual subset of (X, ρ) is called a typical element of X. With these definitions in mind, we recall Baire's theorem on category.

Theorem 2.3. *Let (X, ρ) be a complete metric space, with S a first category subset of X. Then $X - S$ is dense in X.*

Since much of our work will take place in the complete metric space $(C(I, I), \| \circ \|)$, where $\|f - g\| = \sup\{| \ f(x) - g(x) \ |: x \in I\}$, we will have the opportunity to make good use of Baire's Theorem. When working in $(C(I, I), \| \circ \|)$, we let $B_\varepsilon(f) = \{g \in C(I, I) : \|f - g\| < \varepsilon\}$.

3 Category Results

At the Twenty-first Summer Symposium in Real Analysis, Alikhani-Koopaei conjectured that for the typical pair of commuting continuous functions f and g, one will find that $P(f) \cap P(g) = \emptyset$. With this conjecture in mind, we develop a couple of results which give some indication of the likelihood that two continuous functions commute, and that two continuous functions share no periodic points. Given the difficulty of finding two commuting functions that share no fixed points, constructing a pair of commuting functions that have no periodic points in common may be quite challenging. To attempt to prove the existence of such a pair with the Baire Category Theorem seems reasonable.

We begin this section with a summary of the results addressing the likelihood that two continuous functions have disjoint sets of periodic points.

Theorem 3.1. *There exists a residual set $S \subset C(I, I)$ for which $P(f)$ is a first category set, for every $f \in S$. Moreover, for each f in S there exists a residual set $L(f) \subset C(I, I)$ such that $g \in L(f)$ implies*

- *$P(f) \cap P(g) = \emptyset$,*
- *$P(g)$ is of the first category, and*
- *$F_m(g)$ is nowhere dense for each $m \in \mathbb{N}$.*

Before we prove these results, we should note that this theorem complements a result found in [1] for a particular type of contraction mapping so that for every function f in S, there is a dense set of functions $L(f)$ in $C(I,I)$ with which f shares no periodic points. We begin our development of Theorem 3.1 with a characterization of those functions for which $P(f)$ is a first category set.

Lemma 3.1. *Let $f \in C(I,I)$. Then $P(f)$ is of the first category if and only if $P(f)$ does not contain an interval.*

Proof. The necessity of our theorem follows from the Baire Category Theorem. Now, suppose $P(f)$ is not a first category set. Since $P(f) = \cup_{n\in\mathbb{N}} F_n(f)$, there exists an n so that $F_n(f)$ is not nowhere dense. Since $F_n(f)$ is closed and somewhere dense, $F_n(f)$ contains an interval, so that $P(f)$ contains an interval, too. □

With Lemma 3.1 we are now able to prove the existence of the set S found in Theorem 3.1. In order to show that there is a residual set $S \subset C(I,I)$ for which each member has a first category set of periodic points, it suffices to show that only a first category collection of functions in $C(I,I)$ have a set of periodic points containing an interval.

Lemma 3.2. *There exists a residual set $S \subset C(I,I)$ such that $P(f)$ is of the first category for every f in S.*

Proof. We wish to show that the set $\{f \in C(I,I) : P(f) \text{ contains an interval}\}$ is of the first category. Since $P(f) = \cup_{n\in\mathbb{N}} F_n(f)$, it suffices to show that $\{f \in C(I,I) : F_m(f) \text{ contains an interval}\}$ is a first category set. To this end, let Q_n be an enumeration of the rational intervals (a,b), where $a,b \in \mathbb{Q}\cap[0,1]$. We show that $\{f \in C(I,I) : Q_n \subset F_m(f)\}$ is nowhere dense in $C(I,I)$. Let $f \in C(I,I)$ for which $Q_n \subset F_m(f)$, and let $\varepsilon > 0$. Now, take $g \in B_\varepsilon(f)$ such that Q_n is not contained in $F_m(g)$. Then there exists $x \in Q_n$ such that $\mid x - g^m(x) \mid = \sigma > 0$. Choose $\delta > 0$ so that $h \in B_\delta(g)$ implies $\|h^p - g^p\| < \sigma$ for $p = 1, 2, ...m$. Then $\mid x - h^m(x) \mid > 0$, and Q_n is not contained in $F_m(h)$. We conclude that $\{f \in C(I,I) : Q_n \subset F_m(f)\}$ is nowhere dense in $C(I,I)$. □

We now prove the first part of our theorem with the following lemma.

Lemma 3.3. *Let $f : I \longrightarrow I$ be continuous. If $P(f)$ is of the first category, then there exists a residual set $\mathcal{F} \subset C(I,I)$ so that $P(f) \cap P(g) = \emptyset$ for all $g \in \mathcal{F}$.*

PROOF. Let $P(f) = \cup_{n=1}^{\infty} S_n$, where S_n is nowhere dense for each n. It suffices to show that $\Sigma_n = \{g \in C(I,I) : P(g) \cap \overline{S_n} \neq \emptyset\}$ is of the first category, so that, in turn, one need only show that $\Sigma_{nm} = \{g \in C(I,I) : F_m(g) \cap \overline{S_n} \neq \emptyset\}$ is nowhere dense in $C(I,I)$. Let $g \in C(I,I)$ for which $F_m(g) \cap \overline{S_n} \neq \emptyset$, and let $\varepsilon > 0$. Since $\overline{S_n}$ is nowhere dense, there exists $h \in C(I,I)$ such that $\|h - g\| < \varepsilon$, and $F_m(h) \cap \overline{S_n} = \emptyset$. Let $\sigma = \min\{| h^m(x) - x | : x \in \overline{S_n}\}$. Now, choose $\delta > 0$ so that $\|l - h\| < \delta$ implies $\|l^p - h^p\| < \sigma$ for $p = 1, 2, ..., m$. Since $F_m(l) \cap \overline{S_n} = \emptyset$ whenever $l \in B_\delta(h)$, our conclusion follows. □

The proof of the other two parts of Theorem 3.1 now follow easily.

PROOF OF THEOREM 3.1. Let S be the residual set in $C(I,I)$ found in Lemma 3.2, and fix $f \in S$. Now, let $\mathcal{F}$ be that residual set in $C(I,I)$ found in Lemma 3.3. Set $L(f) = S \cap \mathcal{F}$. Then our first two parts follow immediately, with our last assertion a consequence of Lemma 3.1. □

Without the assurance that $P(f)$ is of the first category, Alikhani-Koopaei's conjecture may fail. Suppose we have a commuting pair of continuous functions f and g, and $P(f)$ contains an interval (a,b). If (a,b) contains a stable periodic point of g, then every function sufficiently close to g will also contain a periodic point in (a,b), so that a closed and somewhere dense subset of those functions commuting with f will share at least one periodic point with f.

We now turn our attention to the structure of the set $\mathcal{F} = \{g \in C(I,I) : gf = fg\}$ for a particular continuous function f. We summarize our results in the following theorem.

Theorem 3.2. *If $f : I \longrightarrow I$ is continuous, then the set $\mathcal{F} = \{g \in C(I,I) : gf = fg\}$ is closed in $(C(I,I), \| \circ \|)$, nonempty, and closed under composition. Moreover, if f is not the identity function, then $\mathcal{F}$ is nowhere dense in $(C(I,I), \| \circ \|)$.*

We begin our development of Theorem 3.2 by showing that $\mathcal{F}$ is indeed closed in $C(I,I)$, and nonempty.

Lemma 3.4. *Let $f \in C(I,I)$. The set $\mathcal{F} = \{g \in C(I,I) : gf = fg\}$ is closed in $(C(I,I), \| \circ \|)$, and nonempty.*

PROOF. That $\mathcal{F}$ is nonempty for every $f \in C(I,I)$ follows from the fact that the identity function commutes with every function. Now, suppose $\{g_n\} \subset \mathcal{F}$ such that $g_n \longrightarrow g$. We show that $g \in \mathcal{F}$. Let $x \in I$, and $\varepsilon > 0$. Since f is uniformly continuous, there exists $\delta > 0$ so that $| a - b | < \delta$ implies $| f(a) - f(b) | < \varepsilon$ for every a, b in I. Since $g_n \longrightarrow g$, there exists $N \in \mathbb{N}$ such that $n > N$ implies $\|g_n - g\| < \delta$. Thus, if $n > N$, then $| fg_n(x) - fg(x) | < \varepsilon$, so that $fg_n = g_n f \longrightarrow fg$. Similarly, $g_n f = fg_n \longrightarrow gf$, and $fg = gf$. □

To see that $\mathcal{F}$ is closed under composition, suppose that g and h are in $\mathcal{F}$. Then $f(gh) = (fg)h = (gf)h = g(fh) = g(hf) = (gh)f$, so that gh is an element of $\mathcal{F}$. Similarly, hg is also in $\mathcal{F}$. We now show that $\mathcal{F}$ is nowhere dense for any function f other than the identity function.

Lemma 3.5. *Suppose $f \in C(I,I)$ is not the identity function, and set $\mathcal{F} = \{g \in C(I,I) : gf = fg\}$. Then $\mathcal{F}$ is nowhere dense in $(C(I,I), \| \circ \|)$.*

Proof. Let $\varepsilon > 0, g \in \mathcal{F}$ and choose $x \in I$ so that $f(x) \neq x$. Let $h \in B_\varepsilon(g)$ for which $h(x) = g(x)$, but $h(f(x)) \neq g(f(x))$. Then $hf(x) \neq gf(x) = fg(x) = fh(x)$, so that $h \notin \mathcal{F}$. Set $\sigma = \mid hf(x) - fh(x) \mid$. Since f is uniformly continuous, there exists $\delta > 0$ so that $\mid a - b \mid < \delta$ implies $\mid f(a) - f(b) \mid < \frac{\sigma}{2}$. Let $0 < \alpha < \min\{\frac{\sigma}{2}, \delta\}$. If $l \in B_\alpha(h)$, then $\mid fh(x) - fl(x) \mid < \frac{\sigma}{2}$ and $\mid hf(x) - lf(x) \mid < \frac{\sigma}{2}$, so that $\mid lf(x) - fl(x) \mid \neq 0$, and $l \notin \mathcal{F}$. We conclude that $\mathcal{F}$ is nowhere dense. $\square$

Let us now turn our attention to what, atleast at first glance, looks like a promising method for proving Alikhani-Koopaei's conjecture.

Conjecture 3.1. *Let $f : I \longrightarrow I$ be a continuous function for which $P(f)$ is of the first category. Say $P(f) = \cup_{n=1}^\infty S_n$, where S_n is nowhere dense in $[0,1]$ for each natural number n, and let $\mathcal{F} = \{g \in C(I,I) : gf = fg\}$. Then the typical element $g \in \mathcal{F}$ has the property that $P(f) \cap P(g) = \emptyset$.*

Proof outline. Since our set $\mathcal{F}$ is a closed subset of the complete metric space $(C(I,I), \|\circ\|)$, we can use the Baire Category Theorem. Let $\Sigma_{nm} = \{g \in \mathcal{F} : \overline{S_n} \cap F_m(g) = \emptyset\}$. Since $\{g \in \mathcal{F} : P(f) \cap P(g) \neq \emptyset\} \subset \cup_{n=1}^\infty \cup_{m=1}^\infty \Sigma_{nm}$, it suffices to show that $\Sigma_{nm} \subset \mathcal{F}$ is nowhere dense. Let $\varepsilon > 0$, with $g \in \mathcal{F} \cap \Sigma_{nm}$. This set is nonempty since $\mathcal{F} \cap \Sigma_{nm}$ contains the identity function for all $f \in C(I,I)$, and n, m in $\mathbb{N}$. Now, choose $h \in \mathcal{F}$ so that $\|g - h\| < \varepsilon$, and $\overline{S_n} \cap F_m(h) = \emptyset$. Since $\overline{S_n} \cap F_m(h) = \emptyset$, there exists $\delta > 0$ for which $\|l - h\| < \delta$ implies $\overline{S_n} \cap F_m(l) = \emptyset$, too. It follows, then, that Σ_{nm} is nowhere dense. $\square$

The difficulty in this approach comes in finding an $h \in \mathcal{F}$ for which $\|g - h\| < \varepsilon$, and $\overline{S_n} \cap F_m(h) = \emptyset$. Since $\mathcal{F} = \{g \in C(I,I) : gf = fg\}$ is a nowhere dense set in $C(I,I)$, it is entirely possible that $\mathcal{F}$ has a null intersection with the residual set $\{g \in C(I,I) : P(g) \cap P(f) = \emptyset\}$. Also, given the complicated structure of the constructions in [5] and [6], attempting to construct an appropriate function h is not particularly attractive, either.

4 Iterative properties of commuting functions

We now turn our attention to the iterative properties possessed by commuting pairs of continuous functions. In so doing we are able to find alternate proofs for the main results of [2] and [6], as well as generalize them a bit. We begin with a quick look at some of the interesting relationships that exist between periodic points and ω-limit sets of commuting pairs of continuous functions.

Lemma 4.1. *Suppose f and g in $C(I,I)$ commute. If $x \in P_n(f)$, then $g^{mn}(x) \in F_n(f)$ for all natural numbers m. In particular, if $x \in F(f)$, them $g^m(x) \in F(f)$ for all $m \in \mathbb{N}$.*

PROOF. It suffices to show that $g^n(x) \in F_n(f)$ whenever $x \in P_n(f)$. This follows from the fact that $g^n(x) = g^n(f^n(x)) = (gf)^n(x) = (fg)^n(x) = f^n(g^n(x))$. $\square$

Corollary 4.1. *Suppose f and g in $C(I,I)$ commute. If $x \in P_n(f)$, then $\omega(x,g^n) \subset F_n(f)$. In particular, if $x \in F(f)$, then $\omega(x,g) \subset F(f)$.*

As our next lemma shows, for each $x \in I$, the ω-limit set $\omega(g^n(x),f)$ is a continuous image of $\omega(x,f)$ under g^n, for each natural number n. Thus, we can take $\omega(x,f)$ as a generating set for all the ω-limit sets of the form $\omega(g^n(x),f)$.

Lemma 4.2. *Suppose f and g in $C(I,I)$ commute. Then $g^n(\omega(x,f)) = \omega(g^n(x),f)$ for all $x \in I$, for all $n \in \mathbb{N}$.*

PROOF. Since $fg = gf$, we have that $g(O(x,f)) = O(g(x),f) - \{x\}$, so that $g(\omega(x,f)) = \omega(g(x),f)$. $\square$

Our next result highlights an interesting relationship between $\omega(x,g^n)$, $\omega(x,g)$ and $\omega(x,fg) = \omega(x,gf)$ whenever f and g commute, and $x \in P_n(f)$.

Lemma 4.3. *Suppose f and g in $C(I,I)$ commute. If $x \in P_n(f)$, then $\omega(x,g^n) \subset \omega(x,g) \cap \omega(x,fg)$. Moreover, $\omega(x,g) = \cup_{k=0}^{n-1} g^k(\omega(x,g^n))$ and $\omega(x,fg) = \cup_{k=0}^{n-1}(fg)^k(\omega(x,g^n))$, so that $\omega(x,g^n)$ is periodic with respect to g and fg.*

PROOF. Since

$$O(x,fg) = \cup_{k=0}^{n-1} O((gf)^k(x),g^n) \text{ and } O(x,g) = \cup_{k=0}^{n-1} O(g^k(x),g^n),$$

it follows that

$$\omega(x,fg) = \cup_{k=0}^{n-1}\omega((gf)^k(x),g^n) = \cup_{k=0}^{n-1}(fg)^k(\omega(x,g^n)) \text{ and}$$

$$\omega(x,g) = \cup_{k=0}^{n-1}\omega(g^k(x),g^n) = \cup_{k=0}^{n-1} g^k(\omega(x,g^n)).$$

This completes the proof. $\square$

If we presume that our commuting pair f and g have the additional property that $P(f) \cap P(g) = \emptyset$, then we can say quite a bit more about their particular properties as well as the relationships between their iterative structures. We begin with a rather simple result involving periodic points and ω-limit sets that has some surprising consequences.

Lemma 4.4. *Suppose f and g in $C(I,I)$ commute, and $P(f) \cap P(g) = \emptyset$. If $x \in P(f)$, then $\omega(x,g)$ is uncountable.*

Proof. Suppose, to the contrary, that $x \in P_n(f)$ and $\omega(x,g)$ is countable. Then $\omega(x,g^n) \subset \omega(x,g)$ is countable, so that there exists a finite $\alpha \in \Omega(g^n)$ that is contained in $\omega(x,g^n) \subset F_n(f)$. Thus, $P(f) \cap P(g) \neq \emptyset$, which leads us to a contradiction. □

As a corollary, we get an extension of a result found in [2], which was predicated on the existence of a g-invariant subset A in $F(f)$. As our Corollary 4.1 shows, we can take A to be $\omega(x,g)$ for any $x \in F(f)$, and thus remove this apparent restriction. This gives us the following.

Theorem 4.1. *Suppose f and g in $C(I,I)$ commute, and $P(f) \cap P(g) = \emptyset$. Then both f and g have periodic points of period 2^n for all $n \in \mathbb{N}$.*

We can begin to see why constructing a pair of commuting continuous functions that share no periodic point appears to be so difficult, as each function must have periodic points with arbitrarily long periods. Our next corollary to Lemma 4.4 also generalizes a result found in [2], again by removing the hypothesis that there exists a g-invariant set A in $F(f)$, as discussed above.

Theorem 4.2. *Suppose f and g in $C(I,I)$ commute, and $P(f) \cap P(g) = \emptyset$. Then both $F(f) \cap R(g)$ and $F(g) \cap R(f)$ are uncountable.*

Proof. Let $x \in F(f)$, with $\alpha \in \Omega(g)$ minimal so that $\alpha \subset \omega(x,g) \subset F(f)$. Then α is uncountable, and since α is a minimal ω-limit set, $\alpha \subset R(g)$. Since $\alpha \subset F(f) \cap R(g)$, our conclusion follows. One shows that $F(g) \cap R(f)$ is uncountable in an analogous fashion. □

As a corollary to both Lemma 4.3 and Lemma 4.4, we get the following.

Corollary 4.2. *Suppose f and g in $C(I,I)$ commute, and $P(f) \cap P(g) = \emptyset$. If $x \in P_n(f)$, then $\omega(x,g^n), \omega(x,g)$ and $\omega(x,fg)$ are all uncountable.*

Our last pair of results go back to a 1965 paper of A. J. Schwartz [6]. Our first theorem generalizes his main result, and allows us to develop a nice alternate proof of his theorem as a corollary.

Theorem 4.3. *Suppose f and g in $C(I,I)$ commute, $x \in P(g)$ and $a \in F(f)$.*

- *If $f(x) \geq x, a > x$ and f is nondecreasing on (x,a), then $P(f) \cap P(g) \neq \emptyset$.*
- *If $f(x) \leq x, a < x$ and f is nondecreasing on (a,x), then $P(f) \cap P(g) \neq \emptyset$.*

Proof. We prove the first part, since the proof of the second is similar. Since f is nondecreasing on (x,a), and $f(x) \geq x$, we have that $\lim_{n \longrightarrow \infty} f^n(x) = \min\{y \in F(f) : y \geq x\}$. Thus, $\omega(x,f)$ is finite for $x \in P(g)$, and $P(f) \cap P(g) \neq \emptyset$ by Lemma 4.4. □

Corollary 4.3. *Suppose f and g in $C(I, I)$ commute. If either f or g is continuously differentiable, then $P(f) \cap P(g) \neq \emptyset$.*

Proof. Suppose, to the contrary, that f is continuously differentiable, and $P(f) \cap P(g) = \emptyset$. Choose $\delta > 0$ so that $\mid f'(p) - f'(q) \mid < 1$ whenever $\mid p - q \mid < \delta$. Since $F(f) \cap R(g)$ is uncountable, there is a Cantor set $Q \subset F(f) \cap R(g)$. Let $a, b \in Q$ so that $b - a < \delta$, and there exists $x \in (a, b) \cap P(g)$. This is possible since $\overline{P(g)} = R(g)$. Since $f' \mid Q = 1$, and $b - a < \delta$, it follows that $f'(y) > 0$ for all $y \in (a, b)$, and f is increasing there. From our previous result, then, $P(f) \cap P(g) \neq \emptyset$, a contradiction. $\square$

5 Conclusions

Suppose we were to attempt the construction of a pair of commuting functions f and g for which $P(f) \cap P(g)$ was the empty set. From what we have discerned in the last few pages, both f and g would have to possess 2^n cycles for every $n \in \mathbb{N}$, and neither function could be continuously differentiable. Moreover, both $F(f) \cap R(g)$ and $F(g) \cap R(f)$ would have to be uncountable. Since $R(f) \subset \overline{P(f)}$ for every continuous function f, the periodic points of f and g, while disjoint, would have to be somewhat intertwined. All this makes Alikhani-Koopaei's tact of using a category argument to prove the residual nature of such a pair that much more attractive. The conjecture in section 3 provides one such approach; nevertheless, how one would go about proving the existence of the function h necessary to the outline is not obvious.

References

[1] A. Alikhani-Koopaei, *A counterexample of common periodic points of functions,* preprint.

[2] A. Alikhani-Koopaei, *On common fixed and periodic points of commuting functions,* Intl. J. of Math. and Math. Sci., to appear.

[3] W. M. Boyce, *Commuting functions with no common fixed points*, Trans. Amer. Math. Soc. **137** (1969), 77–92.

[4] L. Block and W. Coppel, *Dynamics in one dimension*, Lecture Notes in Mathematics, vol. 1513, Springer-Verlag, 1991.

[5] J. P. Huneke, *On common fixed points of commuting functions on an interval*, Trans. Amer. Math. Soc. **139** (1969), 371–381.

[6] A. J. Schwartz, *Common periodic points of commuting functions*, Michigan Math. J. **12** (1965), 353–355.