RESEARCH

Real Analysis Exchange
Vol. 35(2), 2009/2010, pp. 463–478

Aparna Vyas, Department of Mathematics, University of Allahabad, Allahabad - 211 002, India. email: aparna.vyaas@gmail.com, dpub@pphmj.com

Rajeshwari Dubey, Department of Mathematics, University of Allahabad, Allahabad - 211 002, India. email: rajeshwari_uni@yahoo.com, dpub@pphmj.com

WAVELET SETS ACCUMULATING AT THE ORIGIN

Abstract

For a natural number n, selecting a $2n$-interval symmetric wavelet set by making use of a result of Arcozzi, Behera and Madan [J. Geom. Anal. **13** (2003), 557-579], we construct a family of symmetric wavelet sets accumulating at the origin. Such a family of wavelet sets is also obtained from a family of symmetric six-interval wavelet sets provided by them in the same paper. Three-interval wavelet sets are employed in having a family of wavelet sets accumulating at the origin which are non-symmetric. Further, we obtain a larger class of H^2-wavelet sets accumulating at the origin, which include the one given by Behera in [Bull. Polish Acad. Sci. Math. **52** (2004), 169-178]. Finally, non-MSF non-MRA wavelets are obtained through the selected family of $2n$-interval symmetric wavelet sets.

1 Introduction.

Since a wavelet set does not contain a nondegenerate interval containing the origin, a natural question asking for the existence of a wavelet set W with the origin as an accumulation point of W, arises. This question is equivalent to the existence of a wavelet set W such that the characteristic function with support

Mathematical Reviews subject classification: Primary: 42C15, 42C40
Key words: wavelets, MSF wavelets, wavelet set, dilation equivalence, translation equivalence, dimension function
Received by the editors September 19, 2009
Communicated by: Alexander Olevskii

W does not vanish in any neighbourhood of the origin. It got affirmatively settled by various workers. Madych [13] gave an example of an MSF wavelet ψ whose Fourier transform does not vanish in any neighbourhood of the origin. Garrigós [9] also constructed a wavelet set with this property. Studying the behaviour of a class of band limited wavelets at the origin, Brandolini, Garrigós, Rzeszotnik and Weiss [6] constructed such wavelet sets K_ϵ, where $\epsilon \in \left(0, \frac{1}{3}\right]$. Garrigós [9] in his Ph. D. thesis asked for the existence or otherwise of a wavelet whose Fourier transform is even, discontinuous at the origin and has compact support. A positive answer to this question has been provided by Arcozzi, Behera and Madan [1] and Behera [3], who constructed a family of bounded symmetric wavelet sets $\left\{K_{r,\epsilon} \,:\, r \in \mathbb{N} \; and \; \epsilon \in \left(0, \frac{2^r-1}{4(2^{r+1}-1)}\right)\right\}$, with the origin as an accumulation point. They did this by selecting for r, a symmetric four interval wavelet set $K_r \equiv K_r^- \cup K_r^+$, where $K_r^- = -K_r^+$, and

$$K_r^+ = \left[\frac{2^{r-1}}{2^{r+1}-1}, \, \frac{1}{2}\right] \cup \left[2^{r-1}, \, \frac{2^{2r}}{2^{r+1}-1}\right]. \tag{1}$$

In this paper, we obtain that such wavelet sets are plentiful. Our construction proceeds as follows:

(i) For an $n \in \mathbb{N}$, we select a $2n$-interval symmetric wavelet set by making use of Theorem 3 of [1], in Section 3. For even n, the wavelet set obtained is denoted by $W_{n,E}$ while for odd n, by $W_{n,O}$.

(ii) We choose a positive number δ_n such that an $\epsilon \in (0, \delta_n)$ provides a symmetric wavelet set $W_{n,E,\epsilon}$ (or $W_{n,O,\epsilon}$) according as n is even (or odd), the characteristic function on which has compact support not vanishing around any neighbourhood of the origin. This is achieved by invoking the technique employed in [9] for obtaining wavelet sets having the origin as its accumulation point, in Section 4.

For $n = 2$, the family $W_{2,E,\epsilon}$ is one amongst the families constructed in [1].

Additionally, we construct such a family of wavelet sets from a given member of the following family of symmetric six interval wavelet sets $K \equiv K(s,t,v) = K^- \cup K^+$, where $K^- = -K^+$,

$$\begin{aligned} K^+ = &\left[\frac{2^s(2t+1)}{2^v-1}, \, \frac{2^{s+2}t}{2^v-2^{s+2}}\right] \cup \left[\frac{2^v t}{2^v-2^{s+2}}, \, \frac{2t+1}{2}\right] \\ &\cup \left[2^s(2t+1), \, \frac{2^{s+v}(2t+1)}{2^v-1}\right], \end{aligned} \tag{2}$$

and s, t, v are non-negative integers such that $t \geq 1$ and $2^v > (2t+1)2^{s+2}$ provided in [1].

Based on the similar technique, we construct a family of wavelet sets in Section 5, which is non-symmetric having the origin as its accumulation point by making use of three-interval wavelet sets, precisely given by

$$W_{j,p} \equiv \left[-\left(1-\frac{2p+1}{2^{j+1}-1}\right), -\frac{1}{2}\left(1-\frac{2p+1}{2^{j+1}-1}\right)\right] \cup \left[\frac{p+1}{2^{j+1}-1}, \frac{2p+1}{2^{j+1}-1}\right] \cup \left[\frac{2^j(2p+1)}{2^{j+1}-1}, \frac{2^{j+1}(p+1)}{2^{j+1}-1}\right], \quad (3)$$

where $j \geq 2$ and an integer p satisfying $1 \leq p \leq 2^j - 2$ [11].

In Section 6, we obtain a larger class of H^2-wavelet sets having the origin as an accumulation point via two interval H^2-wavelet sets which contains the one given by Behera in [2].

Section 2, provides necessary pre-requisites. The last Section 7, is devoted to obtain non-MSF non-MRA wavelets from $2n$-interval wavelet sets $W_{n,E}$ and $W_{n,O}$ as selected in Section 3. The construction of families of non-MSF wavelets have been considered earlier by several workers including Bownik [5], Behera [2], Gu and Han [10] and Vyas [14, 15]. The interest in finding out non-MSF wavelets arose due to a result obtained by Chui and Shi in [4] according to which for the dilation a which satisfies $a^j \notin \mathbb{Q}$, for all positive integers j, there exist only MSF-wavelets.

2 Pre-requisites.

The collection of all Lebesgue integrable functions on $\mathbb{R}$ is denoted by $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ denotes that of all Lebesgue square integrable functions on $\mathbb{R}$. Functions which are equal almost everywhere are identified. With the usual addition and scalar multiplication of functions together with the inner-product $\langle f, g\rangle$ of $f, g \in L^2(\mathbb{R})$ defined by

$$\langle f, g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt,$$

$L^2(\mathbb{R})$ becomes a Hilbert space. The Fourier transform is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi i\xi t}\, dt,$$

where $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. This extends uniquely to an operator on $L^2(\mathbb{R})$.

The classical Hardy space $H^2(\mathbb{R})$ is the collection of all Lebesgue square integrable functions whose Fourier transform is supported in $\mathbb{R}^+$:

$$H^2(\mathbb{R}) := \left\{ f \in L^2(R) :\ \hat{f}(\xi) = 0 \ \ for \ \ a.e. \ \ \xi \leq 0 \right\}.$$

A function $\psi \in L^2(\mathbb{R})$ (resp. $\psi \in H^2(\mathbb{R})$) is called an *orthonormal wavelet* (resp. H^2-*wavelet*) of $L^2(\mathbb{R})$ (resp. $H^2(\mathbb{R})$) if the system

$$\left\{ 2^{j/2}\psi(2^j t - k) :\ j, k \in \mathbb{Z} \right\}$$

forms an orthonormal basis for $L^2(\mathbb{R})$ (resp. $H^2(\mathbb{R})$).

We use the following characterization of an orthonormal wavelet for $L^2(\mathbb{R})$ and $H^2(\mathbb{R})$ available in [12].

Result 2.1. A function $\psi \in L^2(\mathbb{R})$ (resp. $\psi \in H^2(\mathbb{R})$) is an orthonormal wavelet (resp. H^2-wavelet) iff

(i) $||\psi||_2 = 1$,

(ii) $\rho(\xi) = \sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^j\xi)|^2 = \chi_{\mathbb{R}}(\xi)$ (resp. $\chi_{\mathbb{R}^+}(\xi)$), for a.e. $\xi \in \mathbb{R}$,

(iii) $t_q(\xi) = \sum_{j\geq 0} \widehat{\psi}(2^j\xi)\, \overline{\widehat{\psi}(2^j(\xi + q))} = 0$, for a.e. $\xi \in \mathbb{R}$ and for $q \in 2\mathbb{Z}+1$.

A *wavelet set* [7] (resp. H^2-*wavelet set*) is a measurable set W of the real line $\mathbb{R}$ (resp. $\mathbb{R}^+$) such that the characteristic function χ_W on W is equal to the Fourier transform $\hat{\psi}$ for some orthonormal wavelet (resp. H^2-wavelet) ψ in $L^2(\mathbb{R})$ (resp. $H^2(\mathbb{R})$). An MSF *wavelet* (resp. H^2-MSF *wavelet*) ψ is a wavelet (resp. H^2-wavelet) which is associated with a wavelet set (resp. H^2-wavelet set) W in the sense that the support of $\hat{\psi}$ is W [7, 8]. We use the following characterization of a wavelet set (resp. H^2-wavelet set) [7, 12].

Result 2.2. A measurable set $W \subset \mathbb{R}$ (resp. $W \subset \mathbb{R}^+$) is a wavelet set (resp. H^2-wavelet set) if and only if

(i) $\dot{\bigcup}_{n\in\mathbb{Z}}(W + n) = \mathbb{R}$ a.e.,

(ii) $\dot{\bigcup}_{n\in\mathbb{Z}}(2^n W) = \mathbb{R}$ (resp. $\mathbb{R}^+$) a.e.,

where $\dot{\bigcup}$ denotes the disjoint union.

From the above characterization of a wavelet set, we obtain:

Lemma 2.3 ([14]). *Define* $\tau : \mathbb{R} \to [0,\ 1)$ *by* $\tau(x) = x + p$, *where* p *is an integer depending on* x. *Then*

(a) $\tau(E) = \tau(E + k)$, *for* $k \in \mathbb{Z}$ *and* E *is a measurable set in* $\mathbb{R}$,

(b) for any disjoint measurable sets E *and* F *of* $\mathbb{R}$ *contained in a wavelet set* W, $\tau(E) \cap \tau(F) = \phi$.

Definition 2.4. ([12]) A pair $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$ consisting of a family $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ together with a function $\varphi \in V_0$ is called a *multiresolution analysis* (MRA) if it satisfies the following conditions:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,

(b) $f \in V_j$ if and only if $f(2(\cdot)) \in V_{j+1}$ for all $j \in \mathbb{Z}$,

(c) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$,

(d) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$,

(e) $\{\varphi(\cdot - k)\ :\ k \in \mathbb{Z}\}$ is an orthonormal basis for V_0.

The function φ is called a *scaling function* for the given MRA. An MRA determines a function ψ lying in the orthogonal complement of V_0 in V_1 which is an orthonormal wavelet for $L^2(\mathbb{R})$. Such a ψ is called an MRA *wavelet* arising through the MRA $(\{V_j\}_{j\in\mathbb{Z}}, \varphi)$.

A multiresolution analysis for $H^2(\mathbb{R})$ and H^2-MRA wavelet can be described similarly.

For an orthonormal wavelet ψ,

$$D_\psi(\xi) = \sum_{j=1}^{\infty} \sum_{k\in\mathbb{Z}} \left|\hat{\psi}(2^j(\xi + k))\right|^2$$

describes the *dimension function* D_ψ for ψ. We use the following characterization which works for both an MRA wavelet and an H^2-MRA wavelet.

Result 2.5. ([12]) A wavelet $\psi \in L^2(\mathbb{R})$ (resp. $\psi \in H^2(\mathbb{R})$) is an MRA (resp. H^2-MRA) wavelet iff $D_\psi(\xi) = 1$, for almost every $\xi \in \mathbb{R}$.

Definition 2.6. ([7]) A measurable set A is said to be

(a) *translation equivalent* to a measurable set B if there exists a measurable partition $\{A_n\}$ of A and $k_n \in \mathbb{Z}$ such that $\{A_n + k_n\}$ is a partition of B.

(b) *dilation equivalent* to a measurable set B if there exists a measurable partition $\{A'_n\}$ of A and $j_n \in \mathbb{Z}$ such that $\{2^{j_n} A'_n\}$ is a partition of B.

As a consequence of Result 2.2, we have the following:

Corollary 2.7. *Let K and W be subsets of $\mathbb{R}$ (resp. $\mathbb{R}^+$), and W be both translation and dilation equivalent to K. Then W is a wavelet set (resp. H^2-wavelet set) if and only if K is a wavelet set (resp. H^2-wavelet set).*

3 Symmetric Wavelet Sets $W_{n,E}$ and $W_{n,O}$ with $2n$-components.

In this section, we obtain wavelet sets $W_{n,E}$ for even positive integer n and $W_{n,O}$ for odd positive integer n consisting of $2n$-components, based on a result of Arcozzi, Behera and Madan [1, Theorem 3], which we briefly describe below.

Choosing a set $\mathcal{P}$ containing n-elements $P_j \equiv P[\lambda_j,\ m_j] = \left(2^{-\lambda_j},\ 2^{-\lambda_j} m_j\right)$, $j = 1, 2, \ldots, n$, in the Euclidean plane such that $\lambda_j \in \mathbb{Z}$ and $m_j \in \mathbb{N} \cup \{0\}$ satisfying the following:

$$\lambda_1 = 0,\ 4m_1 = 2^{-\lambda_n} \left(2m_n + 1\right)$$

and

$$0 = a_0 < a_1 < a_2 < \ldots < a_{n-1} < a_n = \frac{1}{2},$$

where

$$a_j = -\frac{m_j 2^{-\lambda_j} - m_{j+1} 2^{-\lambda_{j+1}}}{2^{-\lambda_j} - 2^{-\lambda_{j+1}}},$$

Arcozzi, Behera and Madan [1, Theorem 3] obtain the following Theorem.

Theorem 3.1. *For $W_n^+ = I_1^+ \cup I_2^+ \cup \ldots \cup I_n^+$, where*

$$I_j^+ = [a_{j-1},\ a_j] + m_j,\ \ j = 1,\ 2, \ldots, n,$$

$W_n = W_n^- \cup W_n^+$ is a symmetric wavelet set for $L^2(\mathbb{R})$ having $2n$-intervals.

Now, we provide two specific kinds of symmetric wavelet sets, the first has $4m$-intervals while the second has $(4m + 2)$-intervals, where $m \in \mathbb{N}$. The first is obtained by choosing an even positive integer, and the second by choosing an odd positive integer.

Example 3.2. Let $n \in 2\mathbb{N}$. Define λ_j's and m_j's, where $j = 1, 2, \ldots, n$, as follows:

$$\lambda_j = \begin{cases} -\frac{j-1}{2} & \text{for } 1 \leq j \leq n \text{ and } j \text{ odd} \\ \frac{j}{2} - n - 2 & \text{for } 1 < j \leq n \text{ and } j \text{ even} \end{cases}$$

$$m_j = \begin{cases} 2^{(n-j+1)/2} & \text{for } 1 \leq j \leq n \text{ and } j \text{ odd} \\ 0 & \text{for } 1 < j \leq n \text{ and } j \text{ even} \end{cases}$$

With the help of these m_j's and λ_j's, we obtain P_j, where $j = 1, 2, \ldots, n$, as follows:

$$P_j = \begin{cases} P\left[\frac{1-j}{2}, \frac{n-j+1}{2}\right] = \left(2^{\frac{j-1}{2}}, 2^{\frac{n}{2}}\right) & \text{for } 1 \leq j \leq n \text{ and } j \text{ odd} \\ P\left[-n + \frac{j}{2} - 2, 0\right] = \left(2^{n-\frac{j}{2}+2}, 0\right) & \text{for } 1 < j \leq n \text{ and } j \text{ even} \end{cases}$$

Thus, a_j for $j = 1, 2, \ldots, n-1$, comes out to be

$$a_j = \begin{cases} \frac{2^{\frac{n}{2}}}{2^{\frac{2n-j+3}{2}} - 2^{\frac{j-1}{2}}} & \text{for } 1 \leq j \leq n \text{ and } j \text{ odd} \\ \frac{2^{\frac{n}{2}}}{2^{\frac{2n-j+4}{2}} - 2^{\frac{j}{2}}} & \text{for } 1 < j \leq n \text{ and } j \text{ even} \end{cases}$$

Therefore, the positive side $W^+_{n,E}$ of the wavelet set, denoted by $W_{n,E}$, is of the form:

$$W^+_{n,E} = \left[2^{n/2},\ 2^{n/2} + \frac{2^{n/2}}{2^{n+1}-1}\right] \cup \left[\frac{2^{n/2}}{2^{n+1}-1},\ \frac{2^{n/2}}{2^{n+1}-2}\right] \cup \ldots \cup \left[\frac{2}{7},\ \frac{1}{2}\right].$$

Example 3.3. Let $n \in 2\mathbb{N} + 1$. Define λ_j's and m_j's, where $j = 1, 2, \ldots, n$, as follows:

$$\lambda_j = \begin{cases} 0 & \text{for } j = 1 \\ \frac{n-j-6}{2} & \text{for } 1 < j \leq n \text{ and } j \text{ odd} \\ \frac{j-n-9}{2} & \text{for } 1 < j \leq n-1 \text{ and } j \text{ even} \end{cases}$$

$$m_j = \begin{cases} 6 & \text{for } j = 1 \\ 2^{(n-j)/2} & \text{for } 1 < j \leq n \text{ and } j \text{ odd} \\ 0 & \text{for } 1 < j \leq n-1 \text{ and } j \text{ even} \end{cases}$$

With the help of these m_j's and λ_j's, we obtain P_j, where $j = 1, 2, \ldots, n$, as follows:

$$P_j = \begin{cases} P\,[0,6] = (1,6) & \text{for } j=1 \\ P\left[\frac{n-j-6}{2}, 2^{\frac{n-j}{2}}\right] = \left(2^{\frac{6+j-n}{2}}, 8\right) & \text{for } 1 < j \le n \text{ and } j \text{ odd} \\ P\left[\frac{j-9-n}{2}, 0\right] = \left(2^{\frac{9+n-j}{2}}, 0\right) & \text{for } 1 < j \le n-1 \text{ and } j \text{ even} \end{cases}$$

Thus a_j, for $j = 1, 2, \ldots, n-1$, comes out to be

$$a_j = \begin{cases} \frac{6}{2^{\frac{n+7}{2}}-1} & \text{for } j=1 \\ \frac{8\cdot 2^{\frac{n-6-j}{2}}}{2^{n-j+1}-1} & \text{for } 1 \le j \le n \text{ and } j \text{ odd} \\ \frac{8\cdot 2^{\frac{n-7-j}{2}}}{2^{n-j+1}-1} & \text{for } 1 < j \le n-1 \text{ and } j \text{ even} \end{cases}$$

Therefore, the positive side $W^+_{n,O}$ of the wavelet set, denoted by $W_{n,O}$, is of the form:

$$W^+_{n,O} = \left[6,\ 6 + \frac{6}{2^{\frac{n+7}{2}} - 1}\right] \cup \left[\frac{6}{2^{\frac{n+7}{2}} - 1},\ \frac{8 \cdot 2^{\frac{n-9}{2}}}{2^{n-1} - 1}\right] \cup \ldots \cup \left[\frac{4}{3},\ \frac{3}{2}\right].$$

4 Symmetric Wavelet Sets Accumulating at the Origin.

With the help of wavelet sets $W_{n,E}$ and $W_{n,O}$ obtained in the last section, we provide families of bounded symmetric wavelet sets having the origin as their accumulation point. Also, we obtain such a family of wavelet sets considering six-interval wavelet sets as described by (2) in the introduction.

Since the technique employed in these constructions are the same, we provide details in the following theorem only.

Theorem 4.1. *For $n \in 2\mathbb{N}$ and $\epsilon \in (0,\ \delta_n)$, where $\delta_n = \frac{2^{n/2}}{2(2^{n+1}-2)(2^{n+1}-1)}$, there exists a bounded symmetric wavelet set $W_{n,E,\epsilon}$ having the origin as an accumulation point.*

Proof. Selecting $b_n = \frac{2^{n/2}}{2^{n+1}-1}$, we consider the intervals $S_1 = \left[\frac{b_n}{2} + \frac{\epsilon}{2^{n+1}},\ \frac{b_n}{2} + \epsilon\right]$, $S_2 = \left[b_n + 2\epsilon,\ \frac{2^{n/2}}{2^{n+1}-2}\right]$, and $S_3 = \left[2^{n+1}b_n,\ 2^{n+1}b_n + 2\epsilon\right]$. Since $\epsilon \in (0,\ \delta_n)$, S_2 is a non-empty set. Setting

$$E_0 = S_1 + 2^{n/2},\ F_0 = \frac{1}{2^{n+2}} E_0,$$

and for $r \geq 1$,

$$E_r = F_{r-1} + 2^{n/2}, \; F_r = \frac{1}{2^{r+n+2}} E_r,$$

we denote

$$\left(I_1^+ - \bigcup_{r=0}^{\infty} E_r \right) \cup \left(\bigcup_{r=0}^{\infty} F_r \right) \cup (S_1 \cup S_2 \cup S_3) \cup I_3^+ \cup I_4^+ \cup \ldots \cup I_n^+$$

by $W_{n,E,\epsilon}^+$, and define $W_{n,E,\epsilon} = W_{n,E,\epsilon}^- \cup W_{n,E,\epsilon}^+$, where $W_{n,E,\epsilon}^- = -W_{n,E,\epsilon}^+$.

To prove that $W_{n,E,\epsilon}$ is a wavelet set, we make use of Corollary 2.7, according to which $W_{n,E,\epsilon}$ is to be shown translation as well as dilation equivalent to a wavelet set, in general, and hence to the wavelet set $W_{n,E}$, in particular. On account of the symmetry of wavelet sets, it suffices to show that $W_{n,E,\epsilon}^+$ is both translation and dilation equivalent to $W_{n,E}^+$.

First, by induction, we obtain that $E_r \subset I_1^+$, for all $r \geq 0$. Observing that $b_n + 2^{n/2} = 2^{n+1} b_n$, we have $[0, b_n] + 2^{n/2} = \left[2^{n/2}, 2^{n+1} b_n\right] = I_1^+$, and hence $E_0 = S_1 + 2^{n/2} \subset [0, b_n] + 2^{n/2} = I_1^+$. Now, assume that $E_m \subset I_1^+$. Then $F_m = 2^{-(m+n+2)} E_m \subset 2^{-(m+1)} [0, b_n] \subset [0, b_n]$, and hence $E_{m+1} = F_m + 2^{n/2} \subset [0, b_n] + 2^{n/2} = I_1^+$.

As intervals $E_r, r \geq 0$ lie inside the interval I_1^+, and E_{r+1} lies to the left of E_r, for all $r \geq 0$, F_{r+1} lies to the left of F_r, for all $r \geq 0$.

Because the sets $I_3^+, I_4^+, \ldots, I_n^+$ appear in both the partitions of $W_{n,E,\epsilon}^+$ and also of $W_{n,E}^+$, that $W_{n,E,\epsilon}^+$ is dilation and also translation equivalent to $W_{n,E}^+$ follow from (A) and (B), respectively.

(A) (i) $\frac{1}{2^{n+1}} S_3 \;\cup\; 2S_1 \;\cup\; S_2$
$= \left[b_n, b_n + \frac{2\epsilon}{2^{n+1}}\right] \cup \left[b_n + \frac{2\epsilon}{2^{n+1}}, b_n + 2\epsilon\right] \cup \left[b_n + 2\epsilon, \frac{2^{n/2}}{2^{n+1}-2}\right]$
$= \left[b_n, \frac{2^{n/2}}{2^{n+1}-2}\right] = I_2^+$.

(ii) $\left(I_1^+ - \bigcup_{r=0}^{\infty} E_r\right) \cup \left(\bigcup_{r=0}^{\infty} 2^{r+n+2} F_r\right)$
$= \left(I_1^+ - \bigcup_{r=0}^{\infty} E_r\right) \cup \left(\bigcup_{r=0}^{\infty} E_r\right) = I_1^+$.

(B) (i) $(S_3 - 2^{n/2}) \cup S_2$
$= [b_n, b_n + 2\epsilon] \;\cup\; \left[b_n + 2\epsilon, \frac{2^{n/2}}{2^{n+1}-2}\right]$
$= \left[b_n, \frac{2^{n/2}}{2^{n+1}-2}\right] = I_2^+$.

(ii) $\left(I_1^+ - \bigcup_{r=0}^{\infty} E_r\right) \;\cup\; \left(\bigcup_{r=0}^{\infty} \left(F_r + 2^{n/2}\right)\right) \;\cup\; \left(S_1 + 2^{n/2}\right) = I_1^+$.

Further, since a neighbourhood of the origin intersects $\cup_{r=0}^{\infty} F_r$, the origin is an accumulation point of the wavelet set $W_{n,E,\epsilon}$. □

Theorem 4.2. *For $n \in 2\mathbb{N}+1$ and $\epsilon \in (0,\ \delta_n)$, where $\delta_n = \frac{2^{\frac{n-5}{2}}(2^{\frac{n+3}{2}}-1)+3}{(2^{n-1}-1)(2^{\frac{n+7}{2}}-1)}$, there exists a bounded symmetric wavelet set $W_{n,O,\epsilon}$ having the origin as an accumulation point.*

PROOF. With $b_n = \frac{6}{2^{\frac{n+7}{2}}-1}$, we consider the following intervals:

$S_1 = \left[\frac{b_n}{2} + \frac{\epsilon}{2^{\frac{n+7}{2}}},\ \frac{b_n}{2} + \epsilon\right]$, $S_2 = \left[b_n + 2\epsilon,\ \frac{8 \cdot 2^{\frac{n-9}{2}}}{2^{n-1}-1}\right]$, and
$S_3 = \left[2^{\frac{n+7}{2}} b_n,\ 2^{\frac{n+7}{2}} b_n + 2\epsilon\right]$.
That S_2 is a non-empty set follows on account of the choice of ϵ. Setting

$$E_0 = S_1 + 6,\ F_0 = \frac{1}{2^{\frac{n+9}{2}}} E_0,$$

and for $r \geq 1$,

$$E_r = F_{r-1} + 6,\ F_r = \frac{1}{2^{r+\frac{n+9}{2}}} E_r,$$

we denote

$$\left(I_1^+ - \bigcup_{r=0}^{\infty} E_r\right) \cup \left(\bigcup_{r=0}^{\infty} F_r\right) \cup (S_1 \cup S_2 \cup S_3) \cup I_3^+ \cup I_4^+ \cup \ldots \cup I_n^+,$$

by $W_{n,O,\epsilon}^+$. Then

$$W_{n,O,\epsilon} = W_{n,O,\epsilon}^- \cup W_{n,O,\epsilon}^+, \text{ where } W_{n,O,\epsilon}^- = -W_{n,O,\epsilon}^+,$$

is the required wavelet set. □

Recalling wavelet sets with dilation by 2 consisting of six intervals which are symmetric about the origin as provided by (2) in the introduction, we write

$$K^+ = I^+ \cup J^+ \cup H^+, \text{ and } K^- = -K^+,$$

where $I^+ = \left[\frac{2^s(2t+1)}{2^v-1},\ \frac{2^{s+2}t}{2^v-2^{s+2}}\right]$, $J^+ = \left[\frac{2^v t}{2^v-2^{s+2}},\ \frac{2t+1}{2}\right]$, and $H^+ = \left[2^s(2t+1),\ \frac{2^{s+v}(2t+1)}{2^v-1}\right]$. Now, we have the following:

Theorem 4.3. *For non-negative integers s, t, v such that $t \geq 1$, $2^v > (2t+1)2^{s+2}$ and $\epsilon \in (0, \delta_{s,t,v})$, where $\delta_{s,t,v} = \frac{2^{s+v}(2t-1)+2^{s+2}t(2^{s+1}-1)+2^{2s+2}}{2(2^v-2^{s+2})(2^v-1)}$, there exists a bounded symmetric wavelet set $W_{s,t,v,\epsilon}$ having the origin as an accumulation point.*

PROOF. The construction of $W_{s,t,v,\epsilon}$ is given below.

For $a_{s,t,v} = \frac{2^s(2t+1)}{2^v-1}$, we consider the intervals $S_1 = \left[\frac{a_{s,t,v}}{2} + \frac{\epsilon}{2^v},\ \frac{a_{s,t,v}}{2} + \epsilon\right]$, $S_2 = \left[a_{s,t,v} + 2\epsilon,\ \frac{2^{s+2}t}{2^v-2^{s+2}}\right]$, and $S_3 = [2^v a_{s,t,v},\ \ 2^v a_{s,t,v} + 2\epsilon]$.

The choice of ϵ ensures that S_2 is a non-empty set. Setting

$$E_0 = S_1 + 2^s(2t+1),\ F_0 = \frac{1}{2^{v+1}}E_0,$$

and for $r \geq 1$,

$$E_r = F_{r-1} + 2^s(2t+1),\ F_r = \frac{1}{2^{r+v+1}}E_r,$$

we have

$$W^+_{s,t,v,\epsilon} \equiv \left(H^+ - \bigcup_{r=0}^{\infty} E_r\right) \cup \left(\bigcup_{r=0}^{\infty} F_r\right) \cup (S_1 \cup S_2 \cup S_3) \cup J^+,$$

as the portion of $W_{s,t,v,\epsilon}$ on the positive side of the real line. □

5 Wavelet Sets Accumulating at the Origin from Three-interval Wavelet Sets.

In this section, we construct a family of wavelet sets accumulating at the origin from three-interval wavelet sets $W_{j,p}$, where $j \geq 2$ and $1 \leq p \leq 2^j - 2$, and

$$W_{j,p} \equiv I_{j,p}\ \cup J_{j,p}\ \cup H_{j,p},$$

with $I_{j,p} = \left[-\left(1 - \frac{2p+1}{2^{j+1}-1}\right),\ -\frac{1}{2}\left(1 - \frac{2p+1}{2^{j+1}-1}\right)\right]$, $J_{j,p} = \left[\frac{p+1}{2^{j+1}-1},\ \frac{2p+1}{2^{j+1}-1}\right]$, and $H_{j,p} = \left[\frac{2^j(2p+1)}{2^{j+1}-1},\ \frac{2^{j+1}(p+1)}{2^{j+1}-1}\right]$. These wavelet sets are non-symmetric.

Theorem 5.1. *For $j \geq 2$, an integer p satisfying $1 \leq p \leq 2^j - 2$ and $\epsilon \in (0,\ \delta_{j,p})$, where $\delta_{j,p} = \frac{p}{2(2^{j+1}-1)}$, there exists a bounded wavelet set $W_{j,p,\epsilon}$ having the origin as an accumulation point.*

Proof. Taking $a_{j,p} = \frac{(p+1)}{2^{j+1}-1}$, we consider the intervals $S_1 = \left[\frac{a_{j,p}}{2} + \frac{\epsilon}{2^{j+1}}, \frac{a_{j,p}}{2} + \epsilon\right]$, $S_2 = \left[a_{j,p} + 2\epsilon, \frac{2p+1}{2^{j+1}-1}\right]$, and $S_3 = \left[2^{j+1}a_{j,p}, 2^{j+1}a_{j,p} + 2\epsilon\right]$. The choice of ϵ ensures that S_2 is a non-empty set. Setting

$$E_0 = S_1 + (p+1),\ F_0 = \frac{1}{2^{j+2}}E_0,$$

and for $n \geq 1$,

$$E_n = F_{n-1} + (p+1),\ F_n = \frac{1}{2^{n+j+2}}E_n,$$

we obtain

$$W_{j,p,\epsilon} \equiv (H_{j,p} - \textstyle\bigcup_{n=0}^{\infty} E_n) \cup (\textstyle\bigcup_{n=0}^{\infty} F_n) \cup (S_1 \cup S_2 \cup S_3) \cup I_{j,p},$$

to be the required wavelet set. □

6 H^2-Wavelet Sets Accumulating at the Origin.

In this section, we construct a family of H^2-wavelet sets accumulating at the origin by considering certain specific H^2-wavelet sets consisting of two intervals, which are precisely given by

$$K_{r,k} = \left[\frac{k+1}{2^{r+1}-1}, \frac{k}{2^r-1}\right] \cup \left[\frac{2^r k}{2^r-1}, \frac{2^{r+1}(k+1)}{2^{r+1}-1}\right],$$

where $r \in \mathbb{N}$ and k is an integer satisfying $1 \leq k < 2(2^r - 1)$. In fact, we consider two interval H^2-wavelet sets for $r \in \mathbb{N}$ and $k = 2^l - 1,\ 1 \leq l \leq r$, denoted by K_r^l.

We write

$$K_r^l = I_r^l \cup J_r^l,$$

where $I_r^l = \left[\frac{2^l}{2^{r+1}-1}, \frac{2^l-1}{2^r-1}\right]$ and $J_r^l = \left[\frac{2^r(2^l-1)}{2^r-1}, \frac{2^{r+l+1}}{2^{r+1}-1}\right]$.

Theorem 6.1. *For $r \in \mathbb{N}$, an integer l satisfying $1 \leq l \leq r$, and $\epsilon \in (0, \delta_r^l)$, where $\delta_r^l = \frac{2^r(2^l-2)+1}{2(2^r-1)(2^{r+1}-1)}$, there exists bounded H^2-wavelet set $K_{r,\epsilon}^l$ having the origin as an accumulation point.*

Proof. For $a_r^l = \frac{2^l}{2^{r+1}-1}$, we consider the intervals $S_1 = \left[\frac{a_r^l}{2} + \frac{\epsilon}{2^{r+1}},\ \frac{a_r^l}{2} + \epsilon\right]$, $S_2 = \left[a_r^l + 2\epsilon,\ \frac{2^l-1}{2^r-1}\right]$, and $S_3 = \left[2^{r+1}a_r^l,\ \ 2^{r+1}a_r^l + 2\epsilon\right]$. The choice of ϵ ensures that S_2 is a non-empty set. Setting $E_0 = S_1 + 2^l,\ F_0 = \frac{1}{2^{r+2}}E_0$, and for $n \geq 1$, $E_n = F_{n-1} + 2^l,\ F_n = \frac{1}{2^{n+r+2}}E_n$, we obtain

$$K_{r,\epsilon}^l \equiv \left(J_r^l - \bigcup_{n=0}^{\infty} E_n\right) \cup \left(\bigcup_{n=0}^{\infty} F_n\right) \cup (S_1\ \cup\ S_2 \cup S_3)\,,$$

to be the required H^2-wavelet set. □

7 Non-MSF, Non-MRA Wavelets for $L^2(\mathbb{R})$ from $W_{n,E}$ and $W_{n,O}$.

Employing Examples 3.2 and 3.3, we provide non-MSF, non-MRA wavelets in this section. The technique of constructing such wavelets is similar to the one utilized in [14, 15].

7.1 Non-MSF non-MRA wavelets from $W_{n,E}$.

Lemma 7.1. *Under the notation already described, for $(m, n) \in \mathbb{N} \times 2\mathbb{N}$, the following hold:*
(a) $2^{-m}I_2^+ + 2^{n/2} \subset I_1^+$,
(b) $2^{-m}I_2^- - 2^{n/2} \subset I_1^-$,
(c) $I_2^+ + 2^{m+\frac{n}{2}} \subset 2^m I_1^+$,
(d) $I_2^- - 2^{m+\frac{n}{2}} \subset 2^m I_1^-$.

Proof. This is straightforward. □

Theorem 7.2. *For $(m, n) \in \mathbb{N} \times 2\mathbb{N}$, the function $\psi_{m,n}$ defined by*

$$\widehat{\psi}_{m,n}(\xi) = \begin{cases} \frac{1}{\sqrt{2}} & \text{if } \xi \in I_2^+ \cup 2^{-m}I_2^+ \cup (2^{-m}I_2^+ + 2^{n/2}) \cup I_2^- \\ & \quad \cup\, 2^{-m}I_2^- \cup (2^{-m}I_2^- - 2^{n/2}), \\ \frac{-1}{\sqrt{2}} & \text{if } \xi \in (I_2^+ + 2^{m+\frac{n}{2}}) \cup (I_2^- - 2^{m+\frac{n}{2}}), \\ 1 & \text{if } \xi \in (I_1^+ - (2^{-m}I_2^+ + 2^{n/2})) \cup I_3^+ \cup \ldots \cup I_n^+ \cup \\ & \quad (I_1^- - (2^{-m}I_2^- - 2^{n/2})) \cup I_3^- \cup \ldots \cup I_n^-, \\ 0 & \text{otherwise}, \end{cases}$$

is a non-MSF wavelet for $L^2(\mathbb{R})$.

Proof. It is similar to the proof of Theorem 3.4 in [14], by making use of Results 2.1 and 2.2 together with Lemmas 2.3 and 7.1. □

Theorem 7.3. *The wavelet $\psi_{m,n}$, where $(m,n) \in \mathbb{N} \times 2\mathbb{N}$, defined as in Theorem 7.2, is a non-MRA wavelet.*

Proof. To show that $\psi_{m,n}$ is a non-MRA wavelet for $L^2(\mathbb{R})$, we use Result 2.5. For $(m,n) \in \mathbb{N} \times 2\mathbb{N}$, $D_{\psi_{m,n}} \geq 2$, on the interval $2^{-(m+1)}I_2^+$. Indeed,

$$D_{\psi_{m,n}}(\xi) \geq \left|\hat{\psi}_{m,n}(2\xi)\right|^2 + \left|\hat{\psi}_{m,n}(2\xi + 2^{\frac{n}{2}})\right|^2 + \left|\hat{\psi}_{m,n}(2^{m+1}\xi)\right|^2 + \left|\hat{\psi}_{m,n}(2^{m+1}\xi + 2^{m+\frac{n}{2}})\right|^2,$$

and hence, the assertion follows by noting that $2\xi \in 2^{-m}I_2^+$, $2(\xi + 2^{\frac{n-2}{2}}) \in (2^{-m}I_2^+ + 2^{\frac{n}{2}})$, $2^{m+1}\xi \in I_2^+$ and $2^{m+1}(\xi + 2^{\frac{n-2}{2}}) \in I_2^+ + 2^{m+\frac{n}{2}}$, where $\xi \in 2^{-(m+1)}I_2^+$. □

7.2 Non-MSF non-MRA wavelets from $W_{n,O}$.

Lemma 7.4. *Under the notation already described, for $(m,n) \in \mathbb{N} \times 2\mathbb{N}+1$, the following hold:*

(a) $2^{-m}I_2^+ + 6 \subset I_1^+$,

(b) $2^{-m}I_2^- - 6 \subset I_1^-$,

(c) $I_2^+ + 6 \cdot 2^m \subset 2^m I_1^+$,

(d) $I_2^- - 6 \cdot 2^m \subset 2^m I_1^-$.

Proof. This is straightforward. □

Theorem 7.5. *For $(m,n) \in \mathbb{N} \times 2\mathbb{N}+1$, the function $\psi_{m,n}$ defined by*

$$\widehat{\psi}_{m,n}(\xi) = \begin{cases} 1/\sqrt{2} & \text{if } \xi \in I_2^+ \cup 2^{-m}I_2^+ \cup (2^{-m}I_2^+ + 6) \cup I_2^- \\ & \quad \cup\, 2^{-m}I_2^- \cup (2^{-m}I_2^- - 6), \\ -1/\sqrt{2} & \text{if } \xi \in (I_2^+ + 6 \cdot 2^m) \cup (I_2^- - 6 \cdot 2^m), \\ 1 & \text{if } \xi \in (I_1^+ - (2^{-m}I_2^+ + 6)) \cup I_3^+ \cup \ldots \cup I_n^+ \cup \\ & \quad (I_1^- - (2^{-m}I_2^- - 6)) \cup I_3^- \cup \ldots \cup I_n^-, \\ 0 & \text{otherwise,} \end{cases}$$

is a non-MSF wavelet for $L^2(\mathbb{R})$.

Proof. It is similar to that of Theorem 7.2. We have to simply use Results 2.1 and 2.2 together with Lemmas 2.3 and 7.4. □

Theorem 7.6. *The function* $\psi_{m,n}$*, where* $(m,n) \in \mathbb{N} \times 2\mathbb{N}+1$*, defined as in Theorem 7.5, is a non-MRA wavelet.*

Proof. To show that $\psi_{m,n}$ is a non-MRA wavelet for $L^2(\mathbb{R})$, we use Result 2.5. For $(m,n) \in \mathbb{N} \times 2\mathbb{N}+1$, $D_{\psi_{m,n}} \geq 2$, on the interval $2^{-(m+1)} I_2^+$. Indeed,

$$D_{\psi_{m,n}}(\xi) \geq \left|\hat{\psi}_{m,n}(2\xi)\right|^2 + \left|\hat{\psi}_{m,n}(2\xi+6)\right|^2 + \left|\hat{\psi}_{m,n}(2^{m+1}\xi)\right|^2 + \left|\hat{\psi}_{m,n}(2^{m+1}\xi + 6 \cdot 2^m)\right|^2,$$

and hence, the assertion follows by noting that $2\xi \in 2^{-m} I_2^+$, $2(\xi+3) \in (2^{-m} I_2^+ + 6)$, $2^{m+1}\xi \in I_2^+$ and $2^{m+1}(\xi+3) \in I_2^+ + 6 \cdot 2^m$, where $\xi \in 2^{-(m+1)} I_2^+$. □

Acknowledgment. The authors thank the anonymous referee for reading the manuscript carefully and to their supervisor Professor K. K. Azad for his valuable help and guidance.

References

[1] N. Arcozzi, B. Behera and S. Madan, *Large classes of minimally supported frequency wavelets of* $L^2(\mathbb{R})$ *and* $H^2(\mathbb{R})$, J. Geom. Anal., **13** (2003), 557–579.

[2] B. Behera, *Non-MSF wavelets for the Hardy space* $H^2(\mathbb{R})$, Bull. Polish Acad. Sci. Math., **52** (2004), 169–178.

[3] B. Behera, *Wavelets with Fourier transform discontinuous at the origin*, Int. J. Wavelets Multiresolut. Inf. Process., **5** (2007), 679–683.

[4] C. K. Chui and X. Shi, *Orthonormal wavelets and tight frames with arbitrary real dilations*, Appl. Comput. Harmon. Anal., **9** (2000), 243–264.

[5] M. Bownik and D. Speegle, *The wavelet dimension function for real dilations and dilations admitting non-MSF wavelets*, Approximation Theory, X (St. Louis, MO, 2001), 63–85, Innov. Appl. Math., Vanderbilt Univ. Press, Nashville, TN, 2002.

[6] L. Brandolini, G. Garrigós, Z. Rzeszotnik and G. Weiss, *The behaviour at the origin of a class of band-limited wavelets*, Contemp. Math., **247** (1999), 75–91.

[7] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, Mem. Amer. Math. Soc., **134(640)** (1998).

[8] X. Fang and X. Wang, *Construction of minimally-supported-frequency wavelets*, J. Fourier Anal. Appl., **2** (1996), 315–327.

[9] G. Garrigós, *The characterization of wavelets and related functions and the connectivity of* $\alpha-$*localized wavelets on* $\mathbb{R}$, Ph. D. Thesis, Washington University, St. Louis (1998).

[10] Q. Gu and D. Han, *Phases for dyadic orthonormal wavelets*, J. Math. Phys., **43** (2002), 2690–2706.

[11] Y.-H. Ha, H. Kang, J. Lee and J. K. Seo, *Unimodular wavelets for* L^2 *and the Hardy space* H^2, Michigan Math. J., **41** (1994), 345–361.

[12] E. Hernández and G. Weiss, *A First Course on Wavelets*, With a foreword by Yves Meyer, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.

[13] W. R. Madych, *Some elementary properties of multiresolution analyses of* $L^2(R^n)$ *Wavelets*, 259–294, Wavelet Anal. Appl., **2**, Academic Press, Boston, MA, 1992.

[14] A. Vyas, *Construction of non-MSF non-MRA wavelets for* $L^2(\mathbb{R})$ *and* $H^2(\mathbb{R})$ *from MSF wavelets*, Bull. Polish Acad. Sci. Math., **57** (2009), 33–40.

[15] A. Vyas and R. Dubey, *Non-MSF wavelets from six interval MSF wavelets*, Int. J. Wavelets Multiresolut. Inf. Process., (to appear).