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Richárd Balka,* Department of Analysis, Eötvös Loránd University, Pázmány P. s. 1/c, H-1117, Budapest, Hungary. email: balkar@cs.elte.hu

Márton Elekes,† Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, P.O. Box 127, H-1364 Budapest, Hungary and Department of Analysis, Eötvös Loránd University, Pázmány P. s. 1/c, H-1117, Budapest, Hungary. email: emarci@renyi.hu

THE STRUCTURE OF CONTINUOUS RIGID FUNCTIONS OF TWO VARIABLES

Abstract

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *vertically rigid* if $graph(cf)$ is isometric to $graph(f)$ for all $c \neq 0$. In [1] we settled Janković's conjecture by showing that a continuous function $f : \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$. Now we prove that a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid if and only if, after a suitable rotation around the z-axis, $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ $(a, b, d, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous). The problem remains open in higher dimensions.

1 Introduction.

An easy calculation shows that the exponential function $f(x) = e^x$ has the somewhat 'paradoxical' property that cf is a translate of f for every $c > 0$. It is also easy to see that every function of the form $a + be^{kx}$ shares this property. Moreover, for every function of the form $f(x) = a + bx$ the graph of cf is isometric to the graph of f. In [2] Cain, Clark and Rose introduced the notion of vertical rigidity, which we now formulate for functions of several variables.

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Definition 1.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called *vertically rigid*, if $graph(cf)$ is isometric to $graph(f)$ for all $c \in (0, \infty)$. (Clearly, $c \in \mathbb{R} \setminus \{0\}$ would be the same.)

Then D. Janković formulated the following conjecture (see [2]).

Conjecture 1.2. *(D. Janković) A continuous function $f : \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R},\ k \neq 0)$.*

This conjecture, and more, was proved in [1].

Theorem 1.3. *Janković's conjecture holds. (It is actually enough to assume that f is vertically rigid for an uncountable set C, see Definition 1.6 below.)*

Later C. Richter gave generalisations of this theorem in various directions, see [3].

The main goal of the present paper is to give a complete description of the continuous vertically rigid functions of two variables.

Theorem 1.4. ***(Main Theorem)*** *A continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid if and only if, after a suitable rotation around the z-axis, $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ $(a, b, d, k \in \mathbb{R},\ k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous).*

As these classes look somewhat ad hoc, we do not even have conjectures in higher dimensions.

Problem 1.5. *Characterise the continuous vertically rigid functions of n variables for $n \geq 3$.*

In fact, for the proof of the Main Theorem we need the following technical generalisations.

Definition 1.6. If C is a subset of $(0, \infty)$ and $\mathcal{G}$ is a set of isometries of $\mathbb{R}^3$, then we say that f is vertically rigid *for a set $C \subset (0, \infty)$ via elements of $\mathcal{G}$* if for every $c \in C$ there exists a $\varphi \in \mathcal{G}$ such that $\varphi(graph(f)) = graph(cf)$. (If we do not mention C or $\mathcal{G}$ then C is $(0, \infty)$ and $\mathcal{G}$ is the set of all isometries.)

Definition 1.7. Let us say that a set $C \subset (0, \infty)$ *condensates to* ∞ if for every $r \in \mathbb{R}$ the set $C \cap (r, \infty)$ is uncountable.

The Main Theorem will immediately follow from the following, in which we just replace $(0, \infty)$ by a set C condensating to ∞.

Theorem 1.8. ***(Main Theorem, technical form)*** *Let $C \subset (0, \infty)$ be a set condensating to ∞. Then a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for C if and only if after a suitable rotation around the z-axis $f(x, y)$ is of the form $a+bx+dy$, $a+s(y)e^{kx}$ or $a+be^{kx}+dy$ $(a, b, d, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous).*

The structure of the proof will be the following. First we check in Section 2 that functions of the above forms are rigid. (Of course, they are all continuous.) Then we start proving the Main Theorem in more and more general settings. In Section 3 first we show that if all the isometries are horizontal translations then the vertically rigid function $f(x, y)$ is of the form $s(y)e^{kx}$ ($k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous) after a suitable rotation around the z-axis. The punchline here is that we can derive a simple functional equation from vertical rigidity (some sort of 'multiplicativity', see Lemma 3.5). Then we conclude this section by referring to a completely algebraic proof in [1] showing that if we allow arbitrary translations then $f(x, y)$ is of the form $a+s(y)e^{kx}$ ($a, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous) after a suitable rotation around the z-axis.

Then we start working on the case of general isometries. The central idea is to consider the set S_f of directions of segments connecting pairs of points on $graph(f)$ (see Definition 4.1). We collect the necessary properties of this set in Section 4. The set S_f has some sort of rigidity in that the transformation $f \mapsto cf$ distorts the shape of it, but the resulting set has to be isometric to the original one (see Definition 5.1 and Remark 5.2). Using these we determine the possible S_f's in Section 5, then in Section 6 we complete the proof by handling these cases using various methods.

Finally, in Section 7 we collect the open questions.

2 Functions of These Forms are Rigid.

Rotation of the graph around the z-axis does not affect vertical rigidity, so we can assume that f is of the given form without rotations.

Functions of the form $a + bx + dy$ are clearly vertically rigid.

Let $f(x, y) = a + s(y)e^{kx}$ ($a, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous). Then $cf(x, y) = f(x + \frac{\log c}{k}, y) + a(c - 1)$, so f is actually vertically rigid via translations in the xz-plane.

Before checking the third case we need a lemma.

Lemma 2.1. *Let $f(x, y) = g(x) + dy$, where $d > 0$ and let $c > 0$. If we rotate $graph(f)$ around the x-axis by the angle $\alpha_c = \arctan(cd) - \arctan(d)$, then the intersection of this rotated graph with the xy-plane is the graph of a function* *of the form $y = -w_{c,d}g(x)$, where $w_{c,d} > 0$ and the map $c \mapsto w_{c,d}$ is strictly monotone on $(0, \infty)$ for every fixed $d > 0$.*

Remark 2.2. By rather easy and short elementary geometric considerations one can check that for every fixed $d > 0$ the map $c \mapsto w_{c,d}$ is positive and real analytic. It is also very easy to see geometrically that the limit at 0 is ∞, hence it is not constant, therefore countable-to-one. This would suffice for all our purposes, but these arguments are unfortunately very hard to write down rigorously, so we decided to present a less instructive and longer algebraic proof.

PROOF. Using the matrix of the rotation we can write the rotated image of the point of the graph $(x, y_0, g(x) + dy_0)$ as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_c & -\sin\alpha_c \\ 0 & \sin\alpha_c & \cos\alpha_c \end{pmatrix} \begin{pmatrix} x \\ y_0 \\ g(x) + dy_0 \end{pmatrix} = \begin{pmatrix} x \\ y_0(\cos\alpha_c - d\sin\alpha_c) - g(x)\sin\alpha_c \\ y_0(\sin\alpha_c + d\cos\alpha_c) + g(x)\cos\alpha_c \end{pmatrix}. \tag{2.1}$$

Let us now determine the intersection of the rotated graph with the xy-plane. This right hand side of (2.1) is in the xy-plane if and only if the third coordinate vanishes, that is, when $y_0(\sin\alpha_c + d\cos\alpha_c) + g(x)\cos\alpha_c = 0$. This yields

$$y_0 = -\frac{\cos\alpha_c}{\sin\alpha_c + d\cos\alpha_c} g(x). \tag{2.2}$$

Note that the denominator is not zero, as can be seen, e.g., by the computation of $w_{c,d}$ below. In order to complete the proof of the lemma we have to calculate the y-coordinate of the rotated image of the point $(x, y_0, g(x) + dy_0)$, which is the second entry of the right hand side of (2.1). Hence, using (2.2),

$$\begin{aligned} y &= y_0(\cos\alpha_c - d\sin\alpha_c) - g(x)\sin\alpha_c \\ &= -\frac{\cos\alpha_c(\cos\alpha_c - d\sin\alpha_c)}{\sin\alpha_c + d\cos\alpha_c} g(x) - g(x)\sin\alpha_c \\ &= -\frac{\cos^2\alpha_c - d\cos\alpha_c\sin\alpha_c + \sin^2\alpha_c + d\cos\alpha_c\sin\alpha_c}{\sin\alpha_c + d\cos\alpha_c} g(x) \\ &= -\frac{1}{\sin\alpha_c + d\cos\alpha_c} g(x). \end{aligned}$$

Therefore

$$w_{c,d} = \frac{1}{\sin\alpha_c + d\cos\alpha_c} = \left(\sqrt{d^2+1}\left(\sin\alpha_c \frac{1}{\sqrt{d^2+1}} + \cos\alpha_c \frac{d}{\sqrt{d^2+1}}\right)\right)^{-1}.$$

Using the identity

$$\sin\alpha = \frac{\tan\alpha}{\sqrt{\tan^2\alpha + 1}} \quad (\alpha \in (-\pi/2, \pi/2)) \tag{2.3}$$

we obtain $\sin(\arctan(d)) = \frac{d}{\sqrt{d^2+1}}$, which easily implies $\cos(\arctan(d)) = \frac{1}{\sqrt{d^2+1}}$. (Note that $\arctan(d) \in (-\pi/2, \pi/2)$.) So

$$w_{c,d} = \Big(\sqrt{d^2+1}\big(\sin\alpha_c \cos(\arctan(d)) + \cos\alpha_c \sin(\arctan(d))\big)\Big)^{-1}.$$

By the formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and the definition of α_c this equals

$$\Big(\sqrt{d^2+1}\sin(\alpha_c + \arctan(d))\Big)^{-1} = \Big(\sqrt{d^2+1}\sin(\arctan(cd))\Big)^{-1}.$$

Applying (2.3) again yields

$$w_{c,d} = \left(\sqrt{d^2+1}\frac{\tan(\arctan(cd))}{\sqrt{\tan^2(\arctan(cd))+1}}\right)^{-1} = \sqrt{\frac{1}{d^2+1}\left(1 + \frac{1}{(cd)^2}\right)}.$$

From this form it is easy to see that this function is positive and strictly monotone on $(0,\infty)$ for every fixed $d > 0$. $\square$

Let now $f(x,y) = a + be^{kx} + dy$ $(a,b,d,k \in \mathbb{R},\ k \neq 0)$. Rescaling the graph in a homothetic way does not affect vertical rigidity, so we can consider $kf(\frac{x}{k}, \frac{y}{k})$ and assume $k = 1$. We may also assume $b, d \neq 0$, otherwise our function is of one of the previous forms. Adding a constant, reflecting the graph about the xz-plane (needed only if the signs of b and d differ), multiplying by a nonzero constant, as well as a translation in the x-direction do not affect vertical rigidity, so by applying these in this order we can assume that $a = 0$, $bd > 0$, $d = 1$, and $b = 1$.

Hence it suffices to check that $f(x,y) = e^x + y$ is vertically rigid. Let us fix a $c > 0$. In every vertical plane of the form $\{x = x_0\}$ the restriction of f is a straight line of slope 1. Rotation around the x-axis by angle $\alpha_c = \arctan(c) - \frac{\pi}{4}$ takes all these lines to lines of slope c. By applying Lemma 2.1 with $g(x) = e^x$ and $d = 1$, the intersection of the rotated graph and the xy-plane is the graph of the function $y = -w_{c,1}e^x$.

Now, applying a translation in the x-direction we can obtain a function with still all lines of slope c but now with intersection with the xy-plane of the form $y = -e^x$ (note that $w_{c,1} > 0$). But then we are done, since this function clearly agrees with cf. (The intersection of $graph(f)$ and the xy-plane is of the form $y = -e^x$, and all lines in this graph are of slope 1, hence for $graph(cf)$ the intersection is still $y = -e^x$, and all lines are of slope c.) This finishes the proof of vertical rigidity.

3 Vertical Rigidity Via Translations.

Theorem 3.1. *Let $C \subset (0,\infty)$ be an uncountable set. Then a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for C* via horizontal translations *if and only if after a suitable rotation around the z-axis $f(x,y)$ is of the form $s(y)e^{kx}$ ($k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous).*

We already checked the easy direction in the previous section. Before proving the other direction we need some preparation. We will need the following result, which is Theorem 2.5 in [1].

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous vertically rigid function for an uncountable set $C \subset (0,\infty)$ via horizontal translations. Then f is of the form se^{kx} ($s \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\}$).*

The following lemma will be useful throughout the paper. Sometimes we will use it tacitly. The easy proof is left to the reader.

Lemma 3.3. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be vertically rigid for c_0 via φ_0 and for c via φ. Then $c_0 f$ is vertically rigid for $\frac{c}{c_0}$ via $\varphi \circ \varphi_0^{-1}$.*

From now on we will often use the notation $\vec{x}$ for two-dimensional (and sometimes three-dimensional) vectors.

Definition 3.4. For a function $f : \mathbb{R}^2 \to \mathbb{R}$ and a set $C \subset (0,\infty)$ let $T_{f,C} \subset \mathbb{R}^2$ be the additive group generated by the set $T' = \{\vec{t} \in \mathbb{R}^2 : \exists c \in C,\ \forall \vec{x} \in \mathbb{R}^2,\ f(\vec{x}+\vec{t}) = cf(\vec{x})\}$. (We will usually simply write T for $T_{f,C}$.)

Lemma 3.5. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a vertically rigid function for a set $C \subset (0,\infty)$ via horizontal translations such that $f(\vec{0}) = 1$. Then*

$$f(\vec{x}+\vec{t}) = f(\vec{x})f(\vec{t}) \quad \forall \vec{x} \in \mathbb{R}^2\ \forall \vec{t} \in T.$$

Moreover, $f(\vec{t}) > 0$ for every $\vec{t} \in T$, and T' is uncountable if C is.

PROOF. By assumption, for every $c \in C$ there exists $\vec{t_c} \in \mathbb{R}^2$ such that $cf(\vec{x}) = f(\vec{x}+\vec{t_c})$ for every $\vec{x} \in \mathbb{R}^2$. Then $\vec{t_c} \in T'$ for every $c \in C$.

Since T is the group generated by T', every $\vec{t} \in T$ can be written as $\vec{t} = \sum_{i=1}^{m} n_i \vec{t_i}$ ($\vec{t_i} \in T', n_i \in \mathbb{Z}, i = 1, \dots, m$) where $f(\vec{x} + \vec{t_i}) = c_i f(\vec{x})$ ($\vec{x} \in \mathbb{R}^2$, $i = 1, \dots, m$).

From these we easily get

$$f(\vec{x} + \vec{t}) = c_{\vec{t}} f(\vec{x}), \text{ where } c_{\vec{t}} = \prod_{i=1}^{m} c_i^{n_i}, \ \vec{x} \in \mathbb{R}^2, \ \vec{t} \in T. \tag{3.1}$$

Note that $c_{\vec{t}} > 0$ (and also that it is not necessarily a member of C). It suffices to show that $c_{\vec{t}} = f(\vec{t})$ for every $\vec{t} \in T$, but this follows if we substitute $\vec{x} = \vec{0}$ into (3.1).

Since f is not identically zero, $\vec{t_c} \neq \vec{t_{c'}}$ whenever $c, c' \in C$ are distinct. Hence $\{\vec{t_c} : c \in C\}$ is uncountable, so T' is uncountable if C is. $\square$

PROOF. (Thm. 3.1) If f is identically zero, then we are done, so let us assume that this is not the case. The class of continuous, vertically rigid functions for some set condensating to ∞ via horizontal translations, as well as the class of functions of the form $s(y)e^{kx}$ ($k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous) are both closed under horizontal translations and under multiplication by nonzero constants (by Lemma 3.3). Hence we may assume that $f(\vec{0}) = 1$. Then the previous lemma yields that $f(\vec{t_1} + \vec{t_2}) = f(\vec{t_1}) f(\vec{t_2})$ ($\vec{t_1}, \vec{t_2} \in T$), and also that $f|_T > 0$. Then $g(\vec{t}) = \log f(\vec{t})$ is defined for every $\vec{t} \in T$, and g is clearly additive on T.

Let us now consider $\bar{T}$, the closure of T, which is clearly an uncountable closed subgroup of $\mathbb{R}^2$. It is well-known that every closed subgroup of $\mathbb{R}^2$ is a nondegenerate linear image of a group of the form $G_1 \times G_2$, where $G_1, G_2 \in \{\{0\}, \mathbb{Z}, \mathbb{R}\}$. Hence after a suitable rotation around the origin $\bar{T}$ is either $\mathbb{R}^2$ or $\mathbb{R} \times \{0\}$ or $\mathbb{R} \times r\mathbb{Z}$ for some $r > 0$.

Case 1. $\bar{T} = \mathbb{R}^2$.

In this case $T \subset \mathbb{R}^2$ is dense. It is well-known that a continuous additive function on a dense subgroup is of the form $g(x, y) = \alpha x + \beta y$, ($(x, y) \in T$) for some $\alpha, \beta \in \mathbb{R}$. But then $f(x, y) = e^{\alpha x + \beta y}$ on T, and by continuity this holds on the whole plane as well. As the constant 1 function is not vertically rigid via horizontal translations, $\alpha = \beta = 0$ cannot hold. By applying a rotation of angle $\frac{\pi}{2}$ if necessary we may assume that $\alpha \neq 0$. But then by choosing $k = \alpha$, $s(y) = e^{\beta y}$ we are done.

Case 2. $\bar{T} = \mathbb{R} \times \{0\}$.

In this case every $\vec{t_c}$ is of the form $(t_c, 0)$, where $t_c \neq 0$ if $c \neq 1$. (We may assume $1 \notin C$.)

Applying Theorem 3.2 for every fixed y we obtain that $f(x,y) = s(y)e^{k_y x}$ ($s(y), k_y \in \mathbb{R}, k_y \neq 0$). As $s(y) = f(0,y)$, we get that s is continuous. If $s(y) \neq 0$, then it is not hard to see that $k_y = \frac{\log c}{t_c}$, which is independent of y, so for these y's $k_y = k$ is constant. But if $s(y) = 0$ then the value of k_y is irrelevant, so it can be chosen to be the same constant k. Hence without loss of generality $k_y = k$ is constant, and we are done with this case.

Case 3. $\bar{T} = \mathbb{R} \times r\mathbb{Z}$.

As T' is uncountable, there is an $n \in \mathbb{Z}$ so that $T' \cap (\mathbb{R} \times \{rn\})$ is uncountable. Fix an element t_{c_0} of this set. Then Lemma 3.3 yields that $c_0 f$ is vertically rigid for an uncountable set via translations of the form $(t,0)$. Restricting ourselves to these isometries and c's we are done using Case 2, since every uncountable set in $\mathbb{R}$ generates a dense subgroup. □

Now we handle the case of arbitrary translations.

Theorem 3.6. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be an arbitrary function that is vertically rigid for a set $C \subset (0,\infty)$ via translations. Then there exists $a \in \mathbb{R}$ such that $f - a$ is vertically rigid for the same set C via horizontal translations.*

PROOF. The obvious modification of [1, Thm. 2.4] works, just replace all x's and u's by vectors. □

This readily implies the following.

Corollary 3.7. *Let $C \subset (0,\infty)$ be an uncountable set. Then a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for C* via translations *if and only if after a suitable rotation around the z-axis $f(x,y)$ is of the form $a + s(y)e^{kx}$ ($a, k \in \mathbb{R}$, $k \neq 0$, $s : \mathbb{R} \to \mathbb{R}$ continuous).*

4 The Set S_f.

Now we start working on the case of arbitrary isometries.

Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere. For a function $f : \mathbb{R}^2 \to \mathbb{R}$ let S_f be the set of directions between pairs of points on the graph of f; that is,

Definition 4.1.

$$S_f = \left\{ \frac{p-q}{|p-q|} \in \mathbb{S}^2 : p, q \in graph(f),\ p \neq q \right\}.$$

Recall that a *great circle* is a circle line in $\mathbb{R}^3$ of radius 1 centered at the origin. We call it *vertical* if it passes through the points $(0,0,\pm 1)$.

Lemma 4.2. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then*

1. $-S_f = S_f$ *(symmetric about the origin),*
2. $(0, 0, \pm 1) \notin S_f$*,*
3. S_f *is connected,*
4. *every great circle containing* $(0, 0, \pm 1)$ *intersects* S_f *in two (symmetric) nonempty arcs,*
5. $\mathbb{S}^2 \setminus S_f$ *has exactly two connected components, one containing* $(0, 0, 1)$ *and one containing* $(0, 0, -1)$*.*

PROOF. (1.) Obvious.

(2.) Obvious, since f is a function.

(3.) $graph(f)$ is homeomorphic to $\mathbb{R}^2$, hence the square of it minus the (2-dimensional) diagonal is a connected set. Since S_f is the continuous image of this connected set, it is itself connected.

(4.) The intersection of S_f with such a great circle corresponds to restricting our attention to distinct pairs of points $(\vec{x_1}, \vec{x_2}) \in \mathbb{R}^2 \times \mathbb{R}^2$ so that the segment $[\vec{x_1}, \vec{x_2}]$ is parallel to a fixed line $L \subset \mathbb{R}^2$. Now, given two such nondegenerate segments it is easy to move one of them continuously to the other so that along the way it remains nondegenerate and parallel to L. This shows that in both halves of the great circle (separated by $(0, 0, \pm 1)$) S_f is pathwise connected, hence it is an arc.

(5.) By (4.) every point of $\mathbb{S}^2 \setminus S_f$ can be connected with an arc of a vertical great circle either to $(0, 0, 1)$ or to $(0, 0, -1)$ in $\mathbb{S}^2 \setminus S_f$, hence there are at most two connected components.

Now we show that $(0, 0, 1)$ and $(0, 0, -1)$ are in different ones. It suffices to show that there exists a Jordan curve in S_f so that $(0, 0, 1)$ and $(0, 0, -1)$ are in the two distinct components of its complement. Let $\mathbb{S}^1$ denote the unit circle in $\mathbb{R}^2 = \{(x, y, z) : z = 0\}$ and let $\gamma : \mathbb{S}^1 \to S_f$ be given by

$$\gamma(\vec{x}) = \frac{(\vec{x}, f(\vec{x})) - (-\vec{x}, f(-\vec{x}))}{|(\vec{x}, f(\vec{x})) - (-\vec{x}, f(-\vec{x}))|}.$$

In this paragraph the word 'component' will refer to the components of $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$. One can easily check that γ is continuous and injective, hence a Jordan curve. Moreover, it is clearly in S_f, and its intersection with every vertical great circle is a symmetric pair of points. Therefore every point of $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$ can be connected with an arc of a vertical great circle either to

$(0,0,1)$ or to $(0,0,-1)$ in $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$, hence the union of the components of $(0,0,1)$ and $(0,0,-1)$ cover $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$. So $(0,0,1)$ and $(0,0,-1)$ are in different components, otherwise $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$ would be connected, but the complement of a Jordan curve in $\mathbb{S}^2$ has two components. $\square$

The above lemma shows that S_f is something like a 'strip around the sphere'. Now we make this somewhat more precise by defining the top and the bottom 'boundaries' of this strip.

Definition 4.3. Let $h : \mathbb{S}^1 \to \mathbb{S}^2$ be defined as follows. Every $\vec{x} \in \mathbb{S}^1$ is in a unique half great circle connecting $(0,0,1)$ and $(0,0,-1)$. The intersection of S_f with this great circle is an arc, define $h(\vec{x})$ as the top endpoint of this arc.

Clearly, the bottom endpoint of this arc is $-h(-\vec{x})$, so the 'top function bounding the strip S_f is $h(\vec{x})$ and the bottom function is $-h(-\vec{x})$'. The coordinate functions of h are denoted by (h_1, h_2, h_3), where $h_3 : \mathbb{S}^1 \to [-1,1]$ encodes all information about h.

Lemma 4.4. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous, and h be defined as above. Then*

1. *$h(\vec{x}) \neq (0,0,-1)$ for every $\vec{x} \in \mathbb{S}^1$,*
2. *h is lower semicontinuous (in the obvious sense, or equivalently, h_3 is lower semicontinuous),*
3. *h is convex with respect to great circles, that is, if $h(\vec{x})$ and $h(\vec{y})$ determine a unique nonvertical great circle (i.e. there is a subarc of $\mathbb{S}^1$ of length $< \pi$ connecting $\vec{x}$ and $\vec{y}$, and $h(\vec{x}), h(\vec{y}) \neq (0,0,1)$) then on this subarc $graph(h)$ is bounded from above by the great circle.*

PROOF. (1.) Obvious by Lemma 4.2 (2.) and (4.).

(2.) We have to check that if $h_3(\vec{x}) > u$ then the same holds in a neighbourhood of $\vec{x}$. (Note that essentially h_3 is defined as a supremum.) Hence $h_3(\vec{x}) > u$ if and only if there exists a segment $[\vec{a}, \vec{b}] \subset \mathbb{R}^2$ parallel to $\vec{x}$ over which the slope of f is bigger than u. But then by the continuity of f the same holds for segments close enough to $[\vec{a}, \vec{b}]$, in particular to slightly rotated copies, and we are done.

(3.) It is easy to see that for every $\vec{v} \in \mathbb{S}^1$ the slope of f over a segment parallel to $\vec{v}$ is at most the slope of the vector $h(\vec{v})$. Let $\vec{z} \in \mathbb{S}^1$ be an element of the shorter arc connecting $\vec{x}$ and $\vec{y}$ in $\mathbb{S}^1$, let $[\vec{a}, \vec{b}] \subset \mathbb{R}^2$ be a segment parallel to $\vec{z}$, and let $P : \mathbb{R}^2 \to \mathbb{R}$ be the linear map whose graph passes through the origin, $h(\vec{x})$ and $h(\vec{y})$. (Then $graph(P)$ contains the great circle determined by $h(\vec{x})$ and $h(\vec{y})$. Moreover, the slope of P over any vector parallel to $\vec{x}$ is the slope of $h(\vec{x})$, and similarly for $\vec{y}$.) We have to show that the slope of f between $\vec{a}$ and $\vec{b}$ is at most that of P, that is, $f(\vec{b})-f(\vec{a}) \leq P(\vec{b})-P(\vec{a})$. Write $\vec{b}-\vec{a}=\alpha\vec{x}+\beta\vec{y}$ for some $\alpha,\beta>0$. Then by using the definition of P and our first observation for the segments $[\vec{a},\vec{a}+\alpha\vec{x}]$ and $[\vec{a}+\alpha\vec{x},\vec{a}+\alpha\vec{x}+\beta\vec{y}]$, which are parallel to $\vec{x}$ and $\vec{y}$, respectively, we get

$$\begin{aligned}
f(\vec{b})-f(\vec{a}) =& f(\vec{a}+\alpha\vec{x}+\beta\vec{y})-f(\vec{a})\\
=& (f(\vec{a}+\alpha\vec{x}+\beta\vec{y})-f(\vec{a}+\alpha\vec{x}))+(f(\vec{a}+\alpha\vec{x})-f(\vec{a}))\\
\leq& (P(\vec{a}+\alpha\vec{x}+\beta\vec{y})-P(\vec{a}+\alpha\vec{x}))+(P(\vec{a}+\alpha\vec{x})-P(\vec{a}))\\
=& P(\vec{b})-P(\vec{a}).
\end{aligned}$$

□

5 Determining the Possible S_f's.

Definition 5.1. For $c>0$ let $\psi_c:\mathbb{S}^2\to\mathbb{S}^2$ denote the map that 'deforms S_f according to the map $c\mapsto cf$'; that is,

$$\psi_c((x,y,z))=\frac{(x,y,cz)}{|(x,y,cz)|}\quad((x,y,z)\in\mathbb{S}^2).$$

Remark 5.2. Let φ_c be the isometry mapping $graph(f)$ onto $graph(cf)$. Every isometry φ is of the form $\varphi^{trans}\circ\varphi^{ort}$, where φ^{ort} is an orthogonal transformation and φ^{trans} is a translation. Moreover, if φ is orientation-preserving, then φ^{ort} is a rotation around a line passing through the origin. A key observation is the following: The vertical rigidity of f for C implies that $\psi_c(S_f)=\varphi_c^{ort}(S_f)$ for every $c\in C$.

Now we prove the main theorem of this section. For the definition of h_3 see the previous section.

Theorem 5.3. *Let $C\subset(0,\infty)$ be a set condensating to ∞, and let $f:\mathbb{R}^2\to\mathbb{R}$ be a continuous function vertically rigid for C. Then one of the following holds:*

- ***Case A.*** *There is a vertical great circle that intersects S_f in only two points.*
- ***Case B.*** $S_f=\mathbb{S}^2\setminus\{(0,0,1),(0,0,-1)\}$.
- ***Case C.*** *There exists an $\vec{x_0}\in\mathbb{S}^1$ such that $h_3(\vec{x_0})=0$ and $h_3(\vec{x})=1$ for every $\vec{x}\neq\vec{x_0}$; that is, S_f is '$\mathbb{S}^2$ minus two quarters of a great circle'.*

- ***Case D.*** *There exists a closed interval I in $\mathbb{S}^1$ with $0 < length(I) < \pi$ such that $h_3(\vec{x}) = 0$ if $\vec{x} \in I$, and $h_3(\vec{x}) = 1$ if $\vec{x} \notin I$, that is, S_f is '$\mathbb{S}^2$ minus two spherical triangles'.*

Proof. In this proof the word 'component' will refer to the components of $\mathbb{S}^2 \setminus S_f$. We separate two cases according to whether $h_3 \geq 0$ everywhere or not.

First let us suppose that there exists an $\vec{x} \in \mathbb{S}^1$ such that $h_3(\vec{x}) < 0$. This implies that there is a vertical great circle containing two arcs, one in the top component connecting $(0, 0, 1)$ with $\mathbb{S}^1$ and even crossing it, and another one (the symmetric pair in the bottom component) running from the 'South Pole to the Equator' and even above. But then considering geometrically the action of ψ_c one can easily check that if we choose larger and larger c's (tending to ∞) then we obtain that $\psi_c(S_f)$ contains in the two components two symmetrical arcs on the same great circle which are only leaving out two small gaps of length tending to 0. But then, by Remark 5.2, S_f also contains two such arcs in the two components on some (not necessarily vertical) great circle, hence the distance of the components is 0.

Let $\vec{p_n}$ and $\vec{q_n}$ be sequences in the top and bottom components, respectively, so that $\operatorname{dist}(\vec{p_n}, \vec{q_n}) \to 0$. By compactness we may assume $\vec{p_n}, \vec{q_n} \to \vec{p} \in \mathbb{S}^2$. We claim that $\vec{p_n} \to \vec{p}$ implies $\vec{p} \neq (0, 0, -1)$. (And similarly $\vec{q_n} \to \vec{p}$ implies $\vec{p} \neq (0, 0, 1)$.) Indeed, let $\vec{x}_n \in \mathbb{S}^1$ be so that $\vec{x}_n$ and $\vec{p_n}$ lay on the same vertical great circle. We may assume by compactness that $\vec{x}_n \to \vec{x}$ for some $\vec{x} \in \mathbb{S}^1$. Using the fact $h(\vec{x}) \neq (0, 0, -1)$ and the lower semicontinuity of h at $\vec{x}$ (Lemma 4.4 (1.) and (2.)) we are done.

Using the lower semicontinuity of h at $\vec{x}$ again (and $\vec{p_n} \to \vec{p}$) we get that $h(\vec{x})$ cannot be above $\vec{p}$. Similarly, $-h(-\vec{x})$ cannot be below $\vec{p}$. But $h(\vec{x})$ is always above $-h(-\vec{x})$, so the only option is $h(\vec{x}) = -h(-\vec{x})$, hence there is a vertical great circle whose intersection with S_f is just a (symmetric) pair of points, so Case A holds, and hence we are done with the first half of the proof.

Now let us assume that $h_3 \geq 0$ everywhere. First we prove that $h_3(\vec{x}) \in \{0, 1\}$ for Lebesgue almost every $\vec{x} \in \mathbb{S}^1$. Indeed, fix an arbitrary $c \in C \setminus \{1\}$. By rigidity the (equal) measure of the two components remains the same after applying ψ_c. Since $h_3 \geq 0$, the intersection of the top component with the vertical great circle containing an $\vec{x}$ shrinks if $c > 1$ and grows if $c < 1$, unless $h_3(\vec{x}) = 0$ or 1. Hence we are done, since the measure of the top component can be calculated from the lengths of these arcs.

Now we show that $\{\vec{x} : h_3(\vec{x}) = 0\}$ is either empty, or a pair of points of the form $\{\vec{x_0}, -\vec{x_0}\}$, or a closed interval in $\mathbb{S}^1$ (possibly degenerate or the whole $\mathbb{S}^1$). So we have to show that if $\vec{x}, \vec{y} \in \mathbb{S}^1$ are so that there is an arc connecting them shorter than π, and $h_3(\vec{x}) = h_3(\vec{y}) = 0$ then $h_3(\vec{z}) = 0$ for every $\vec{z}$ in this arc. But $h_3(\vec{z}) \geq 0$ by assumption, and $h_3(\vec{z}) \leq 0$ by the convexity of h applied to $h(\vec{x}) = \vec{x}$ and $h(\vec{y}) = \vec{y}$. The fact that the endpoints are also contained in $\{\vec{x} : h_3(\vec{x}) = 0\}$ easily follows from the semicontinuity.

If $\{\vec{x} : h_3(\vec{x}) = 0\}$ is a symmetrical pair of points or a closed interval of length at least π, then it is easy to see that Case A holds. Hence we may assume that it is empty, or a singleton, or a closed interval I with $0 < length(I) < \pi$.

Case 1. $\{\vec{x} : h_3(\vec{x}) = 0\} = \emptyset$.

In this case, $h_3 > 0$ everywhere, and hence $h_3 = 1$ almost everywhere. Therefore one can easily see (using the convexity) that $h_3 = 1$ everywhere but possibly at at most two points of the form $\{\vec{x_0}, -\vec{x_0}\}$. We claim that actually $h_3 = 1$ everywhere. We know already that S_f is $\mathbb{S}^2$ minus two symmetric arcs on the same vertical great circle. The arcs contain $(0, 0, 1)$ and $(0, 0, -1)$, respectively, and they do not reach the 'Equator', since $h_3 > 0$. Let us fix an arbitrary $c \in C \setminus \{1\}$. By rigidity the (equal) length of the arcs should not change when applying ψ_c, but it clearly changes, a contradiction.

Hence $S_f = \mathbb{S}^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$, so Case B holds.

Case 2. $\{\vec{x} : h_3(\vec{x}) = 0\}$ is a singleton.

Let $\{\vec{x_0}\} = \{\vec{x} : h_3(\vec{x}) = 0\}$. Similarly as above, $h_3 = 1$ almost everywhere. Then convexity easily implies that $h_3(\vec{x}) = 1$ whenever $\vec{x} \notin \{\vec{x_0}, -\vec{x_0}\}$. Again similarly, the length of the arcs is unchanged by ψ_c only if $h_3(-\vec{x}_0) = 1$, so S_f is $\mathbb{S}^2$ minus two symmetric quarter arcs starting from the 'Poles' on a vertical great circle, so Case C holds.

Case 3. $\{\vec{x} : h_3(\vec{x}) = 0\}$ is a closed interval in $\mathbb{S}^1$ with $0 < length(I) < \pi$.

Let $I = \{\vec{x} : h_3(\vec{x}) = 0\}$. As $h_3 = 0$ or 1 almost everywhere, convexity readily implies that $h_3 = 1$ on $\mathbb{S}^1 \setminus I$. Hence S_f is '$\mathbb{S}^2$ minus two spherical triangles', and Case D holds.

This concludes the proof. □

6 The End of the Proof.

Now we complete the proof of the technical form of the Main Theorem. We repeat the statement here.

Theorem 6.1. *(Main Theorem, technical form) Let $C \subset (0, \infty)$ be a set condensating to ∞. Then a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for C if and only if after a suitable rotation around the z-axis $f(x, y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ $(a, b, d, k \in \mathbb{R},\ k \neq 0,\ s : \mathbb{R} \to \mathbb{R}$ continuous).*

Proof. By Theorem 5.3 it suffices to consider Cases A-D.

Case A. There is a vertical great circle that intersects S_f in only two points.

We may assume using a suitable rotation around the z-axis that the vertical great circle is in the yz-plane, hence $f(x,y)$ is of the form $g(x)+dy$. The continuity of f implies that g is also continuous.

Subcase A1. $d=0$.

Let $c \in C$ be fixed, and let φ_c be the corresponding isometry. The graph of cf is invariant under translations parallel to the y-axis. As the same holds for f, by rigidity, cf is also invariant under translations parallel to the φ_c-image of the y-axis. If these two directions are nonparallel, then $graph(cf)$ is a plane, and hence so is $graph(f)$, so we are done since $f(x,y)$ is of the form $a+bx$ (note that there is no '$+dy$' since f does not depend on y). Therefore we may assume that all lines parallel to the y-axis are taken to lines parallel to the y-axis, but then all planes parallel to the xz-plane are taken to planes parallel to the xz-plane. But this shows (by considering the intersections of the graphs with the xz-plane) that g is vertically rigid for c, hence by Theorem 1.3 $g(x)$ is of the from $a+bx$ or $a+be^{kx}$ $(a,b,k \in \mathbb{R},\ k \neq 0)$, and we are done.

Subcase A2. $d \neq 0$.

We may assume that $d>0$, since otherwise we may consider $-f$.

For every $c \in C$ let φ_c be the corresponding isometry. We claim that we may assume that all these are orientation-preserving. We are finished by shrinking C if $\{c \in C : \varphi_c \text{ is orientation-preserving}\}$ condensates to ∞, otherwise we may assume that they are all orientation-reversing (note that if we split C into two pieces then at least one of them still condensates to ∞). Let us fix a $c_0 \in C$ and consider $c_0 f$ instead of f. By Lemma 3.3 this function is rigid for a set condensating to ∞ with all isometries orientation-preserving, and if it is of the desired form then so is f, so we are done.

We may assume $1 \notin C$. Let us fix a $c \in C$. Similarly as in the previous subcase, we may assume that lines parallel to $(0,1,d)$ are taken to lines parallel to $(0,1,cd)$ as follows. The special form of f implies that $graph(f)$ is invariant under translations in the $(0,1,d)$-direction, hence $graph(cf)$ is invariant under translations in the $(0,1,cd)$-direction; moreover, by rigidity, $graph(cf)$ is also invariant under translations parallel to the φ_c-image of the lines of direction $(0,1,d)$. If these two latter directions do not coincide, then $graph(cf)$ is a plane, and we are done.

Therefore the image of every line parallel to $(0,1,d)$ is a line parallel to $(0,1,cd)$ under the orientation-preserving isometry φ_c. As in Remark 5.2, write $\varphi_c = \varphi_c^{trans} \circ \varphi_c^{ort}$, where φ_c^{ort} is a rotation about a line containing the origin and φ_c^{trans} is a translation. Since the translation does not affect directions, the rotation φ_c^{ort} takes the direction $(0,1,d)$ to the nonparallel direction

$(0,1,cd)$ $(d \neq 0)$, therefore the axis of the rotation has to be orthogonal to the plane spanned by these two directions. Hence the axis has to be the x-axis. Moreover, the angle of the rotation is easily seen to be $\arctan(cd) - \arctan(d)$.

We now show that we may assume that φ_c^{trans} is a horizontal translation. Decompose the translation as $\varphi_c^{trans} = \varphi_c^{\vec{u}} \circ \varphi_c^{\vec{v}}$, where $\varphi_c^{\vec{v}}$ is a horizontal translation and $\varphi_c^{\vec{u}}$ is a translation in the $(0,1,cd)$-direction. Since $\varphi_c^{ort}(graph(f))$ is invariant under translations in the $(0,1,cd)$-direction, so is $\varphi_c^{\vec{v}} \circ \varphi_c^{ort}(graph(f))$, hence

$$\varphi_c^{\vec{v}} \circ \varphi_c^{ort}(graph(f)) = \varphi_c^{\vec{u}} \circ \varphi_c^{\vec{v}} \circ \varphi_c^{ort}(graph(f)) = \varphi_c(graph(f)) = graph(cf),$$

so we can assume $\varphi_c = \varphi_c^{\vec{v}} \circ \varphi_c^{ort}$, and we are done.

We will now complete the proof of this subcase by showing that the function $-\frac{1}{d}g$ is rigid for an uncountable set. Indeed, this suffices by Theorem 1.3 and by the special form of f.

Let us denote the xy-plane by $\{z=0\}$ and consider the intersection of both sides of the equation $\varphi_c(graph(f)) = graph(cf)$ with $\{z=0\}$. On the one hand, $\{z=0\} \cap \varphi_c(graph(f)) = \{z=0\} \cap \varphi_c^{\vec{v}} \circ \varphi_c^{ort}(graph(f)) = \varphi_c^{\vec{v}}(\{z=0\} \cap \varphi_c^{ort}(graph(f))) = \varphi_c^{\vec{v}}(graph(-w_{c,d}g)) = \varphi_c^{\vec{v}}\left(graph\left((w_{c,d}d)(-\frac{1}{d}g)\right)\right)$, where we used the fact that $\varphi_c^{\vec{v}}$ is horizontal and Lemma 2.1. On the other hand, it is easy to calculate that $\{z=0\} \cap graph(cf) = graph\left(-\frac{1}{d}g\right)$. Therefore $graph\left(-\frac{1}{d}g\right) = \varphi_c^{\vec{v}}\left(graph\left((w_{c,d}d)(-\frac{1}{d}g)\right)\right)$ and hence $-\frac{1}{d}g$ is rigid for $w_{c,d}d$ for every $c>0$. The map $c \mapsto w_{c,d}d$ is strictly monotone for every fixed d, hence the range of C is uncountable. So $-\frac{1}{d}g$ is rigid for an uncountable set, and we are done.

Case B. $S_f = \mathbb{S}^2 \setminus \{(0,0,1),(0,0,-1)\}$.

S_f is invariant under every ψ_c, and hence it is under every φ_c^{ort}. Then clearly $\varphi_c^{ort}((0,0,1)) = (0,0,1)$ or $\varphi_c^{ort}((0,0,1)) = (0,0,-1)$ for every $c \in C$. By the same argument as above we can assume that the former holds for every $c \in C$. Using the argument again we can assume that all φ_c's are orientation-preserving. But then each of these is a rotation around the z-axis followed by a translation, in other words, an orientation-preserving transformation in the xy-plane followed by a translation in the z-direction. An orientation-preserving transformation in the plane is either a translation or a rotation. If it is a translation for every c, then we are done by Corollary 3.7. So let us assume that there exists a c such that φ_c is a proper rotation around $\vec{x} \in \mathbb{R}^2$ followed by a vertical translation. We claim that then f is constant, which will contradict that S_f is nearly the full sphere, finishing the proof of this case. We will actually show that f is constant on every closed disc $B(\vec{x},R)$ centered at $\vec{x}$. Indeed, consider $\max_{B(\vec{x},R)} f - \min_{B(\vec{x},R)} f$. This is unchanged by the rotation around $\vec{x}$ as well as by the vertical translation, hence by φ_c.

But the map $f \mapsto cf$ multiplies this amount by $c \neq 1$, so the only option is $\max_{B(\vec{x},R)} f - \min_{B(\vec{x},R)} f = 0$, and we are done.

Case C. There exists an $\vec{x_0} \in \mathbb{S}^1$ such that $h_3(\vec{x_0}) = 0$ and $h_3(\vec{x}) = 1$ for every $\vec{x} \neq \vec{x_0}$, that is, S_f is '$\mathbb{S}^2$ minus two quarters of a great circle'.

So S_f is invariant under every ψ_c, and hence so is under every φ_c^{ort}. Hence φ_c^{ort} maps $(0,0,1)$ to one of the four endpoints of the two arcs. Therefore we can assume by splitting C into four pieces according to the image of $(0,0,1)$ and applying Lemma 3.3 that $(0,0,1)$ is a fixed point of every φ_c^{ort}. But then the two arcs are also fixed, and actually φ_c^{ort} is the identity. Hence every φ_c is a translation, and we are done by Corollary 3.7.

Case D. There exists a closed interval I in $\mathbb{S}^1$ with $0 < length(I) < \pi$ such that $h_3(\vec{x}) = 0$ if $\vec{x} \in I$ and $h_3(\vec{x}) = 1$ if $\vec{x} \notin I$, that is, S_f is '$\mathbb{S}^2$ minus two spherical triangles'.

As S_f is invariant under every φ_c^{ort}, vertices of the triangles are mapped to vertices. Hence we may assume (by splitting C into six pieces) that $(0,0,1)$ is fixed. But then the triangles are also fixed sets, and every φ_c^{ort} is the identity, so we are done as in the previous case.

This finishes the proof of the Main Theorem. $\square$

7 Open Questions.

Question 7.1. *In the Main Theorem what can we say if we relax the assumption of continuity to Lebesgue measurability, Baire measurability, Borel measurability, Baire class one, separate continuity or at least one point of continuity?*

Question 7.2. *Which notion of largeness of C suffices for the various results of this paper? For example, does the Main Theorem hold if we only assume that C contains three elements that pairwise generate dense multiplicative subgroups of $(0, \infty)$?*

Remark 7.3. It was shown in [3] that two such elements suffice for the analogous one-variable result. However, two independent elements are not enough here, since if g is vertically rigid for c_1 via a translation and h is vertically rigid for c_2 via a translation then $f(x,y) = g(x)h(y)$ is vertically rigid for both.

Moreover, the main point in that proof in [3] is to replace 'splitting C' by alternative arguments, and we were unable to do so here.

The following question is rather vague.

Question 7.4. *Let us call a set $H \subset \mathbb{S}^2$ rigid if $\psi_c(H)$ is isometric to H for every $c > 0$. Is there a simple description of rigid sets? Or if we assume some regularity?*

And finally, the most intriguing problem.

Question 7.5. *What can we say if there are more than two variables?*

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