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ON REVERSE WEAK $(1,1)$ TYPE INEQUALITIES FOR MAXIMAL OPERATORS WITH RESPECT TO ARBITRARY MEASURES

Abstract

Necessary and sufficient conditions on a measure are obtained for the corresponding maximal operators to be of reverse weak $(1,1)$ type.

Let ν be a locally finite non-negative Borel measure on the real line $\mathbb{R}$. For any locally integrable (with respect to ν) function f, $f \in L_{\text{loc}}(\nu)$, the maximal functions $M_\nu^+(f)$ and $M_\nu(f)$ are defined by

$$M_\nu^+(f)(x) = \sup_{x<b,\nu[x,b)>0} \frac{1}{\nu[x,b)} \int_{[x,b)} |f|\, d\nu,$$
$$M_\nu(f)(x) = \sup_{a<x<b,\nu(a,b)>0} \frac{1}{\nu(a,b)} \int_{(a,b)} |f|\, d\nu, \quad x \in \mathbb{R}.$$

An operator $T : L_{\text{loc}}(\nu) \to L_0(\mathbb{R})$ (The latter notation stands for the class of measurable functions.) is said to be of (locally) reverse weak (1,1) type if there exists an independent constant C such that

$$\nu\big\{(T(\chi_I f) > \lambda) \cap I\big\} \geq \frac{1}{\lambda \cdot C} \int_{(T(\chi_I f)>\lambda)\cap I} |f|\, d\nu$$

for every $f \in L_{\text{loc}}(\nu)$ and interval $I = (\alpha,\beta)$, whenever $\lambda > \max\big(T(\chi_I f)(\alpha), T(\chi_I f)(\beta)\big)$.

That the maximal operators M_ν and M_ν^+ are of reverse weak (1,1) type when ν is the Lebesgue measure was proved in [1], [2]. It is also well-known that in general M_ν and M_ν^+ may not be of this type. Theorems 1 and 2 below give necessary and sufficient conditions on the measure ν for the corresponding maximal operators to be of reverse (1,1) type.

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Theorem 1. *There exists a constant C such that*

$$\nu\big\{(M_\nu^+(\chi_I f) > \lambda) \cap I\big\} \geq \frac{1}{\lambda \cdot C} \int_{(M_\nu^+(\chi_I f) > \lambda) \cap I} |f|\, d\nu \tag{1}$$

for every $f \in L_{\mathrm{loc}}(\nu)$ and $I = (\alpha, \beta)$ whenever

$$\lambda > M_\nu^+(\chi_I f)(\alpha) \tag{2}$$

if and only if

$$\sup_{\nu(a,b)>0} \frac{\nu[a,b)}{\nu(a,b)} \leq C. \tag{3}$$

For any $f \in L_{\mathrm{loc}}(\nu)$, if $x \in (M_\nu^+(f) > \lambda) \equiv G_\lambda^+$, then there is $\delta_x > 0$ such that $y \in G_\lambda^+$ for each $y \in (x - \delta_x, x]$. Thus the connected components of G_λ^+ will necessarily be the intervals open from the left.

We need the following Lemma.

Lemma. *Let $f \in L_{\mathrm{loc}}(\nu)$. If an interval $]a, b|$ is a connected component of G_λ^+ (the sign $|$ next to b indicates that b either belongs or does not belong to G_λ^+), then $\nu]a, b| > 0$.*

Proof. If $b \in G_\lambda^+$, i.e. $]a, b| = (a, b]$, then there exists a sequence b_n, $n = 1, 2, \dots$, from $\mathbb{R} \backslash G_\lambda^+$ which tends to b from the right. Assuming that b' is a number greater than b for which $\frac{1}{\nu[b,b')} \int_{[b,b')} |f|\, d\nu > \lambda$, we will get

$$\frac{1}{\nu[b_n, b')} \int_{[b_n, b')} |f|\, d\nu \nrightarrow \frac{1}{\nu[b, b')} \int_{[b,b')} |f|\, d\nu.$$

Thus $\nu\{b\} > 0$.

If $b \notin G_\lambda^+$, then we can consider any $x \in (a, b)$. Since we know that there exists $x' > x$ such that

$$\int_{[x,x')} |f|\, dv > \lambda \nu[x, x') \tag{4}$$

and $\int_{[b,x')} |f| dv \leq \lambda \nu[b, x')$ whenever $x' > b$, we can conclude that (4) holds for some $x' \in (a, b]$. Hence $\nu(a, b) > 0$. □

Proof of Theorem 1. Suppose there exists a constant $C > 1$ such that (3) holds. Assume $f \in L_{\mathrm{loc}}(\nu)$, $I = (\alpha, \beta)$ and λ is sufficiently large so that inequality (2) holds. Since $(-\infty, \alpha] \cap (M_\nu^+(\chi_I f) > \lambda) = \varnothing$, to prove inequality (1) it is sufficient to show that

$$\nu]a, b| \geq \frac{1}{C\lambda} \int_{]a,b|} |f|\, d\nu \tag{5}$$

holds, where $]a,b|$ is a connected component of $(M_\nu^+(\chi_I f) > \lambda) \equiv G_\lambda^+(\chi_I f)$. Indeed,

$$\nu[a,b| \geq \frac{1}{\lambda}\int_{[a,b|} |f|\, d\nu, \tag{6}$$

since $a \notin G_\lambda^+(\chi_I f)$, and by virtue of the Lemma and inequality (3) we have $\nu]a,b| > 0$ and

$$\nu[a,b| \leq C\nu]a,b|. \tag{7}$$

Hence (6) and (7) imply (5). The sufficient part of the theorem is proved.

If (a,b) is an interval such that $\nu(a,b) > 0$ and $\frac{\nu[a,b)}{\nu(a,b)} > C$ for some constant $C > 1$, then we can take $f = \chi_{(a,b)}$. We will have

$$M_\nu^+(f)(a) = \sup_{a<x\leq b} \frac{\nu(a,x)}{\nu[a,x)} = \frac{\nu(a,b)}{\nu[a,b)}.$$

So

$$M_\nu^+(f)(x) \leq M_\nu^+(f)(a) < \frac{1}{C},$$

when $x \leq a$, and $M_\nu^+(f)(x) = 1$ when $a < x < b$ and $\nu[x,b) > 0$. Thus for each $\lambda \in \big(M_\nu^+(f)(a), \frac{1}{C}\big)$ we have

$$\nu(M_\nu^+(f) > \lambda) = \nu(a,b) = \int_{(|f|>\lambda)} f\, d\nu,$$

and inequality (1) fails to hold (I is assumed to be (a,b)). □

Remark. Theorem 1 asserts that if ν has some atom and M_ν^+ is of reverse weak (1,1) type, then starting from this point ν necessarily consists of isolated atoms.

Theorem 2. *There exists a constant C such that*

$$\nu\big\{(M_\nu(\chi_I f) > \lambda) \cap I\big\} \geq \frac{1}{\lambda \cdot C}\int_{(M_\nu(\chi_I f)>\lambda)\cap I} |f|\, d\nu \tag{8}$$

for every $f \in L_{\mathrm{loc}}(\nu)$ and $I = (\alpha,\beta)$ whenever

$$\lambda > \max(M_\nu(\chi_I f)(\alpha), M_\nu(\chi_I f)(\beta)) \tag{9}$$

if and only if

$$\sup_{\nu(a,b)>0} \min\left(\frac{\nu[a,b)}{\nu(a,b)}, \frac{\nu(a,b]}{\nu(a,b)}\right) \leq C. \tag{10}$$

PROOF. Clearly, for the operator M_ν the set $(M_\nu(f) > \lambda) \equiv G_\lambda$ is now open and similarly to the Lemma

$$\nu(a,b) > 0 \tag{11}$$

if (a,b) is a connected component of G_λ. Suppose there exists a constant $C > 1$ such that (10) holds. Assume $f \in L_{\mathrm{loc}}(\nu)$, $I = (\alpha, \beta)$ and λ is sufficiently large so that inequality (9) holds. Just as in Theorem 1, to prove inequality (8) it is sufficient to show that

$$\nu(a,b) \geq \frac{1}{C\lambda} \int_{(a,b)} |f|\, d\nu, \tag{12}$$

where (a,b) is a connected component of $(M_\nu(\chi_I f) > \lambda)$. Since $M_\nu(\chi_I f)(x) \leq \lambda$ for $x = a, b$, we readily have

$$\nu[a,b) \geq \frac{1}{\lambda} \int_{(a,b)} |f|\, d\nu, \quad \nu(a,b] \geq \frac{1}{\lambda} \int_{(a,b)} |f|\, d\nu. \tag{13}$$

It follows from (10) and (13) that (12) holds.

If now (a,b) is an interval such that (11) holds and

$$\min\Big(\frac{\nu[a,b)}{\nu(a,b)}, \frac{\nu(a,b]}{\nu(a,b)}\Big) > C > 1,$$

then one can consider the function $f = \chi_{(a,b)}$ just like in Theorem 1. We have

$$M_\nu(f)(a) = \frac{\nu(a,b)}{\nu[a,b)} \quad \text{and} \quad M_\nu(f)(b) = \frac{\nu(a,b)}{\nu(a,b]}.$$

Obviously, $M_\nu(f)(x) \leq \min(M_\nu(f)(a), M_\nu(f)(b))$ when $x \in \mathbb{R} \backslash (a,b)$ and $M_\nu(f)(x) = 1$ when $x \in (a,b)$. Thus for each $\lambda \in (\max, \frac{1}{C})$ we have

$$\nu(M_\nu(f) > \lambda) = \nu(a,b) = \int_{(|f|>\lambda)} f\, d\nu$$

and inequality (8) fails to hold. □

References

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