**Real Analysis Exchange**
Vol. 24(2), 1998/9, pp. 695–702

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# SEPARATELY CONTINUOUS FUNCTIONS IN A NEW SENSE ARE CONTINUOUS

**Abstract**

Two new notions of separate continuity either of which is equivalent to continuity are introduced.

## 1 Introduction

Let us denote the set of real numbers by $\mathbb{R}$. $\mathbb{R}^n$ $(n \geq 2)$ will denote the set of all vectors $x = (x_1, \ldots, x_n)$ with elements $x_i \in \mathbb{R}$ $(i = 1, \ldots, n)$ and a norm $\|x\| = \big(\sum\limits_{i=1}^{n} x_i^2\big)^{1/2}$. By $U(x^0)$ will be denoted a neighborhood of the point $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$. We shall consider functions whose values are in $\mathbb{R}$.

Recall that a function $f$, defined in the neighborhood $U(x^0)$, is called continuous at the point $x^0$ (sometimes it is called jointly continuous at the point $x^0$) if

$$\lim_{x \to x^0} f(x) = f(x^0). \tag{1.1}$$

In addition to the notion of continuity at the point, for functions of several variables we also have the notion of separate continuity at a given point. A function $f$ is called continuous at $x^0$ with respect to the variable $x_k$ if the relation

$$\lim_{x_k \to x_k^0} f(x_1^0, \ldots, x_{k-1}^0, x_k, x_{k+1}^0, \ldots, x_n^0) = f(x^0) \tag{1.2}$$

holds.

If a function $f$ is continuous at $x^0$ with respect to each variable $x_k$, $k = 1, \ldots, n$, then $f$ is called separately continuous at the point $x^0$. It is well known that separate continuity of the function $f$ at the point $x^0$ does not

Key Words: Separate continuity in the strong sense, separate continuity in the angular sense, continuity in the wide sense.
Mathematical Reviews subject classification: 26B05
Received by the editors July 3, 1998

imply, generally speaking, the continuity of $f$ at $x^0$. Therefore the continuity of a function of two variables at the point $(x_1^0, x_2^0)$ along the two straight lines $x_2 = x_2^0$ and $x_1 = x_1^0$ does not imply its continuity at $(x_1^0, x_2^0)$. Moreover, there exists a function of two variables discontinuous at the point $(0,0)$ which is continuous along every straight line passing through $(0,0)$ ([2], p. 404–5; [1], p. 48; [3], p. 464). Among such functions there are both bounded and unbounded ones ([7], Chapter 4, Exercise 8). What is more, H. Lebesgue called attention to the fact that a function of two variables can be discontinuous at the point $(x_1^0, x_2^0)$ even if it is continuous along every analytic curve through $(x_1^0, x_2^0)$ [4]. Later, A. Rosenthal established the following fact. If the single-valued function $f(x_1, x_2)$ is continuous at the point $P_0 = (x_1^0, x_2^0)$ along every convex curve through $P_0$ which is (at least) once differentiable, then $f$ is also continuous at $P_0$ as a function of $(x_1, x_2)$. Yet $f$ can be continuous along every curve through $P_0$ which is (at least) twice differentiable without being continuous at $P_0$ as a function of $(x_1, x_2)$ ([6], Theorem 1. Here a generalization of this theorem is also given for $n > 2$ variables with the term "convex curve" being replaced by "primitive curve" (Theorem 4)).

The main purpose in this paper is to find necessary and sufficient conditions for the function $f(x_1, \dots, x_n)$ to be continuous at the point $x^0 = (x_1^0, \dots, x_n^0)$, which would give information on the properties of the function $f$ at $x^0$ with respect to each independent variable. To this end, below we shall give the required definitions.

## 2 Separate Continuity in the Strong Sense

**Definition 2.1.** A function $f$ defined in the neighborhood $U(x^0)$, will be called continuous in the strong sense, at $x^0$, with respect to the variable $x_k$ if

$$\begin{aligned} \lim_{x \to x^0} \big[ & f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) \\ & - f(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n) \big] = 0. \end{aligned} \tag{2.1}$$

**Definition 2.2.** A function $f$, defined on $U(x^0)$, will be called separately continuous in the strong sense, at the point $x^0$, if $f$ is continuous in the strong sense, at $x^0$, with respect to each variable $x_k$, $k = 1, \dots, n$.

We have the following assertion.

**Theorem 2.1.** *For a function $f$ to be continuous at the point $x^0$ it is necessary and sufficient that it be separately continuous in the strong sense at $x^0$.*

PROOF. To show the necessity we take any number $k$, $1 \le k \le n$, and write the obvious equality

$$\begin{aligned} f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) - f(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n) \\ = \big[f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) - f(x^0)\big] \\ + \big[f(x^0) - f(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n)\big]. \end{aligned} \tag{2.2}$$

Since the function $f$ is continuous at the point $x^0$, the square-bracketed expressions on the right-hand side of equality (2.2) are arbitrarily small. Therefore the left-hand side of (2.2) is also arbitrarily small. This is equivalent to equality (2.1) for the number $k$. But $k$ was chosen arbitrarily and therefore the function $f$ is separately continuous in the strong sense at the point $x^0$.

To prove the sufficiency, we first recall the fact that the function $f$ is strongly continuous at the point $x^0$ with respect to the variable $x_1$ and thus we have the equality

$$\lim_{x \to x^0} \big[f(x_1, x_2, \dots, x_n) - f(x_1^0, x_2, \dots, x_n)\big] = 0. \tag{2.3$_1$}$$

Likewise, strong continuity of $f$ at the point $x^0$ with respect to the variable $x_2$ is equivalent to the equality

$$\lim_{x \to x^0} \big[f(x_1, x_2, x_3, \dots, x_n) - f(x_1, x_2^0, x_3, \dots, x_n)\big] = 0,$$

which in the particular case $x_1 = x_1^0$ takes the form

$$\lim_{x \to x^0} \big[f(x_1^0, x_2, x_3, \dots, x_n) - f(x_1^0, x_2^0, x_3, \dots, x_n)\big] = 0. \tag{2.3$_2$}$$

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By the strong continuity of the function $f$ at the point $x^0$ with respect to the variable $x_n$, this process eventually results in the equality

$$\lim_{x \to x^0} \big[f(x_1, \dots, x_{n-1}, x_n) - f(x_1, \dots, x_{n-1}, x_n^0)\big] = 0,$$

which for particular values $x_j = x_j^0$, $j = 1, \dots, n-1$, takes the form

$$\lim_{x \to x^0} \big[f(x_1^0, \dots, x_{n-1}^0, x_n) - f(x_1^0, \dots, x_{n-1}^0, x_n^0)\big] = 0. \tag{2.3$_n$}$$

Summing up equalities (2.3$_1$)–(2.3$_n$), we obtain

$$\lim_{x \to x^0} \big[f(x_1, \dots, x_n) - f(x_1^0, \dots, x_n^0)\big] = 0,$$

which is equivalent to equality (1.1). The theorem is proved. □

Equality (2.2) indicates that the following theorem is valid.

**Theorem 2.2.** *For a function $f(x_1, \dots, x_n)$ to be continuous at the point $x^0 = (x_1^0, \dots, x_n^0)$ it is necessary and sufficient that $f(x_1, \dots, x_n)$ be strongly continuous at the point $x^0$ with respect to only one of the variables and continuous at $x^0$ with respect to all other variables collectively.*

This theorem is especially simple for functions of two variables and is formulated as follows.

**Corollary 2.1.** *For a function of two variables $\varphi(x_1, x_2)$ to be continuous at the point $x^0 = (x_1^0, x_2^0)$, it is necessary and sufficient that $\varphi(x_1, x_2)$ be continuous in the strong sense at the point $x^0$ with respect to one of the variables and continuous at $x^0$ with respect to the other variable.*

This corollary yields the following assertion.

**Corollary 2.2.** *Let a function of two variables $\varphi(x_1, x_2)$ be separately continuous at the point $x^0 = (x_1^0, x_2^0)$. For $\varphi(x_1, x_2)$ to be continuous at the point $x^0$, it is necessary and sufficient that $\varphi(x_1, x_2)$ be continuous in the strong sense at $x^0$ with respect to only one of the variables.*

Corollary 2.1 and Theorem 2.1 imply this corollary.

**Corollary 2.3.** *If a function of two variables $\varphi(x_1, x_2)$ is continuous in the strong sense at the point $x^0 = (x_1^0, x_2^0)$ with respect to one of the variables and continuous at $x^0$ with respect to the other variable, then $\varphi(x_1, x_2)$ will be continuous in the strong sense at the point $x^0$ even with respect to that other variable.*

## 3 Separate Continuity in the Angular Sense

**Definition 3.1.** A function $f$ defined in the neighborhood $U(x^0)$, will be called continuous in the angular sense at $x^0$ with respect to the variable $x_k$ if for every collection of positive constants $c = (c_1, \dots, c_n)$

$$\lim_{\substack{x \to x^0 \\ |x_j - x_j^0| \le c_j |x_k - x_k^0| \\ j \ne k}} \Big[ f(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_n) - \\ - f(x_1, \dots, x_{k-1}, x_k^0, x_{k+1}, \dots, x_n) \Big] = 0. \tag{3.1}$$

Obviously, the fact that the function at $x^0$ is continuous in the strong sense at the point $x^0$ with respect to the variable $x_k$ implies that this function is continuous in the angular sense at $x^0$ with respect to the same variable $x_k$.

**Definition 3.2.** A function $f$, defined on $U(x^0)$, will be called separately continuous in the angular sense at the point $x^0$ if $f$ is continuous in the angular sense at $x^0$ with respect to each variable $x_k$, $k = 1, \dots, n$.

**Theorem 3.1.** *For a function $f$ to be continuous at the point $x^0$ it is necessary and sufficient that it be separately continuous in the angular sense at $x^0$.*

PROOF. The necessity follows from the fact that the continuity of the function $f$ at the point $x^0$ implies that the function $f$ is continuous in the strong sense at $x^0$ with respect to each variable (see Theorem 2.1). This in turn implies that the function $f$ is continuous in the angular sense at the point $x^0$ with respect to each variable. Therefore the function $f$ is separately continuous in the angular sense at the point $x^0$.

Let us now prove the sufficiency. Let the function $f$ be separately continuous in the angular sense at the point $x^0$. Then $f$ will be continuous in the angular sense at the point $x^0$ with respect to each variable. Hence relation (3.1) will be fulfilled with respect to each variable $x_k$ $(k = 1, \dots, n)$ for every collection of positive constants. In particular, it will be fulfilled for $c_j = 1$, $j = 1, \dots, n$.

Let us represent the space $\mathbb{R}^n$ as the union of pyramids $\Delta_1, \dots, \Delta_n$ having the common vertices at $x^0$. Each pyramid $\Delta_k$ is defined by a system of inequalities $|x_j - x_j^0| \le |x_k - x_k^0|$ for all $j \ne k$. The pyramid $\Delta_k$ contains a straight line passing through the point $x^0$ and parallel to the real $Ox_k$-axis. (Here the pyramid is assumed to be two-sheeted; i. e., stretching infinitely on both sides of the vertex $x^0$.)

To show that equality (1.1) holds, it is sufficient to prove the validity of the relation

$$\lim_{\substack{x \to x^0 \\ x \in \Delta_k}} f(x) = f(x^0) \tag{3.2}$$

for each $k = 1, \dots, n$.

Without loss of generality, we consider $k = 1$ and shall adapt our further reasoning to the case $(x_1, \dots, x_n) \in \Delta_1$. Since $f$ is continuous in the angular sense at the point $x^0$ with respect to the variable $x_1$, we have the equality

$$\lim_{\substack{x \to x^0 \\ |x_j - x_j^0| \le |x_1 - x_1^0| \\ j=2,\dots,n}} \big[f(x_1, x_2, \dots, x_n) - f(x_1^0, x_2, \dots, x_n)\big] = 0. \tag{3.3$_1$}$$

Since $f$ is continuous in the angular sense at $x^0$ also with respect to the variable $x_2$, we obtain

$$\lim_{\substack{x\to x^0\\|x_j-x_j^0|\le|x_2-x_2^0|\\j\ne 2}}\big[f(x_1,x_2,x_3,\dots,x_n)-f(x_1,x_2^0,x_3,\dots,x_n)\big]=0.$$

Hence for the particular case $x_1=x_1^0$ we have the equality

$$\lim_{\substack{x\to x^0\\|x_j-x_j^0|\le|x_2-x_2^0|\\j=3,\dots,n}}\big[f(x_1^0,x_2,x_3,\dots,x_n)-f(x_1^0,x_2^0,x_3,\dots,x_n)\big]=0, \qquad (3.3_2)$$

and so on.

Finally, by the continuity of $f$ in the angular sense at the point $x^0$ with respect to the variable $x_n$, we have the equality

$$\lim_{\substack{x\to x^0\\|x_j-x_j^0|\le|x_n-x_n^0|\\j=1,\dots,n-1}}\big[f(x_1,\dots,x_{n-1},x_n)-f(x_1,\dots,x_{n-1},x_n^0)\big]=0,$$

which for $x_j=x_j^0$, $j=1,\dots,n-1$, takes the form

$$\lim_{x\to x^0}\big[f(x_1^0,\dots,x_{n-1}^0,x_n)-f(x_1^0,\dots,x_{n-1}^0,x_n^0)\big]=0. \qquad (3.3_n)$$

Combining equalities $(3.3_1)$–$(3.3_n)$ we obtain continuity. □

Theorems 2.1 and 3.1 give rise to the following fact.

**Theorem 3.2.** *Separate continuity in the strong sense of the function $f$ of $n$ variables at the point $x^0=(x_1^0,\dots,x_n^0)$ and separate continuity in the angular sense of the same function $f$ at $x^0$ are equivalent to each other, and either of them is equivalent to the continuity of the function $f$ at the point $x^0$.*

## 4 Continuity in the Wide Sense

For functions of several variables it is possible to form differences of another type which figure in the definition of functions of bounded variation (see, e. g., [2], §254; [5], Ch. II, §4, Section 42, and Ch. III, §2, Section 59). For a function $f$ of $n$ variables the difference $\Delta_{x_0}^h f$ is defined as $\Delta_{x_1^0}^{h_1}(\Delta_{x_2^0,\dots,x_n^0}^{h_2,\dots,h_n}f)=\cdots=\Delta_{x_n^0}^{h_n}(\Delta_{x_1^0,\dots,x_{n-1}^0}^{h_1,\dots,h_{n-1}}f)=\Delta_{x_1^0}^{h_1}(\Delta_{x_2^0}^{h_2}(\cdots\Delta_{x_n^0}^{h_n}f))$, where $x=(x_1,\dots,x_n)$, $x^0=(x_1^0,\dots,x_n^0)$, $h=(h_1,\dots,h_n)$, $\Delta_{x_k^0}^{h_k}f(x_1,\dots,x_n)=f(x_1,\dots,x_{k-1},x_k^0+h_k,x_{k+1},\dots,x_n)-f(x_1,\dots,x_{k-1},x_k^0,x_{k+1},\dots,x_n)$.

Thus,

$$\begin{aligned}\Delta^{h_1,h_2}_{x_1^0,x_2^0}\varphi(x_1,x_2) =&\varphi(x_1^0+h_1,x_2^0+h_2)-\varphi(x_1^0,x_2^0+h_2)\\ &-\varphi(x_1^0+h_1,x_2^0)+\varphi(x_1^0,x_2^0),\end{aligned}\tag{4.1}$$

$$\begin{aligned}\Delta^{h_1,h_2,h_3}_{x_1^0,x_2^0,x_3^0}\psi(x_1,x_2,x_3) = \big[&\psi(x_1^0+h_1,x_2^0+h_2,x_3^0+h_3)\\ &-\psi(x_1^0,x_2^0+h_2,x_3^0+h_3)-\\ &-\psi(x_1^0+h_1,x_2^0,x_3^0+h_3)+\psi(x_1^0,x_2^0,x_3^0+h_3)\big]\\ &-\big[\psi(x_1^0+h_1,x_2^0+h_2,x_3^0)-\psi(x_1^0,x_2^0+h_2,x_3^0)\\ &-\psi(x_1^0+h_1,x_2^0,x_3^0)+\psi(x_1^0,x_2^0,x_3^0)\big].\end{aligned}\tag{4.2}$$

Now we introduce the following notion.

**Definition 4.1.** A function $f(x)$, $x=(x_1,\dots,x_n)\in U(x^0)$ will be called continuous in the wide sense at the point $x^0=(x_1^0,\dots,x_n^0)$ if

$$\lim_{h\to 0}\Delta^h_{x^0}f(x)=0.\tag{4.3}$$

**Theorem 4.1.** *If a function $\psi(x)$, $x=(x_1,\dots,x_n)\in U(x^0)$, is continuous in the strong sense at the point $x^0=(x_1^0,\dots,x_n^0)$ with respect to only one variable, then $\psi(x)$ is continuous in the wide sense at $x^0$.*

Proof. Without loss of generality we shall verify that this theorem is valid for functions of three variables. Let the function $\psi(x_1,x_2,x_3)$ be continuous in the strong sense at the point $x^0=(x_1^0,x_2^0,x_3^0)$ with respect to any one variable, say, with respect to $x_2$. Then we can regroup the terms in equality (4.2) as follows

$$\begin{aligned}\Delta^{h_1,h_2,h_3}_{x_1^0,x_2^0,x_3^0}\psi(x_1,x_2,x_3) =&\big[\psi(x_1^0+h_1,x_2^0+h_2,x_3^0+h_3)\\ &-\psi(x_1^0+h_1,x_2^0,x_3^0+h_3)\big]\\ &-\big[\psi(x_1^0,x_2^0+h_2,x_3^0+h_3)-\psi(x_1^0,x_2^0,x_3^0+h_3)\big]\\ &-\big[\psi(x_1^0+h_1,x_2^0+h_2,x_3^0)-\psi(x_1^0+h_1,x_2^0,x_3^0)\big]\\ &+\big[\psi(x_1^0,x_2^0+h_2,x_3^0)-\psi(x_1^0,x_2^0,x_3^0)\big].\end{aligned}$$

The expressions within the square brackets tend to zero when $(h_1,h_2,h_3)\to(0,0,0)$ (see equality (2.1) for $n=3$ and $k=2$). Therefore

$$\lim_{(h_1,h_2,h_3)\to(0,0,0)}\Delta^{h_1,h_2,h_3}_{x_1^0,x_2^0,x_3^0}\psi(x_1,x_2,x_3)=0$$

and the theorem is thereby proved. □

Theorems 2.1 and 4.1 give rise to the following result.

**Corollary 4.1.** *If a function $f$, defined on $U(x^0)$, is continuous at the point $x^0 = (x_1^0, \dots, x_n^0)$, then $f$ is continuous in the wide sense at $x^0$.*

**Remark 4.1.** Generally speaking, continuity in the wide sense does not imply the continuity. Indeed, for any finite functions $\alpha(x_1)$ and $\beta(x_2)$ we have the equality $\Delta_{x_1^0,x_2^0}^{h_1,h_2}\varphi(x_1,x_2) = 0$, where $\varphi(x_1,x_2) = \alpha(x_1) + \beta(x_2)$. Therefore $\alpha(x_1) + \beta(x_2)$ is continuous in the wide sense at $(x_1^0, x_2^0)$. For this function to be continuous at some point $(x_1^0, x_2^0)$ it is necessary and sufficient (by Theorem 2.1) that $\varphi(x_1, x_2)$ be continuous in the strong sense at $(x_1^0, x_2^0)$ with respect to both $x_1$ and $x_2$. Hence the differences

$$\varphi(x_1^0 + h_1, x_2^0 + h_2) - \varphi(x_1^0, x_2^0 + h_2) = \alpha(x_1^0 + h_1) - \alpha(x_1^0),$$
$$\varphi(x_1^0 + h_1, x_2^0 + h_2) - \varphi(x_1^0 + h_1, x_2^0) = \beta(x_2^0 + h_2) - \beta(x_2^0)$$

must tend to zero when $(h_1, h_2) \to (0, 0)$. Therefore $\alpha(x_1)$ and $\beta(x_2)$ are continuous at $x_1^0$ and $x_2^0$, respectively.

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