RESEARCH

Real Analysis Exchange
Vol. 35(1), 2009/2010, pp. 121–138

N. K. Shukla,* Department of Mathematics, University of Allahabad, Allahabad - 211 002, India. email: o.nirajshukla@gmail.com

G. C. S. Yadav,† Department of Mathematics, University of Allahabad, Allahabad - 211 002, India. email: gcsyadav@gmail.com

A CHARACTERIZATION OF THREE-INTERVAL SCALING SETS

Abstract

In this paper, we characterize scaling sets consisting of three intervals. In addition, we provide a procedure to obtain scaling sets possessing finitely many intervals.

1 Introduction.

Observing that a minimally supported frequency (MSF) wavelet ψ arises from a multiresolution analysis (MRA) with scaling function φ iff there is a measurable set S in the real line $\mathbb{R}$ such that $|\hat{\varphi}| = \chi_S$, the notion of a scaling set has been developed in [2, 6]. A measurable set S of $\mathbb{R}$ containing a neighborhood of zero and contained in $2S$ is a *scaling set* if each element of S uniquely corresponds with an element of $[a, a+2\pi)$, $a \in \mathbb{R}$, by a 2π-integral translate and vice versa. In case an MSF wavelet arises from a generalized multiresolution analysis (GMRA) with scaling function φ and there is a measurable set S in $\mathbb{R}$ such that $|\hat{\varphi}| = \chi_S$, S has been called to be a *generalized scaling set* [2, 10].

The notion of a generalized scaling set provides a method to obtain wavelet sets. A measurable set W of the real line $\mathbb{R}$ is called a *wavelet set* if the characteristic function on W is $\sqrt{2\pi}$ times the modulus of the Fourier transform of an orthonormal wavelet ψ of $L^2(\mathbb{R})$ [3]. By an *orthonormal wavelet* ψ, we mean a function in $L^2(\mathbb{R})$ whose successive dilates by a scalar d of all integral

Mathematical Reviews subject classification: Primary: 42C15, 42C40
Key words: scaling set, generalized scaling set, wavelet set
Received by the editors January 21, 2009
Communicated by: Alexander Olevskii
*Supported by the CSIR, New Delhi.
†Supported by the UGC, New Delhi.

translates form an orthonormal basis for $L^2(\mathbb{R})$. These notions have their versions in higher dimensions as well [2, 4, 5, 10], and these have been extensively studied in many research papers besides those already referred. Ha, Kang, Lee and Seo [7] characterized wavelet sets in $\mathbb{R}$ which are unions of three disjoint intervals. These are precisely

$$\Big[-2(1-\frac{2p+1}{2^{j+1}-1})\pi,\ -(1-\frac{2p+1}{2^{j+1}-1})\pi\Big] \cup \Big[\frac{2(p+1)\pi}{2^{j+1}-1}, \frac{2(2p+1)\pi}{2^{j+1}-1}\Big]\cup \Big[\frac{2^{j+1}(2p+1)\pi}{2^{j+1}-1}, \frac{2^{j+2}(p+1)\pi}{2^{j+1}-1}\Big] \qquad (1)$$

for natural numbers j and p, where $j \geq 2$ and $1 \leq p \leq 2^j - 2$.

Determination of wavelet sets of $\mathbb{R}$ which are unions of pairwise disjoint intervals attracted several workers who made significant contributions towards this end [1, 3, 7, 8, 10].

The purpose of this paper is to characterize three-interval scaling sets of $\mathbb{R}$ by selecting three distinct and increasing points in the circle S^1, or equivalently, in $[0, 2\pi)$. Certain examples of wavelet sets arising from three-interval scaling sets are provided. In the end, we give a procedure to obtain scaling sets of $\mathbb{R}$ consisting of finitely many intervals.

2 Notation and Preliminaries.

For a set W of the real line $\mathbb{R}$, W^+ denotes $W \cap (0,\infty)$ and W^- denotes $W \cap (-\infty, 0)$. Also, we denote $(0,\infty)$ by $\mathbb{R}^+$ and $(-\infty, 0)$ by $\mathbb{R}^-$. Let $a \in \mathbb{R}$. Then for $x \in \mathbb{R}$, there is a unique integer k such that $a \leq x + 2k\pi < a + 2\pi$. Next, let $b \in \mathbb{R}^+$. Then for $x > 0$, there is a unique integer j such that $b \leq 2^j x < 2b$. These observations provide the following maps for $a \in \mathbb{R}$, $b \in \mathbb{R}^+$ and $c \in \mathbb{R}^-$:

(i) $\tau_a : \mathbb{R} \longrightarrow [a,\, a + 2\pi)$, defined by

$$\tau_a(x) = x + 2k\pi,\ x \in \mathbb{R},$$

(ii) $\delta_b : \mathbb{R}^+ \longrightarrow [b,\, 2b)$, defined by

$$\delta_b(x) = 2^j x,\ x \in \mathbb{R}^+,$$

(iii) $\delta_c : \mathbb{R}^- \longrightarrow [2c,\, c)$, defined by

$$\delta_c(x) = -\delta_{-c}(-x),\ x \in \mathbb{R}^-.$$

It has been proved in [7, Theorem 3.6], that W is a wavelet set of $\mathbb{R}$ iff τ_a, δ_b and δ_c are measurable bijections for some $a \in \mathbb{R}$, $b \in \mathbb{R}^+$ and $c \in \mathbb{R}^-$, when restricted to W, W^+ and W^-, respectively. It is pertinent to mention that there are several other criteria for wavelet sets [2, 3, 4, 5, 9].

In the sequel, we shall frequently make use of the consequence of Lemma 3 in [6] stated below.

Lemma 2.1 (6; Lemma 3)**.** Let A be a real expansive $n \times n$ matrix such that $|detA| = 2$ and $A\mathbb{Z}^n \subset \mathbb{Z}^n$. Let S be a measurable subset of $\mathbb{R}^n$ such that S contains a neighborhood of zero. If $S \subset A^tS$ and S is, modulo null sets, 2π-translation congruent to $[-\pi, \pi)^n$, then $W = A^tS - S$ is an A-dilation MRA wavelet.

As its consequence, we have

Result 2.2. Let S be a measurable set in $\mathbb{R}$ which contains a neighborhood of zero and satisfies $S \subset 2S$. If S is 2π-translation congruent to $[-\pi, \pi)$, or equivalently to $[\alpha, \alpha + 2\pi)$, where $\alpha \in \mathbb{R}$, modulo a null set, then $W = 2S - S$ is a wavelet set associated with a multiresolution analysis.

Definition 2.3. A measurable set S in $\mathbb{R}$ is called a *generalized scaling set* for the dilation 2 if $S = \cup_{j<0} 2^j W$, for some wavelet set W [10, Definition 1]. This is equivalent to saying that S is a generalized scaling set of $\mathbb{R}$ [2] iff

(i) $S \subset 2S$, and
(ii) $W \equiv 2S - S$ is a wavelet set of $\mathbb{R}$.

A scaling set of $\mathbb{R}$ is a generalized scaling set but the converse is not necessarily true as is shown in the following example.

Example 2.4. The set

$$S = [\tfrac{-4\pi}{7}, \tfrac{4\pi}{7}) \cup [\tfrac{6\pi}{7}, \tfrac{8\pi}{7}) \cup [\tfrac{12\pi}{7}, \tfrac{16\pi}{7}),$$

is a generalized scaling set because

(i) $S \subset 2S$, and
(ii) $W \equiv 2S - S = [\frac{-8\pi}{7}, \frac{-4\pi}{7}) \cup [\frac{4\pi}{7}, \frac{6\pi}{7}) \cup [\frac{24\pi}{7}, \frac{32\pi}{7})$

is a wavelet set, in view of (1) in the Introduction, for $j = 2$ and $p = 1$. However, S is not a scaling set.

In the light of Result 2.2 and Definition 2.3, we have the following result:

Result 2.5. A measurable set S of $\mathbb{R}$ which contains a neighborhood of zero and satisfies:
(1) $S \subset 2S$, and
(2) S is 2π-translation congruent to $[a,\, a+2\pi)$, where $a \in \mathbb{R}$,
is a scaling set of $\mathbb{R}$.

Consider the map $p : \mathbb{R} \longrightarrow S^1$ which sends $t \in \mathbb{R}$ to $e^{it} \in S^1$. We identify $t \in [0,\, 2\pi)$ with e^{it}. For $\alpha_1,\, \alpha_2,\, \alpha_3, ..., \alpha_n$ in S^1, or equivalently, in $[0,\, 2\pi)$ such that $0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_n < 2\pi$, $p^{\leftarrow}(\alpha_1,\, \alpha_2,\, \alpha_3, ..., \alpha_n)$ denotes the set $[\alpha_1,\, \alpha_2) \cup [\alpha_2,\, \alpha_3) \cup ... \cup [\alpha_n,\, \alpha_1 + 2\pi)$ in $\mathbb{R}$. By an *n-interval scaling set* (*generalized scaling set*), we mean a scaling set (generalized scaling set) consisting of n intervals of $\mathbb{R}$. Similarly, we define an *n-interval wavelet set*. We say an n-interval scaling (generalized scaling, wavelet) set to consist of n components.

The next section is devoted to the characterization of three-interval scaling sets of $\mathbb{R}$.

3 A Characterization of Three-Interval Scaling Sets.

Let α, β, γ in S^1, or equivalently, in $[0, 2\pi)$ be such that $0 < \alpha < \beta < \gamma < 2\pi$. Recall that $p^{\leftarrow}(\alpha,\, \beta,\, \gamma)$ denotes the set $[\alpha,\, \beta) \cup [\beta,\, \gamma) \cup [\gamma,\, \alpha+2\pi)$ in $\mathbb{R}$. We now proceed to obtain three-interval scaling sets of $\mathbb{R}$ by translating $[\alpha,\, \beta)$, $[\beta,\, \gamma)$ and $[\gamma,\, \alpha + 2\pi)$ by integral multiples of 2π. Since a scaling set of $\mathbb{R}$ has to contain a neighborhood of zero, we have to translate the interval $[\gamma,\, \alpha + 2\pi)$ by -2π. Translate $[\alpha,\, \beta)$ by $2n\pi$ and $[\beta,\, \gamma)$ by $2m\pi$ such that the three intervals $[\gamma - 2\pi,\, \alpha)$, $[\alpha + 2n\pi,\, \beta + 2n\pi)$ and $[\beta + 2m\pi,\, \gamma + 2m\pi)$ are mutually separated; that is to say that the closure of one does not meet the other.

Let $m > 0$. Then, in light of the fact that we are concerned with three-interval scaling sets, we have to discard the cases when $n = 0$ and also when $n = m$. For $n > 0$, consider the two possibilities: (i) $m > n$, and (ii) $n > m$. In both the possibilities, the components remain mutually separated. However, $S \not\subset 2S$ in these situations, where

$$S = [\gamma - 2\pi,\, \alpha) \cup [\alpha + 2n\pi,\, \beta + 2n\pi) \cup [\beta + 2m\pi,\, \gamma + 2m\pi).$$

For $n < 0$, $m > n$, it can again be seen that $S \not\subset 2S$. Thus S cannot be a scaling set, when $m > 0$.

Next, let $m < 0$. To have components in S mutually separated, $m \neq -1$ and also, $n \neq 0$. Therefore $m \leq -2$. Suppose $m < -2$. If $n > 0$, then $n > m$, and in this situation, although the three-components of S remain mutually separated, $S \not\subset 2S$. For $n < 0$, the possibilities (i) $n > m$, and (ii)

$n < m$, provide $S \not\subset 2S$, while $n = m$ reduces the number of components in S. Thus only $m \in \{-2, 0\}$ can provide scaling sets. When $m = -2$, and $n \in \mathbb{Z} - \{-1\}$, $S \not\subset 2S$. Likewise, when $m = 0$ and $n \in \mathbb{Z} - \{-1, 1\}$, $S \not\subset 2S$. Therefore, to have S as a scaling set of $\mathbb{R}$, we are left with following choices:

Choice I. $m = 0$ and $n = 1$,

Choice II. $m = 0$ and $n = -1$, and

Choice III. $m = -2$ and $n = -1$.

Choice I gives

$$S = [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma) \cup [\alpha + 2\pi,\, \beta + 2\pi).$$

The requirement that $S \subset 2S$ for S to be a scaling set holds iff the following conditions are satisfied:

(a) $2\alpha \geq \gamma$,

(b) $\alpha + 2\pi \geq 2\beta$, and

(c) $2\gamma \geq \beta + 2\pi$.

By construction, S is 2π-translation congruent to $[\alpha,\, \alpha+2\pi)$. Hence, by Result 2.5, it follows that S is a scaling set.

Given below is an alternative proof for S to be a scaling set. Note that

$$\begin{aligned} W =& \big[2(\gamma - 2\pi),\, \gamma - 2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[2\beta,\, \alpha + 2\pi\big) \cup \\ & \big[\beta + 2\pi,\, 2\gamma\big) \cup \big[2(\alpha + 2\pi),\, 2(\beta + 2\pi)\big), \\ \equiv&\ I_1 \cup I_2 \cup I_3 \cup I_4 \cup I_5 \cup I_6 \qquad (say). \end{aligned}$$

That W is a wavelet set follows by observing that the maps τ_α, $\delta_{\gamma-2\pi}$ and δ_α are bijections. Indeed,

(i) $\tau_\alpha : W \longrightarrow [\alpha,\, \alpha + 2\pi)$ is defined by

$$\tau_\alpha(x) = \begin{cases} x + 2\pi & \text{if } x \in I_1, \\ x & \text{if } x \in I_2 \cup I_3 \cup I_4, \\ x - 2\pi & \text{if } x \in I_5, \\ x - 4\pi & \text{if } x \in I_6. \end{cases}$$

(ii) $\delta_{\gamma-2\pi} : W^- \longrightarrow \big[2(\gamma - 2\pi),\, \gamma - 2\pi\big)$, where $W^- \equiv I_1$, is defined by

$$\delta_{\gamma-2\pi}(x) = x.$$

(iii) $\delta_\alpha : W^+ \longrightarrow [\alpha,\, 2\alpha)$, where $W^+ \equiv I_2 \cup I_3 \cup I_4 \cup I_5 \cup I_6$, is defined by

$$\delta_\alpha(x) = \begin{cases} x & \text{if } x \in I_2 \cup I_3, \\ \frac{x}{2} & \text{if } x \in I_4 \cup I_5, \\ \frac{x}{4} & \text{if } x \in I_6. \end{cases}$$

Thus S is a generalized scaling set. Further, in the light of Corollary 3.4 of [2] according to which a measurable set S in $\mathbb{R}$ is a scaling set iff it is a generalized scaling set of order $d - 1 \equiv 2 - 1 = 1$ and $\sum_{k\in\mathbb{Z}} \chi_S(\xi + 2k\pi) = 1$, almost everywhere, it follows that S is a scaling set.

Choice II gives

$$S = [\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma).$$

The requirement that $S \subset 2S$ for S to be a scaling set holds iff the following conditions are satisfied:

(a) $2\alpha \geq \gamma$, and

(b) $\alpha + 2\pi \geq 2\gamma$.

By construction, S is 2π-translation congruent to $[\alpha,\, \alpha+2\pi)$. Hence, by Result 2.5, it follows that S is a scaling set.

Choice III gives

$$S = [\beta - 4\pi,\, \gamma - 4\pi) \cup [\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha).$$

The requirement that $S \subset 2S$ for S to be a scaling set holds iff the following conditions are satisfied:

(a) $2\gamma \leq \alpha + 2\pi$,

(b) $\gamma \leq 2\beta$, and

(c) $2\alpha \leq \beta$.

By construction, S is 2π-translation congruent to $[\alpha,\, \alpha+2\pi)$. Hence, by Result 2.5, it follows that S is a scaling set.

We sum up the above in the following:

Theorem 3.1. *A triple $(\alpha,\, \beta,\, \gamma)$, where α, β, γ in S^1, or equivalently, in $[0,\, 2\pi)$ such that $0 < \alpha < \beta < \gamma < 2\pi$ provides exactly three kinds of three-interval scaling sets described as follows:*

(i) $S = [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma) \cup [\alpha + 2\pi,\, \beta + 2\pi)$, *where*

$$(a)\ 2\alpha \geq \gamma,\ (b)\ \alpha + 2\pi \geq 2\beta,\ \textit{and}\ (c)\ 2\gamma \geq \beta + 2\pi.$$

(ii) $S = [\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma)$, *where*

$$(a)\ 2\alpha \geq \gamma,\ \textit{and}\ (b)\ \alpha + 2\pi \geq 2\gamma.$$

(iii) $S = [\beta - 4\pi,\, \gamma - 4\pi) \cup [\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha)$, *where*

$$(a)\ 2\gamma \leq \alpha + 2\pi,\ (b)\ \gamma \leq 2\beta,\quad \textit{and}\ (c)\ 2\alpha \leq \beta.$$

The converse of Theorem 3.1 is also true.

Theorem 3.2. *Suppose S is a three-interval scaling set of $\mathbb{R}$. Then there are three elements α, β, γ in S^1, or equivalently, in $[0,\, 2\pi)$ such that $0 < \alpha < \beta < \gamma < 2\pi$ for which S is of the form (i) or (ii) or (iii) as described in Theorem 3.1.*

Proof. Suppose $S = I_1 \cup I_2 \cup I_3$, where $I_1 = [a, b)$, $I_2 = [c, d)$, and $I_3 = [e, f)$ are three mutually separated intervals of $\mathbb{R}$. Since a scaling set is associated with an MRA, S is 2π-translation congruent to an interval of $\mathbb{R}$ of measure 2π. Also, one of the components of S, say, I_1 contains a neighborhood of zero.

(1) Suppose $a < b < c < d < e < f$. As a scaling set satisfies $S \subset 2S$, we shall have either of the following cases:

(i) $f \leq 2b$.

(ii) (1) $d \leq 2b$, (2) $2c \leq e$, and (3) $f \leq 2d$.

When restricted to S the map $\tau_a : S \longrightarrow [a,\, a + 2\pi)$ is the identity on I_1. Furthermore, from the fact that neither I_2 nor I_3 can contain an integer multiple of 2π, we have either of the following situations arising from τ_a :

$$(A)\quad \begin{aligned} b &= c + 2m\pi \\ d + 2m\pi &= e + 2n\pi \\ f + 2n\pi &= a + 2\pi \end{aligned} \qquad\qquad (B)\quad \begin{aligned} b &= e + 2n\pi \\ f + 2n\pi &= c + 2m\pi \\ a + 2\pi &= d + 2m\pi \end{aligned}$$

First we take up Case (i) with (A). Since $b < 2\pi$, from $b = c + 2m\pi$, we deduce that $m < 0$ and $c \in (-2m\pi,\, 2(1 - m)\pi)$. As $f \leq 2b$ and $c < f$, we have $c < 4\pi$. Thus $m = -1$. This gives $c = b + 2\pi$ and hence $2b < c$. This contradicts the fact that $S \subset 2S$. Considering Case (i) with (B), we arrive at a similar conclusion and therefore Case (i) cannot occur.

Next, we consider Case (ii) with (A). Since $b < 2\pi$, from $b = c + 2m\pi$, we deduce that $m < 0$ and $c \in \big(-2m\pi,\, 2(1-m)\pi\big)$. As $d < 4\pi$ and $c < d$, $c < 4\pi$. Thus $m = -1$. This gives $c = b + 2\pi$ and hence $2b < c$. This contradicts the fact that $S \subset 2S$.

Now consider Case (ii) with (B). Since $d \le 2b$, $d < 4\pi$. Also, $f \le 2d$ gives $f < 8\pi$. From $b = e + 2n\pi$, we get $n < 0$ and $e \in \big(-2n\pi,\, 2(1-n)\pi\big)$. As $f < 8\pi$ and $e < f$, $e < 8\pi$. Thus $n = -1$, or -2, or -3. From $a + 2\pi = d + 2m\pi$, we get $m < 1$ and $d \in \big(-2m\pi,\, 2(1-m)\pi\big)$. As $d < 4\pi$, $m = 0$, or -1. If $m = -1$, then $d = a + 4\pi$, which gives $2b < d$. This contradicts the fact that $S \subset 2S$. If $m = 0$ and $n \in \{-2, -3\}$, $2d < f$ which again contradicts the fact that $S \subset 2S$. Finally, $m = 0$ and $n = -1$ give

$$S = [a, b) \cup [c,\, a + 2\pi) \cup [b + 2\pi,\, c + 2\pi),$$

showing that S arises via **Choice I** on taking $\alpha = b$, $\beta = c$ and $\gamma = a + 2\pi$.

(2) Suppose $e < f < a < b < c < d$. Since $S \subset 2S$, we have $2a \le e$, and $d \le 2b$. When restricted to S the map $\tau_a : S \longrightarrow [a,\, a + 2\pi)$ is identity on I_1. Furthermore, from the fact that neither I_2 nor I_3 can contain an integral multiple of 2π, we have either of the following situations arising from τ_a :

$$(A) \quad \begin{aligned} b &= c + 2m\pi \\ d + 2m\pi &= e + 2n\pi \\ f + 2n\pi &= a + 2\pi \end{aligned} \qquad\qquad (B) \quad \begin{aligned} b &= e + 2n\pi \\ f + 2n\pi &= c + 2m\pi \\ a + 2\pi &= d + 2m\pi \end{aligned}$$

First consider (A). From $b = c + 2m\pi$, we get $m < 0$ and $c \in (-2m\pi,\, 2(1-m)\pi)$. As $d < 4\pi$ and $c < d$, we have $c < 4\pi$. Thus $m = -1$, and $c = b + 2\pi$. This provides $2b < c$ which is a contradiction to the fact that $S \subset 2S$.

Next, consider (B). From $b = e + 2n\pi$, we deduce that $n > 0$ and $e \in (-2n\pi,\, 2(1-n)\pi)$. As $2a \le e$ and $-2\pi < a$, we have $-4\pi < e$. Thus $n = 1$, or 2. From $a + 2\pi = d + 2m\pi$, we obtain that $m < 1$ and $d \in (-2m\pi,\, 2(1-m)\pi)$. As $d \le 2b$ and $b < 2\pi$, we have $d < 4\pi$. Thus $m = 0$, or -1. If $m = -1$, then $d = a + 4\pi$ which gives $2b < d$. This contradicts the fact that $S \subset 2S$. If $n = 2$, then $e = b - 4\pi$ which provides $2a > e$, again a contradiction to the fact $S \subset 2S$. If $m = 0$ and $n = 1$, then

$$S = [b - 2\pi,\, c - 2\pi) \cup [a,\, b) \cup [c,\, a + 2\pi),$$

which arises via **Choice II** by taking $\alpha = b,\ \beta = c,\ \gamma = a + 2\pi$.

(3) In case, when $e < f < c < d < a < b$, it can be seen that S arises from **Choice III**, in a way similar to (1) for $\alpha = b$, $\beta = d + 2\pi$ and $\gamma = a + 2\pi$. □

Remark 3.3. Denoting the scaling set S obtained in (i) of Theorem 3.1 by $S(\mathrm{I};\, \alpha,\, \beta,\, \gamma)$ and that in (iii) by $S(\mathrm{III};\, \alpha,\, \beta,\, \gamma)$, it is seen that

$$S(\mathrm{III}; \alpha,\, \beta,\, \gamma) = -S(\mathrm{I}; 2\pi - \gamma,\, 2\pi - \beta,\, 2\pi - \alpha).$$

Remark 3.4. The generalized scaling set in Example 2.4 is not a scaling set because S is not of the form (i), or (ii), or (iii) in Theorem 3.1 for any $\alpha,\, \beta,\, \gamma \in [0, 2\pi)$.

4 Examples of Three-Interval Scaling Sets.

In this section, from the conditions obtained on α, β and γ so that $p^{\leftarrow}(\alpha,\, \beta,\, \gamma)$ furnishes scaling sets are discussed. Certain three-interval scaling sets are obtained and the number of components in the associated wavelet sets is seen to be between 3 and 6 in case of **Choice I** as well as in case of **Choice III** and between 4 and 6 in case of **Choice II**.

We first consider scaling sets obtained via **Choice I**.

(a) When $2\alpha = \gamma,\ 2\beta = \alpha + 2\pi$ and $2\gamma = \beta + 2\pi$, we have $\alpha = \frac{6\pi}{7},\ \beta = \frac{10\pi}{7}$ and $\gamma = \frac{12\pi}{7}$ so that the scaling set obtained is given by

$$S = \left[-\frac{2\pi}{7},\, \frac{6\pi}{7}\right) \cup \left[\frac{10\pi}{7},\, \frac{12\pi}{7}\right) \cup \left[\frac{20\pi}{7},\, \frac{24\pi}{7}\right).$$

The corresponding wavelet set is

$$W \equiv 2S - S = \left[-\frac{4\pi}{7},\, -\frac{2\pi}{7}\right) \cup \left[\frac{6\pi}{7},\, \frac{10\pi}{7}\right) \cup \left[\frac{40\pi}{7},\, \frac{48\pi}{7}\right).$$

Notice that W is a three-interval wavelet set.

(b) When $2\alpha = \gamma,\ 2\beta = \alpha + 2\pi$ and $2\gamma > \beta + 2\pi$, we have $\alpha > \frac{6\pi}{7},\ \beta > \frac{10\pi}{7}$ and $\gamma > \frac{12\pi}{7}$ so that the scaling sets obtained are

$$S = [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma) \cup [\alpha + 2\pi,\, \beta + 2\pi).$$

The corresponding wavelet sets are

$$W = \big[2(\gamma - 2\pi),\, \gamma - 2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[\beta + 2\pi,\, 2\gamma\big) \cup \big[2(\alpha + 2\pi),\, 2(\beta + 2\pi)\big).$$

These corresponding wavelet sets are four-interval wavelet sets. For illustration, choosing $\alpha = \frac{13\pi}{14}$, we have $\beta = \frac{41\pi}{28}$ and $\gamma = \frac{13\pi}{7}$ so that the scaling set thus obtained is

$$S = \left[-\frac{\pi}{7},\, \frac{13\pi}{14}\right) \cup \left[\frac{41\pi}{28},\, \frac{13\pi}{7}\right) \cup \left[\frac{41\pi}{14},\, \frac{97\pi}{28}\right),$$

and the corresponding wavelet set is

$$W=\left[-\frac{2\pi}{7},-\frac{\pi}{7}\right)\cup\left[\frac{13\pi}{14},\frac{41\pi}{28}\right)\cup\left[\frac{97\pi}{28},\frac{26\pi}{7}\right)\cup\left[\frac{41\pi}{7},\frac{97\pi}{14}\right)$$

which has four components.

Proceeding as in **(b)**, we obtain scaling sets in cases

(i) $2\alpha=\gamma,\ 2\beta<\alpha+2\pi$ and $2\gamma=\beta+2\pi$,

(ii) $2\alpha>\gamma,\ 2\beta=\alpha+2\pi$ and $2\gamma=\beta+2\pi$,

in each of which the associated wavelet sets have four components.

(c) When $2\alpha=\gamma,\ 2\beta<\alpha+2\pi$ and $2\gamma>\beta+2\pi$, we have scaling sets given by

$$S=[\gamma-2\pi,\,\alpha)\cup[\beta,\,\gamma)\cup[\alpha+2\pi,\,\beta+2\pi),$$

and the corresponding wavelet sets are given by

$$\begin{aligned}W=&\big[2(\gamma-2\pi),\,\gamma-2\pi\big)\cup\big[\alpha,\,\beta\big)\cup\big[2\beta,\,\alpha+2\pi\big)\cup\big[\beta+2\pi,\,2\gamma\big)\cup\\&\big[2(\alpha+2\pi),\,2(\beta+2\pi)\big).\end{aligned}$$

These corresponding wavelet sets are five-interval wavelet sets. For illustration, choosing $\alpha=\frac{13\pi}{14}$, $\beta=\frac{10\pi}{7}$ and $\gamma=\frac{13\pi}{7}$, we have the scaling set S as

$$S=\left[-\frac{\pi}{7},\frac{13\pi}{14}\right)\cup\left[\frac{10\pi}{7},\frac{13\pi}{7}\right)\cup\left[\frac{41\pi}{14},\frac{24\pi}{7}\right),$$

while the corresponding wavelet set is

$$W=\left[-\frac{2\pi}{7},-\frac{\pi}{7}\right)\cup\left[\frac{13\pi}{14},\frac{10\pi}{7}\right)\cup\left[\frac{20\pi}{7},\frac{41\pi}{14}\right)\cup\left[\frac{24\pi}{7},\frac{26\pi}{7}\right)\cup\left[\frac{41\pi}{7},\frac{48\pi}{7}\right)$$

which possesses five components.

Proceeding as in **(c)**, we obtain scaling sets in cases

(i) $2\alpha>\gamma,\ 2\beta<\alpha+2\pi$ and $2\gamma=\beta+2\pi$,

(ii) $2\alpha>\gamma,\ 2\beta=\alpha+2\pi$ and $2\gamma>\beta+2\pi$,

in each of which the associated wavelet sets have five components.

(d) When $2\alpha>\gamma,\ 2\beta<\alpha+2\pi$ and $2\gamma>\beta+2\pi$, we have $\alpha>\frac{2\pi}{3}$ and $\gamma>\frac{4\pi}{3}$ so that the scaling sets thus obtained are

$$S=[\gamma-2\pi,\,\alpha)\cup[\beta,\,\gamma)\cup[\alpha+2\pi,\,\beta+2\pi),$$

and the corresponding wavelet sets are

$$W = \big[2(\gamma - 2\pi),\, \gamma - 2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[2\beta,\, \alpha + 2\pi\big) \cup \big[\beta + 2\pi,\, 2\gamma\big) \cup \big[2(\alpha + 2\pi),\, 2(\beta + 2\pi)\big).$$

These corresponding wavelet sets are six-interval wavelet sets. For illustration, choosing $\alpha = \frac{13\pi}{14}$, $\beta = \frac{9\pi}{7}$ and $\gamma = \frac{12\pi}{7}$, we have the scaling set

$$S = \left[-\frac{2\pi}{7},\, \frac{13\pi}{14}\right) \cup \left[\frac{9\pi}{7},\, \frac{12\pi}{7}\right) \cup \left[\frac{41\pi}{14},\, \frac{23\pi}{7}\right),$$

and the corresponding wavelet set

$$W = \left[-\frac{4\pi}{7},\, -\frac{2\pi}{7}\right) \cup \left[\frac{13\pi}{14},\, \frac{9\pi}{7}\right) \cup \left[\frac{12\pi}{7},\, \frac{13\pi}{7}\right) \cup \left[\frac{18\pi}{7},\, \frac{41\pi}{14}\right) \cup \left[\frac{23\pi}{7},\, \frac{24\pi}{7}\right) \cup \left[\frac{41\pi}{7},\, \frac{46\pi}{7}\right)$$

which has six components.

Next, we consider certain scaling sets obtained via **Choice II**.

(a) When $\gamma = 2\alpha$ and $2\gamma = \alpha + 2\pi$, we have $\alpha = \frac{2\pi}{3}$, $\beta \in (\frac{2\pi}{3}, \frac{4\pi}{3})$ and $\gamma = \frac{4\pi}{3}$ so that the scaling sets obtained are

$$S = \left[-\frac{4\pi}{3},\, \beta - 2\pi\right) \cup \left[-\frac{2\pi}{3},\, \frac{2\pi}{3}\right) \cup \left[\beta,\, \frac{4\pi}{3}\right),$$

and the corresponding wavelet sets are

$$W = \left[-\frac{8\pi}{3},\, 2(\beta - 2\pi)\right) \cup \left[\beta - 2\pi,\, -\frac{2\pi}{3}\right) \cup \left[\frac{2\pi}{3},\, \beta\right) \cup \left[2\beta,\, \frac{8\pi}{3}\right).$$

These corresponding wavelet sets are four-interval wavelet sets.

(b) When $2\alpha = \gamma$ and $2\gamma < \alpha + 2\pi$, we have $\alpha < \frac{2\pi}{3}$ and $\gamma < \frac{4\pi}{3}$ so that the scaling sets obtained are

$$S = [\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma),$$

while the corresponding wavelet sets are

$$W = \big[2(\alpha - 2\pi),\, 2(\beta - 2\pi)\big) \cup \big[2(\gamma - 2\pi),\, \alpha - 2\pi\big) \cup \big[\beta - 2\pi,\, \gamma - 2\pi\big) \cup \big[\alpha,\, \beta\big) \cup \big[2\beta,\, 2\gamma\big).$$

These corresponding wavelet sets are five-interval wavelet sets.

Proceeding as in **(b)**, we obtain scaling sets in case $2\alpha > \gamma$ and $2\gamma = \alpha+2\pi$ for which the associated wavelet sets possess five components.

(c) When $2\alpha > \gamma$ and $2\gamma < \alpha + 2\pi$, we have the scaling sets as

$$S = [\alpha - 2\pi,\, \beta - 2\pi) \cup [\gamma - 2\pi,\, \alpha) \cup [\beta,\, \gamma),$$

and the corresponding wavelet sets as

$$\begin{aligned} W =& \big[2(\alpha - 2\pi),\, 2(\beta - 2\pi)\big) \cup \big[2(\gamma - 2\pi),\, \alpha - 2\pi\big) \cup \big[\beta - 2\pi,\, \gamma - 2\pi\big) \cup \\ & \big[\alpha,\, \beta\big) \cup \big[\gamma,\, 2\alpha\big) \cup \big[2\beta,\, 2\gamma\big). \end{aligned}$$

These corresponding wavelet sets are six-interval wavelet sets. For illustration, choosing $\alpha = \pi$, $\beta = \frac{9\pi}{8}$ and $\gamma = \frac{5\pi}{4}$, we have the scaling set

$$S = \left[-\pi,\, -\frac{7\pi}{8}\right) \cup \left[-\frac{3\pi}{4},\, \pi\right) \cup \left[\frac{9\pi}{8},\, \frac{5\pi}{4}\right),$$

whose corresponding wavelet set is

$$\begin{aligned} W =& \left[-2\pi,\, -\frac{7\pi}{4}\right) \cup \left[-\frac{3\pi}{2},\, -\pi\right) \cup \left[-\frac{7\pi}{8},\, -\frac{3\pi}{4}\right) \cup \left[\pi,\, \frac{9\pi}{8}\right) \cup \\ & \left[\frac{5\pi}{4},\, 2\pi\right) \cup \left[\frac{9\pi}{4},\, \frac{5\pi}{2}\right) \end{aligned}$$

having six components.

A similar discussion as for **Choice I** can be made for **Choice III** to have particular scaling sets.

5 n-Interval Scaling Sets.

Consider $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ in S^1, or equivalently, in $[0,\, 2\pi)$ such that $0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_n < 2\pi$. Recall that $p^{\leftarrow}(\alpha_1,\, \alpha_2,\, \alpha_3, ..., \alpha_n)$ denotes the set $[\alpha_1,\, \alpha_2) \cup [\alpha_2,\, \alpha_3) \cup ... \cup [\alpha_n,\, \alpha_1 + 2\pi)$ in $\mathbb{R}$. In this section, we provide n-interval scaling sets with the help of these points. First, assume that n is an odd natural number greater than 1. For n=1, $p^{\leftarrow}(\alpha_1) - 2\pi$ becomes a scaling set. Translate the interval $[\alpha_n,\, \alpha_1 + 2\pi)$ by -2π and intervals $[\alpha_1,\, \alpha_2)$, $[\alpha_3,\, \alpha_4)$, $[\alpha_5,\, \alpha_6), ..., [\alpha_{n-2},\, \alpha_{n-1})$, each by 2π, to have

$$S = \big[\alpha_n - 2\pi,\, \alpha_1\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \big[\alpha_{2m},\, \alpha_{2m+1}\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \big[\alpha_{2m-1} + 2\pi,\, \alpha_{2m} + 2\pi\big).$$

The requirement that $S \subset 2S$ for S to be a scaling set holds if the following conditions are satisfied:

(i) $\alpha_n \leq 2\alpha_1$,

(ii) $2\alpha_{2m} \leq \alpha_{2m-1} + 2\pi$,

(iii) $\alpha_{2m} + 2\pi \leq 2\alpha_{2m+1}$,

where $m \in \{1, 2, 3, ..., \frac{n-1}{2}\}$.
By construction, S is 2π-translation congruent to $[\alpha,\ \alpha+2\pi)$. Hence, by Result 2.5, it follows that S is a scaling set.

Below is given an alternative proof for S to be a scaling set. Note that

$$\begin{aligned} W =& \big[2(\alpha_n - 2\pi), \alpha_n - 2\pi\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \big[\alpha_{2m-1}, \alpha_{2m}\big) \cup \big[\alpha_n, 2\alpha_1\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \\ & \big[2\alpha_{2m}, \alpha_{2m-1} + 2\pi\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \big[\alpha_{2m} + 2\pi, 2\alpha_{2m+1}\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \\ & \big[2(\alpha_{2m-1} + 2\pi), 2(\alpha_{2m} + 2\pi)\big), \\ \equiv& I_1 \cup \bigcup_{m=1}^{\frac{n-1}{2}} I_{2,m} \cup I_3 \cup \bigcup_{m=1}^{\frac{n-1}{2}} I_{4,m} \cup \bigcup_{m=1}^{\frac{n-1}{2}} I_{5,m} \cup \bigcup_{m=1}^{\frac{n-1}{2}} I_{6,m} \qquad (say), \end{aligned}$$

where

$$\begin{aligned} I_1 =& \big[2(\alpha_n - 2\pi), \alpha_n - 2\pi\big), \\ I_{2,m} =& \big[\alpha_{2m-1}, \alpha_{2m}\big), \\ I_3 =& \big[\alpha_n, 2\alpha_1\big), \\ I_{4,m} =& \big[2\alpha_{2m}, \alpha_{2m-1} + 2\pi\big), \\ I_{5,m} =& \big[\alpha_{2m} + 2\pi, 2\alpha_{2m+1}\big), \\ I_{6,m} =& \big[2(\alpha_{2m-1} + 2\pi), 2(\alpha_{2m} + 2\pi)\big). \end{aligned}$$

That W is a wavelet set follows by observing that the maps τ_{α_1}, $\delta_{\alpha_n - 2\pi}$ and δ_{α_1} are bijections. Indeed,

(i) $\tau_{\alpha_1} : W \longrightarrow [\alpha_1,\ \alpha_1 + 2\pi)$ is defined by

$$\tau_{\alpha_1}(x) = \begin{cases} x & \text{if } x \in \bigcup_{m=1}^{\frac{n-1}{2}} I_{2,\,m} \cup I_3 \cup I_{4,\,1}, \\ x - 2\pi & \text{if } x \in \bigcup_{m=2}^{\frac{n-1}{2}} I_{4,\,m} \cup \bigcup_{m=1}^{\frac{n-1}{2}} I_{5,\,m}, \\ x - 4\pi & \text{if } x \in I_{6,\,1}, \\ x - 6\pi & \text{if } x \in \bigcup_{m=2}^{\frac{n-1}{2}} I_{6,\,m}, \\ x + 2\pi & \text{if } x \in I_1. \end{cases}$$

(ii) $\delta_{\alpha_n-2\pi} : W^- \longrightarrow \big[2(\alpha_n - 2\pi),\, \alpha_n - 2\pi\big)$, where $W^- \equiv I_1$, is defined by

$$\delta_{\alpha_n-2\pi}(x) = x.$$

(iii) $\delta_{\alpha_1} : W^+ \longrightarrow [\alpha_1,\, 2\alpha_1)$, where $W^+ \equiv W - W^-$, is defined by

$$\delta_{\alpha_1}(x) = \begin{cases} x & \text{if } x \in \bigcup_{m=1}^{\frac{n-1}{2}} I_{2,\,m} \cup I_3, \\ \frac{x}{2} & \text{if } x \in \bigcup_{m=1}^{\frac{n-1}{2}} I_{4,\,m} \cup \bigcup_{m=1}^{\frac{n-1}{2}} I_{5,\,m}, \\ \frac{x}{4} & \text{if } x \in \bigcup_{m=1}^{\frac{n-1}{2}} I_{6,\,m}. \end{cases}$$

Next, assume that n is an even natural number greater than 2. Translate the intervals $[\alpha_n,\, \alpha_1 + 2\pi)$ and $[\alpha_{n-2},\, \alpha_{n-1})$ by -2π and intervals $[\alpha_1,\, \alpha_2)$, $[\alpha_3,\, \alpha_4), [\alpha_5,\, \alpha_6), \ldots, [\alpha_{n-3},\, \alpha_{n-2})$, each by 2π, to have

$$S = \big[\alpha_{n-2} - 2\pi,\, \alpha_{n-1} - 2\pi\big) \cup \big[\alpha_n - 2\pi,\, \alpha_1\big) \cup \bigcup_{m=1}^{\frac{n-4}{2}} \big[\alpha_{2m},\, \alpha_{2m+1}\big) \cup \big[\alpha_{n-1},\, \alpha_n\big) \cup \bigcup_{m=1}^{\frac{n-2}{2}} \big[\alpha_{2m-1} + 2\pi,\, \alpha_{2m} + 2\pi\big).$$

The requirement that $S \subset 2S$ for S to be a scaling set holds if the following conditions are satisfied:

(i) $2\alpha_n = \alpha_{n-2} + 2\pi$,

(ii) $\alpha_n \le 2\alpha_1$,

(iii) $2\alpha_{n-1} \le \alpha_{n-3} + 2\pi$,

(iv) $2\alpha_{2m} \le \alpha_{2m-1} + 2\pi$,

(v) $\alpha_{2m} + 2\pi \le 2\alpha_{2m+1}$,

where $m \in \{1, 2, 3, ..., \frac{n-4}{2}\}$. In case $n = 4$, we require (i), (ii) and (iii), as others do not arise. By construction, S is 2π-translation congruent to $[\alpha,\, \alpha + 2\pi)$. Hence, by Result 2.5, it follows that S is a scaling set.

Note that

$$\begin{aligned} W =& \big[2(\alpha_{n-2} - 2\pi),\, 2(\alpha_{n-1} - 2\pi)\big) \cup \big[\alpha_{n-1} - 2\pi,\, \alpha_n - 2\pi\big) \cup \\ & \bigcup_{m=1}^{\frac{n-4}{2}} \big[\alpha_{2m-1},\, \alpha_{2m}\big) \cup \big[\alpha_{n-3},\, \alpha_{n-1}\big) \cup \big[\alpha_n,\, 2\alpha_1\big) \cup \\ & \bigcup_{m=1}^{\frac{n-4}{2}} \big[2\alpha_{2m},\, \alpha_{2m-1} + 2\pi\big) \cup \big[2\alpha_{n-1},\, \alpha_{n-3} + 2\pi\big) \cup \\ & \bigcup_{m=1}^{\frac{n-4}{2}} \big[\alpha_{2m} + 2\pi,\, 2\alpha_{2m+1}\big) \cup \bigcup_{m=1}^{\frac{n-2}{2}} \big[2(\alpha_{2m-1} + 2\pi),\, 2(\alpha_{2m} + 2\pi)\big). \end{aligned}$$

The above discussion is summed up below:

Result 5.1. Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ in S^1, or equivalently, in $[0, 2\pi)$ be such that $0 < \alpha_1 < \alpha_2 < \alpha_3 < ... < \alpha_n < 2\pi$, where n is a natural number. Then

(a) for n=1, $[\alpha_1 - 2\pi,\, \alpha_1)$ is a scaling set,

(b) for odd $n > 1$,

$$S = \big[\alpha_n - 2\pi,\, \alpha_1\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \big[\alpha_{2m},\, \alpha_{2m+1}\big) \cup \bigcup_{m=1}^{\frac{n-1}{2}} \big[\alpha_{2m-1} + 2\pi,\, \alpha_{2m} + 2\pi\big),$$

is a scaling set under the conditions:

(i) $\alpha_n \le 2\alpha_1$,

(ii) $2\alpha_{2m} \le \alpha_{2m-1} + 2\pi$,

(iii) $\alpha_{2m} + 2\pi \le 2\alpha_{2m+1}$,

where $m \in \{1, 2, 3, ..., \frac{n-1}{2}\}$.

(c) for even $n > 2$,

$$\begin{aligned} S =& \big[\alpha_{n-2} - 2\pi,\, \alpha_{n-1} - 2\pi\big) \cup \big[\alpha_n - 2\pi,\, \alpha_1\big) \cup \bigcup_{m=1}^{\frac{n-4}{2}} \big[\alpha_{2m},\, \alpha_{2m+1}\big) \cup \\ & \big[\alpha_{n-1},\, \alpha_n\big) \cup \bigcup_{m=1}^{\frac{n-2}{2}} \big[\alpha_{2m-1} + 2\pi, \alpha_{2m} + 2\pi\big), \end{aligned}$$

is a scaling set under the conditions:

(i) $2\alpha_n = \alpha_{n-2} + 2\pi$,

(ii) $\alpha_n \leq 2\alpha_1$,

(iii) $2\alpha_{n-1} \leq \alpha_{n-3} + 2\pi$,

(iv) $2\alpha_{2m} \leq \alpha_{2m-1} + 2\pi$,

(v) $\alpha_{2m} + 2\pi \leq 2\alpha_{2m+1}$,

where $m \in \{1, 2, 3, ..., \frac{n-4}{2}\}$. In case $n = 4$, (iv) and (v) do not arise.

Remark 5.2. In case $n = 2$, the condition $S \subset 2S$ is not satisfied and hence there is no two-interval scaling set.

Acknowledgment. The authors thank the referee for his valuable remarks and suggestions. They also thank their supervisor Professor K.K. Azad for his help and encouragement.

References

[1] L. Baggett, H. Medina and K. D. Merrill, *Generalized multiresolution analyses, and a construction procedure for all wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl., **5** (1999), 563–573.

[2] M. Bownik, Z. Rzeszotnik and D. Speegle, *A characterization of dimension functions of wavelets*, Appl. Comput. Harmon. Anal., **10** (2001), 71–92.

[3] X. Dai and D. Larson, *Wandering vectors for unitary systems and orthogonal wavelets*, (English summary) Mem. Amer. Math. Soc., **134(640)** (1998).

[4] X. Dai, D. Larson and D. Speegle, *Wavelet sets in* $\mathbb{R}^n$, J. Fourier Anal. Appl., **3** (1997), 451–456.

[5] X. Dai, D. Larson and D. Speegle, *Wavelet sets in* $\mathbb{R}^n$*, II*, Contemp. Math., **216** (1998), 15–40.

[6] Qing Gu and Deguang Han, *On Multiresolution Analysis (MRA) Wavelets in* $\mathbb{R}^n$, J. Fourier Anal. Appl., **6** (2000), 437–447.

[7] Young-Hwa Ha, Hyeonbae Kang, Jungseob Lee and Jin Keun Seo, *Unimodular wavelets for* L^2 *and the Hardy Space* H^2, Michigan Math. J., **41** (1994), 345–361.

[8] E. Hernández and G. L. Weiss, *A first course on wavelets*, English summary, With a foreword by Yves Meyer, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1996.

[9] E. Ionascu, D. Larson and C. Pearcy, *On wavelet sets,* J. Fourier Anal. Appl., **4** (1998), 711–721.

[10] K. D. Merrill, *Simple wavelet sets for scalar dilations in* $\mathbb{R}^2$, (English summary), Representations, wavelets, and frames, 177–192, Appl. Numer. Harmon. Anal., Birkhäuser Boston, Boston, MA, 2008.