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ON LUNINA'S 7-TUPLES FOR IDEAL CONVERGENCE

Abstract

We prove the ideal versions of Lunina's Theorem on convergence and divergence sets of real continuous functions defined on a metric space for F_σ-ideals and ideals with Baire property.

Let (M,ρ) be a metric space. For a sequence of continuous real functions $\vec{f} = (f_n)_n$ defined on M we consider 7 types of sets of convergence or divergence of that sequence:

$$\begin{aligned}
E_1^{\vec{f}} &= \{x : -\infty < \lim_{n\to\infty} f_n(x) < +\infty\},\\
E_2^{\vec{f}} &= \{x : \lim_{n\to\infty} f_n(x) = +\infty\},\\
E_3^{\vec{f}} &= \{x : \lim_{n\to\infty} f_n(x) = -\infty\},\\
E_4^{\vec{f}} &= \left\{x : -\infty < \underline{\lim}_{n\to\infty} f_n(x) < \overline{\lim}_{n\to\infty} f_n(x) < +\infty\right\},\\
E_5^{\vec{f}} &= \left\{x : -\infty < \underline{\lim}_{n\to\infty} f_n(x) < \overline{\lim}_{n\to\infty} f_n(x) = +\infty\right\},\\
E_6^{\vec{f}} &= \left\{x : -\infty = \underline{\lim}_{n\to\infty} f_n(x) < \overline{\lim}_{n\to\infty} f_n(x) < +\infty\right\},\\
E_7^{\vec{f}} &= \left\{x : -\infty = \underline{\lim}_{n\to\infty} f_n(x) \ \wedge\ \overline{\lim}_{n\to\infty} f_n(x) = +\infty\right\}.
\end{aligned}$$

Moreover, let

$$\begin{aligned}
E_8^{\vec{f}} &= \left\{x : \overline{\lim}_{n\to\infty} f_n(x) = +\infty\right\},\\
E_9^{\vec{f}} &= \{x : \underline{\lim}_{n\to\infty} f_n(x) = -\infty\}.
\end{aligned}$$

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Observe that $E_8^{\vec{f}} = E_2^{\vec{f}} \cup E_5^{\vec{f}} \cup E_7^{\vec{f}}$ and $E_9^{\vec{f}} = E_3^{\vec{f}} \cup E_6^{\vec{f}} \cup E_7^{\vec{f}}$

Theorem 1 (Lunina, [6])**.** *Suppose that a metric space M is a union of 7 disjoint sets $E_1, E_2 \ldots E_7$ such that E_1, E_2, E_3 is $F_{\sigma\delta}$ in M and $E_2 \cup E_5 \cup E_7$, $E_3 \cup E_6 \cup E_7$ are G_δ in M. Then there exists the sequence of real-valued continuous functions $\vec{f} = (f_n)_n$ on M so that $E_i = E_i^{\vec{f}}$ for $i = 1, 2, \ldots, 7$.*

This completely describes the defined sets because it was known that for a given sequence of continuous functions $\vec{f}$ on a metric space M, the sets satisfy the assumption of the theorem. We will call $(E_1, \ldots, E_7)$ *Lunina's 7-tuple* in M if there exists a sequence of real-valued continuous functions $\vec{f} = (f_n)_n$ on M such that $E_i = E_i^{\vec{f}}$ for $i = 1, 2, \ldots, 7$. Let us denote $\Lambda 7(M) = \{(E_1, \ldots, E_7) : (E_1, \ldots, E_7)$ is Lunina's 7-tuple in M$\}$. So Lunina's theorem can be expressed in the following way:

$$\Lambda 7(M) = \{(E_1, \ldots, E_7) : (E_1, \ldots, E_7) \text{ is a partition of M and } E_1, E_2, E_3 \in F_{\sigma\delta}(M) \text{ and } E_2 \cup E_5 \cup E_7, E_3 \cup E_6 \cup E_7 \in G_\delta(M)\}$$

for a metric space M.

In this paper we are going to prove some results which generalize Lunina's Theorem (however using it) for ideal convergence for ideals with Baire property (inclusion) and F_σ-ideals (equality). The notion of ideal convergence ($\mathcal{I}$-convergence) is a generalization of the notion of convergence (in the case of the ordinary convergence the ideal $\mathcal{I}$ is equal to the ideal of finite subsets of $\omega = \{0, 1, 2, \ldots\}$). It was first considered in the case of the ideal of sets of statistical density 0 by Steinhaus and Fast [4] (in such a case ideal convergence is equivalent to statistical convergence.) In its general form it appears in the work of Bernstein [1] (for maximal ideals) and M. Katětov [5], where both authors use dual notions of filter convergence.

A family of sets of integers $\mathcal{I} \subset P(\omega)$ is an ideal if $\omega \notin \mathcal{I}$ and it is closed under finite unions and taking subsets. Throughout the paper assume that $\mathcal{I}$ contains the ideal of finite subsets Fin. Since we can identify a set of integers with it's characteristic function we can identify $P(\omega)$ with the Cantor space. In this sense ideals can be F_σ-subsets or have Baire property in the space $P(\omega)$. The ideal Fin is an F_σ-ideal. Let us give two more less trivial examples of F_σ-ideals. The ideal $\mathcal{I}_{1/n} = \{A \subset \omega : \Sigma_{n \in A} 1/n < \infty\}$ and the van der Waerden ideal, an ideal of sets which do not contain arbitrarily long arithmetic progressions. It is known that the ideal of sets of statistical density 0 is an $F_{\sigma\delta}$-ideal but not an F_σ-ideal.

Definition 2. *Let $\mathcal{I}$ be an ideal on ω and $(x_n)_n$ be a sequence of real numbers* *and* $x \in \mathbb{R}$. *Then*

$$\begin{aligned}
&\mathcal{I}\text{--}\lim_{n\to\infty} x_n = x \Leftrightarrow \forall l \in \mathbb{N}_+ \ \left\{n \in \omega \colon |x_n - x| > \frac{1}{l}\right\} \in \mathcal{I},\\
&\mathcal{I}\text{--}\lim_{n\to\infty} x_n = -\infty \Leftrightarrow \forall l \in \mathbb{Z} \ \{n \in \omega \colon x_n > l\} \in \mathcal{I},\\
&\mathcal{I}\text{--}\lim_{n\to\infty} x_n = +\infty \Leftrightarrow \forall l \in \mathbb{Z} \ \{n \in \omega \colon x_n < l\} \in \mathcal{I},\\
&\mathcal{I}\text{--}\overline{\lim}\, x_n = \inf\{\alpha \colon \{n \colon x_n > \alpha\} \in \mathcal{I}\},\\
&\mathcal{I}\text{--}\underline{\lim}\, x_n = \sup\{\alpha \colon \{n \colon x_n < \alpha\} \in \mathcal{I}\}.
\end{aligned}$$

Observe that to define the first three parts of the definition it is enough to use only the last two parts, simply defining $\mathcal{I} - \lim_{n\to\infty} x_n = x$ if $\mathcal{I} - \overline{\lim}\, x_n = \mathcal{I} - \underline{\lim}\, x_n = x$, and $\mathcal{I} - \lim_{n\to\infty} x_n = -\infty$ if $\mathcal{I} - \overline{\lim}\, x_n = -\infty$, and $\mathcal{I} - \lim_{n\to\infty} x_n = \infty$ if $\mathcal{I} - \underline{\lim}\, x_n = \infty$

Let $\vec{f} = (f_n)_n$ be a sequence of continuous functions such that $f_n \colon M \to \mathbb{R}$ for all $n = 1, 2, 3, \ldots$ Suppose that $\mathcal{I}$ is an ideal on ω. Let us introduce the following notation:

$$\begin{aligned}
&E_1^{\vec{f}}(\mathcal{I}) = \left\{x : -\infty < \mathcal{I}\text{--}\lim_{n\to\infty} f_n(x) < +\infty\right\},\\
&E_2^{\vec{f}}(\mathcal{I}) = \left\{x : \mathcal{I}\text{--}\lim_{n\to\infty} f_n(x) = +\infty\right\},\\
&E_3^{\vec{f}}(\mathcal{I}) = \left\{x : \mathcal{I}\text{--}\lim_{n\to\infty} f_n(x) = -\infty\right\},\\
&E_4^{\vec{f}}(\mathcal{I}) = \left\{x : -\infty < \mathcal{I}\text{--}\underline{\lim}_{n\to\infty} f_n(x) < \mathcal{I}\text{--}\overline{\lim}_{n\to\infty} f_n(x) < +\infty\right\},\\
&E_5^{\vec{f}}(\mathcal{I}) = \left\{x : -\infty < \mathcal{I}\text{--}\underline{\lim}_{n\to\infty} f_n(x) < \mathcal{I}\text{--}\overline{\lim}_{n\to\infty} f_n(x) = +\infty\right\},\\
&E_6^{\vec{f}}(\mathcal{I}) = \left\{x : -\infty = \mathcal{I}\text{--}\underline{\lim}_{n\to\infty} f_n(x) < \mathcal{I}\text{--}\overline{\lim}_{n\to\infty} f_n(x) < +\infty\right\},\\
&E_7^{\vec{f}}(\mathcal{I}) = \left\{x : -\infty = \mathcal{I}\text{--}\underline{\lim}_{n\to\infty} f_n(x) \wedge \mathcal{I}\text{--}\overline{\lim}_{n\to\infty} f_n(x) = +\infty\right\}.
\end{aligned}$$

Moreover, let

$$\begin{aligned}
&E_8^{\vec{f}}(\mathcal{I}) = \left\{x : \mathcal{I}\text{--}\overline{\lim}_{n\to\infty} f_n(x) = +\infty\right\},\\
&E_9^{\vec{f}}(\mathcal{I}) = \left\{x : \mathcal{I}\text{--}\underline{\lim}_{n\to\infty} f_n(x) = -\infty\right\}.
\end{aligned}$$

Since standard convergence coincides with the ideal convergence with respect to Fin, for the ideal Fin we have $E_i^{\vec{f}} = E_i^{\vec{f}}(\mathrm{Fin})$ for $i = 1, 2, \ldots, 9$.

We will call $(E_1,\dots,E_7)$ *Lunina's 7-tuple in M for $\mathcal{I}$* if there exists a sequence of real-valued continuous functions $\vec{f}=(f_n)_n$ on M so that $E_i = E_i^{\vec{f}}(\mathcal{I})$ for $i=1,2,\dots,7$. Let us denote $\Lambda 7(M,\mathcal{I}) = \{(E_1,\dots,E_7) : (E_1,\dots,E_7)$ is Lunina's 7-tuple in M for $\mathcal{I}\}$.

Let us recall *the Rudin-Keisler ordering* for ideals. $\mathcal{I} \leq_{RK} \mathcal{J}$ if there is a function $h:\omega\to\omega$ such that $A\in\mathcal{I}$ iff $h^{-1}[A]\in\mathcal{J}$.

Theorem 3. *If $\mathcal{I} \leq_{RK} \mathcal{J}$ then $\Lambda 7(M,\mathcal{I}) \subset \Lambda 7(M,\mathcal{J})$.*

Proof. Let $h:\omega\to\omega$ be a function such that for each $A\subset\omega$ $A\in\mathcal{I}$ iff $h^{-1}[A]\in\mathcal{J}$ and let $(E_1,\dots,E_7)\in\Lambda 7(M,\mathcal{I})$. Then there exists a sequence of continuous functions $\vec{f}=(f_n)_n, f_n : M\to\mathbb{R}$ so that $E_i = E_i^{\vec{f}}(\mathcal{I})$ for $i=1,2,\dots,7$. We define a sequence of functions $\vec{g}=(g_n)_n,\ g_n\colon M\to\mathbb{R}$ such that $g_k=f_n$ for $k\in h^{-1}[\{n\}]$. To show that $E_i=E_i^{\vec{g}}(\mathcal{J})$ for $i=1,2,\dots,7$, it is enough to show the following for all $x\in M$:

1. $\mathcal{I}-\underline{\lim}_{n\to\infty} f_n(x) = \mathcal{J}-\underline{\lim}_{n\to\infty} g_n(x)$,
2. $\mathcal{I}-\overline{\lim}_{n\to\infty} f_n(x) = \mathcal{J}-\overline{\lim}_{n\to\infty} g_n(x)$,
3. $\mathcal{I}-\lim_{n\to\infty} f_n(x) = \mathcal{J}-\lim_{n\to\infty} g_n(x)$.

Observe first that $\{n : f_n(x)\in Z\}\in\mathcal{I}$ iff $\{k : g_k(x)\in Z\}\in\mathcal{J}$ for fixed $x\in M$ and $Z\subset\mathbb{R}$, simply because $\{k : g_k(x)\in Z\} = h^{-1}[\{n : f_n(x)\in Z\}]$. Then $\{\alpha : \{n : f_n(x)>\alpha\}\in\mathcal{I}\} = \{\alpha : \{k : g_k(x)>\alpha\}\in\mathcal{J}\}$ and $\{\alpha : \{n : f_n(x)<\alpha\}\in\mathcal{I}\} = \{\alpha : \{k : g_k(x)<\alpha\}\in\mathcal{J}\}$ as well as their suprema and infima.

□

Corollary 4. *If $\mathcal{I}$ is an ideal with the Baire property then $\Lambda 7(M)\subset\Lambda 7(M,\mathcal{I})$.*

Proof. M. Talagrand ([8] or [3], Corollary 3.10.2) proved that if $\mathcal{I}$ has the Baire property then $\mathrm{Fin}\leq_{RK}\mathcal{I}$. □

Next we are going to show the inverse of the above inclusion for F_σ-ideals. We start with the following characterization of F_σ-ideals. A map $\Phi : P(\omega)\to[0,\infty]$ is a submeasure on ω if $\Phi(\emptyset)=0$, and $\Phi(A)\leq\Phi(A\cup B)\leq\Phi(A)+\Phi(B)$, for all $A,B\subset\omega$. It is lower semicontinuous if for all $A\subset\omega$ we have $\Phi(A)=\lim_n\Phi(A\cap\{0,\dots,n\})$.

Theorem 5 (K. Mazur, [7], Lemma 1.2 or [3], Theorem 1.2.5)**.** *Let $\mathcal{I}$ be an ideal on ω. Then $\mathcal{I}$ is F_σ if and only if $\mathcal{I}=\mathrm{Fin}(\phi)$ for some lower semicontinuous submeasure ϕ, where* $\mathrm{Fin}(\phi)=\{A\subseteq\omega\colon\ \phi(A)<\infty\}$.

Lemma 6. *Assume that $\mathcal{I} = Fin(\phi)$ for some lower semicontinuous submeasure ϕ. Then $A \in \mathcal{I}$ if and only if there exists a natural number n so that* $(\forall B \in Fin)\,(\phi\,(B) > n \Rightarrow \exists m \in B \quad m \notin A)$.

Proof. If $A \in \mathcal{I}$ then let $n = \phi(A)$. If $\phi(B) > n$ then B cannot be contained in A. And conversely, assume that $A \notin \mathcal{I}$. Then $\phi(A) = \infty$ so from lower semicontinuity of ϕ for each n there is $B \subset A$ finite with $\phi(B) > n$. □

Theorem 7. *Let M be a metric space. If $\mathcal{I}$ is F_σ-ideal then $\Lambda 7(M) = \Lambda 7(M, \mathcal{I})$*

Proof. Let ϕ be a lower semicontinuous submeasure with $\mathcal{I} = \mathrm{Fin}(\phi)$. Fix a sequence $\vec{f} = (f_n)_n$ of continuous functions $f_n : M \to \mathbb{R}$. From [2] (Proposition 1, Theorem 2) we use Cauchy-like characterization of ideal convergence and we get

$$E_1^{\vec{f}}\,(\mathcal{I}) = \left\{x\colon\ \forall k \in \mathbb{N}_+\ \exists l \in \mathbb{N}\ \left\{n\colon\ |f_l(x) - f_n(x)| > \frac{1}{k}\right\} \in \mathcal{I}\right\}.$$

By Lemma 6

$$\begin{aligned} E_1^{\vec{f}}\,(\mathcal{I}) = \{x\colon\ &\forall k \in \mathbb{N}_+\ \exists l \in \mathbb{N}\ \exists m \in \mathbb{N}\ \forall B \in \mathrm{Fin} \\ &(\phi(B) > m \Rightarrow \exists b \in B\ \ |f_l(x) - f_b(x)| \leqslant \tfrac{1}{k})\} = \end{aligned}$$
$$\bigcap_{k \in \mathbb{N}_+} \bigcup_{l \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcap_{B \in \mathrm{Fin}, \phi(B) > m} \bigcup_{b \in B} \left\{x\colon\ |f_l(x) - f_b(x)| \leqslant \frac{1}{k}\right\}.$$

Since $\left\{x\colon\ |f_l(x) - f_b(x)| \leqslant \frac{1}{k}\right\}$ is a closed subset of M, therefore $E_1^{\vec{f}}\,(\mathcal{I})$ is $F_{\sigma\delta}$.

In the next case $E_2^{\vec{f}}\,(\mathcal{I}) = \{x\colon\ \forall l \in \mathbb{Z}\ \ \{n\colon\ f_n\,(x) < l\} \in \mathcal{I}\}$. Applying Lemma 6 we get

$$E_2^{\vec{f}} = \{x\colon\ \forall l \in \mathbb{Z}\ \ \exists m \in \mathbb{N}\ \ \forall B \in \mathrm{Fin}\,(\phi\,(B) > m \Rightarrow \exists b \in B\ \ f_b\,(x) \geq l)\} =$$
$$\bigcap_{l \in \mathbb{Z}} \bigcup_{m \in \mathbb{N}} \bigcap_{B \in \mathrm{Fin}, \phi(B) > m} \bigcup_{b \in B} \{x\colon\ f_b\,(x) \geq l\}.$$

Since $\{x\colon\ f_b\,(x) \geq l\}$ is a closed subset of M thus $E_2^{\vec{f}}\,(\mathcal{I})$ is $F_{\sigma\delta}$.

Next we consider the set $E_8^{\vec{f}}\,(\mathcal{I}) = \{x\colon\ \forall l \in \mathbb{Z}\ \ \{n\colon\ f_n\,(x) > l\} \notin \mathcal{I}\}$. Again, applying Lemma 6 we have

$$E_8^{\vec{f}}(\mathcal{I}) = \bigcap_{l \in \mathbb{Z}} \bigcap_{m \in \mathbb{N}} \bigcup_{\substack{B \in \mathrm{Fin} \\ \phi(B) > m}} \bigcap_{b \in B} \{x\colon\ f_b(x) > l\}.$$

Therefore we see that $E_8^{\vec{f}}(\mathcal{I})$ is G_δ, because $f_b^{-1}[(l,+\infty)]$ is an open subset of M. Similarly, we show that $E_3^{\vec{f}}(\mathcal{I})$ is $F_{\sigma\delta}$ and $E_9^{\vec{f}}(\mathcal{I})$ is G_δ. So we have proven that $\Lambda 7(M) \supset \Lambda 7(M,\mathcal{I})$. The inverse inclusion follows from the fact that F_σ sets have the Baire Property and Corollary 4. □

For some spaces the previous theorem can be inverted.

Theorem 8. *Let M be a metric space containing a subspace homeomorphic to the Cantor space. If $\Lambda 7(M) = \Lambda 7(M,\mathcal{I})$ then $\mathcal{I}$ is F_σ-ideal.*

Proof. Assume that $\mathcal{I}$ is not F_σ-ideal. Let us define a sequence of continuous functions $f_n : P(\omega) \to \mathbb{R}$ by the formula

$$f_n(A) = \begin{cases} 0 & \text{if } n \notin A \\ n & \text{otherwise} \end{cases}.$$

Observe that if $A \in \mathcal{I}$ then $\mathcal{I} - \lim f_n(A) = 0$, and if $A \notin \mathcal{I}$ then for each k $\{n : f_n(A) > k\} = A \setminus \{0,\dots,k\} \notin \mathcal{I}$ so $\mathcal{I} - \overline{\lim}_{n\to\infty} f_n(A) = \infty$ so $E_8^{\vec{f}}(\mathcal{I}) = P(\omega) \setminus \mathcal{I}$ is not G_δ. Now assume that $P(\omega)$ is a homeomorphic subset of M. Since $P(\omega)$ is a compact space it is also a closed subset of M. So we can extend functions f_n to continuous functions $g_n : M \to \mathbb{R}$. Observe that $E_8^{\vec{f}}(\mathcal{I}) = E_8^{\vec{g}}(\mathcal{I}) \cap P(\omega)$ so if $E_8^{\vec{f}}(\mathcal{I})$ is not G_δ in $P(\omega)$ then $E_8^{\vec{g}}(\mathcal{I})$ is not G_δ in M. □

Corollary 9. *Let $\mathcal{I}$ be an F_σ-ideal. Then*

$$F_{\sigma\delta}(M) = \left\{A \subseteq M\colon\ A = E_i^{\vec{f}}(\mathcal{I}) \ \text{for } \vec{f} \in C(M)^\omega\right\} \ \text{for } i = 1,2,3,$$
$$G_\delta(M) = \left\{A \subseteq M\colon\ A = E_i^{\vec{f}}(\mathcal{I}) \ \text{for } \vec{f} \in C(M)^\omega\right\} \ \text{for } i = 8,9.$$

where $C(M)$ denotes the set of all real-valued continuous functions defined on M.

Proof. For $A \in F_{\sigma\delta}(M)$ we apply Theorems 1 and 7 for 7-tuples:

$$(A,\emptyset,\emptyset,M\setminus A,\emptyset,\emptyset,\emptyset) \text{ for } i = 1,$$
$$(\emptyset,A,\emptyset,\emptyset,M\setminus A,\emptyset,\emptyset) \text{ for } i = 2,$$
$$(\emptyset,\emptyset,A,\emptyset,\emptyset,M\setminus A,\emptyset) \text{ for } i = 3.$$

For $A \in G_\delta(M)$ we take $(\emptyset,\emptyset,\emptyset,M\setminus A,\emptyset,\emptyset,A)$ for $i = 8,9$.

□

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