RESEARCH

Real Analysis Exchange
Vol. 35(1), 2009/2010, pp. 91–120

Nicholas Freeman, St Anne's College, Woodstock Road, Oxford, OX2 6HS.
email: nicholas.freeman@st-annes.ox.ac.uk

PERTURBED ITERATED FUNCTION SYSTEMS AND THE EXACT HAUSDORFF MEASURE OF THEIR ATTRACTORS

Abstract

We define a perturbed iterated function system (pIFS) in $\mathbb{R}^d$ as, loosely speaking, a sequence of iterated function systems (IFSs) whose constituent transformations converge towards some limiting IFS. We define the attractor of such a system in a similar style to that of an IFS, and prove that such a set exists uniquely. We define a partially perturbed IFS (ppIFS) to be a perturbed IFS with a constant tail. In a setup with similitudes and the strong separation condition we show that a pIFS attractor can be approximated by a sequence of ppIFS attractors in such a way that the Hausdorff measure is preserved in the limit. We use this result to calculate the exact Hausdorff measure of the pIFS attractor from that of the limiting IFS.

1 Introduction.

Perturbed Cantor sets, obtained by repeated deletion of varying (typically non-middle-third) intervals from $[0,1]$ have been studied in [2]. In this paper we develop a similar construction in $\mathbb{R}^d$ using arbitrary Lipschitz contractions.

We consider a sequence $\{\{S_t^j\}_{t=1}^p\}_{j\in\mathbb{N}}$ of IFSs (on $\mathbb{R}^d$) with a limiting IFS $\{S_t\}_{t=1}^p$ such that $S_t^j \to S_t$ in some sense as $j \to \infty$. Let T^j denote the application of $\{S_t^j\}_{t=1}^p$, so as $T^j(R) = \bigcup_{t=1}^{p} S_t^j(R)$ for $R \subseteq \mathbb{R}^d$. We define the

Mathematical Reviews subject classification: Primary: 28A78; Secondary: 28A80
Key words: Hausdorff dimension, Hausdorff measure, fractal, Cantor set, deranged Cantor set, iterated function system, IFS, perturbed iterated function system, pIFS
Received by the editors October 18, 2008
Communicated by: R. Daniel Mauldin

attractor of the system to be the unique non-empty compact set such that

$$E = \lim_{j\to\infty} T^1 \circ \ldots \circ T^j(E)$$

(in the Hausdorff metric). Reasons why the T^j are applied apparently in reverse order become obvious from drawing the first few prefractals.

Our first section is concerned with the precise statement and proof that the attractor of a pIFS is well defined and unique.

When looking for the Hausdorff measure of the attractors we do not consider all pIFSs and our second section is devoted to defining the setup and notation we work in. This is essentially the set of pIFSs whose constituent transformations are similitudes with a mild convergence condition. We are only concerned with Hausdorff measure, but various properties of perturbed Cantor sets have been studied for other measures, see [2] and [4] for some examples.

The bulk of our results come in two parts. Firstly, we show that given any pIFS satisfying the conditions above, we can approximate its attractor with a sequence of attractors of related, but much simpler, pIFSs. Further, the measure of the pIFS attractor is the limit of the measures of the attractors of these systems. Secondly, we use this result to obtain a formula for the ratio of the measures of the attractor of the pIFS and it's associated limiting IFS.

Our proofs are 'bare hands' methods using elementary real analysis. The only prerequisite definitions are those of an IFS and Hausdorff dimension and measure; These can be found in [1].

1.1 Our Results.

The following theorems summarise our results in self contained statements.

Theorem 1.1 (Generalising the fundamental theorem of IFSs)**.** *We work in* $\mathbb{R}^d$*. Let* $\{\{S_t^j\}_{t=1}^p\}_j$ *be a sequence of IFSs and* $\{S_t\}_{t=1}^p$ *an IFS. Suppose there exists a non-empty compact set* C *such that* $S_t^j(C), S_t(C) \subseteq C$*, and suppose that when restricted to* C*,* $S_t^j \to S_t$ *in the supremum metric.*

Define the attractor of the pIFS $\{\{S_t^j\}_{t=1}^p\}_j\}$ *to be the non-empty compact set* E *such that*

$$E = \lim_{j\to\infty} T^1 \circ \ldots \circ T^j(E)$$

(in the Hausdorff metric).

Then the attractor E *of the pIFS is well defined, unique to the pIFS, non-empty, compact and independent of the choice of* C*. Further, if* C' *is another non-empty compact set such that* $S_t^j(C'), S_t(C') \subseteq C'$ *then* $E = \lim_{j\to\infty} T^1 \circ \ldots \circ T^j(E')$ *for every non-empty compact* $E' \subseteq C'$*.*

Theorem 1.2 (On the measure of the attractors of pIFSs)**.** *In the setup of the preceding theorem, let*

$$S_t^j(x) = c_t^j + A_t^j(x) \quad \textit{and} \quad S_t(x) = c_t + A_t(x)$$

where $c_t^j, c_t \in \mathbb{R}^d$ *and* A_t^j, A_t *are linear transformations of* $\mathbb{R}^d$*. Suppose for each* A_t^j, A_t *there exists* a_t^j, a_t *such that*

$$||A_t^j(x)|| = a_t^j||x|| \quad \textit{and} \quad ||A_t(x)|| = a_t||x||.$$

Assume the following conditions are met:

(1) There exists $\lambda < 1$ *and* $\eta > 0$ *such that for all* j, t, $\eta \leq a_t^j, a_t \leq \lambda$.

(2) For all t, $c_t^j \to c_t$ *in* $\mathbb{R}^d$.

(3) For all t, $A_t^j \to A_t$ *in the* $|| \cdot ||_\infty$ *norm on some ball (independent of* t*) about* 0.

(4) For all t, $\sum_{j=1}^{\infty} |a_t^j - a_t| < \infty$.

(5) For each $m \in \mathbb{N}$*, for no pair* $\mathbf{i} = \{i_1, i_2, \ldots\}$ *and* $\mathbf{j} = \{j_1, j_2, \ldots\}$ *with* $\mathbf{i} \neq \mathbf{j}$ *do either of*

$$\lim_{n\to\infty} S_{i_1}^1 \circ \ldots \circ S_{i_n}^n(0) = \lim_{n\to\infty} S_{j_1}^1 \circ \ldots \circ S_{j_n}^n(0)$$

$$\lim_{n\to\infty} S_{i_1}^1 \circ \ldots \circ S_{i_m}^m S_{i_{m+1}} \ldots S_{i_n}(0) = \lim_{n\to\infty} S_{j_1}^1 \circ \ldots \circ S_{j_m}^m S_{j_{m+1}} \ldots S_{j_n}(0)$$

occur.

Then the preceding theorem applies. Let F *denote the attractor of the IFS* $\{S_t\}_{t=1}^p$*, and* E *the attractor of the pIFS* $\{\{S_t^j\}_{t=1}^p\}_j$*. Further, with* $s = \dim_{\mathcal{H}}(F)$ *(i.e.* $\sum_{t=1}^{p}(a_t)^s = 1$*),*

$$\mathcal{H}^s(E) = \mathcal{H}^s(F) \times \prod_{n=1}^{\infty} \sum_{t=1}^{p} (a_t^j)^s \tag{1}$$

where $\dim_{\mathcal{H}}$ *denotes Hausdorff dimension and* $\mathcal{H}^s$ *denotes s-dimensional Hausdorff measure. Finally,* $0 < \prod_{n=1}^{\infty} \sum_{t=1}^{p} (a_t^j)^s < \infty$*, and hence* $\dim_{\mathcal{H}}(F) = \dim_{\mathcal{H}}(E)$*.*

Condition *(5)*, upon which the proofs given here are heavily reliant, implies the strong separation condition on the limiting IFS $\{S_t\}_{t=1}^p$ (i.e. there exists a compact set R containing the attractor of $\{S_t\}_{t=1}^p$ such that the images $S_t(R)$ are positively separated). Weakening this condition seems to cause the precise result of 1 to fail; see comment 7.1. Weakening *(5)* can cause the dimension of the pIFS attractor to vary from that of the limiting IFS. This is of interest in it's own right, but is not studied here.

Whilst quite natural, condition *(5)* is difficult to verify and our final section proves that it holds in a setup covering many canonical examples of IFSs we might perturb. The method (which constructs a decreasing sequence of sets converging to E) serves as a template for checking the condition in other cases. The result is as follows:

Theorem 1.3 (Satisfying condition *(5)*)**.** *In the setup of the preceding theorem, assuming only conditions (1)-(3), suppose we have the strong separation condition on* $\{S_t\}_{t=1}^p$ *and also the following:*

(5a) For all n, $\max_t \frac{||c_t^n||}{1-a_t^n} \leq \max_t ||\frac{c_t}{1-a_t}|| < \min_{s\neq t} \frac{||c_s^n - c_t^n||}{a_s^n + a_t^n}$.

Then condition (5) holds.

The above results correspond to Theorems 2.3, 5.5 and 6.2 (in the text these do not have self contained statements).

1.2 Notation.

What follows will involve very many superscripts and subscripts. To accommodate this we write sequences as superscripts and reserve subscripts for indexes. Of course $(x)^n$ and $|x|^n$ still refer to powers. Our perturbed systems will be denoted by ^{j} as opposed to a blank superscript, and we write $^{(j)}$ when we wish to specify a condition holding both in a perturbed and unperturbed form. For example, $a_t^{(j)} = 0$ serves as shorthand for $a_t^j = a_t = 0$ for all t, j. On rare occasions we will need to raise to a power s, where s denotes Hausdorff dimension. The distinction between ^{j} and ^{s} will be apparent from both the symbol and context, and should not cause confusion.

We write $||\,||$ as the Euclidean norm, $||\,||_\infty$ as the supremum norm, and $d(\,,\,)$ as the Hausdorff metric. We denote the diameter function $\text{diam}(R)$ by $|R|$. We write the Lipschitz constant as

$$\text{lip}(S) = \inf\{\kappa : \forall x, y \in \text{the domain of } S, ||Sx - Sy|| \leq \kappa||x-y||\}$$

and the distance function as

$$\text{dist}(X,Y) = \inf\{||x-y|| : x \in X, y \in Y\};$$

conv(R) denotes the convex hull of R.

For our purposes, an iterated function system consists of a finite number of Lipschitz contractions defined on $\mathbb{R}^d$. The strong separation condition (SSC) is said to hold on the IFS $\{S_t\}_{t=1}^p$ when there exists a non-empty compact set R, containing the attractor of $\{S_t\}_{t=1}^p$, such that the images $S_t(R)$ are positively separated.

2 Perturbed Iterated Function Systems.

We begin by defining precisely what we mean by a pIFS.

Fix $p \in \mathbb{N}$. For each $j \in \mathbb{N}$, let $\{S_t^j\}_{t=1}^p$ be an IFS. Then $\{\{S_t^j\}_{t=1}^p\}_j$ is a sequence of IFSs. Let $\{S_t\}_{t=1}^p$ be an IFS and let $\mathcal{A}(\{S_t\}_{t=1}^p)$ denote it's attractor. Suppose

(i) *There exists a non-empty compact set $C \subseteq \mathbb{R}^d$ such that for all $1 \leq t \leq p$ and j, $S_t^{(j)}(C) \subseteq C$.*
Define $\mathcal{S}_t^j$ and $\mathcal{S}_t$ to be the restrictions of S_t^j and S_t to C. Then $\{\mathcal{S}_t^j, \mathcal{S}_t : t = 1, \ldots p,\ j \in \mathbb{N}\}$ is a normed space with the $|| \, ||_\infty$ norm (on $\mathbb{R}^d$). Suppose also,

(ii) $\mathcal{S}_t^j \to \mathcal{S}_t$ *in* $|| \, ||_\infty$.

Then we say $\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j\}$ is a perturbed iterated function system (pIFS), in particular a perturbation of $\{S_t\}_{t=1}^p$. We say $\{S_t\}_{t=1}^p$ is the limiting IFS of $\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j\}$.

We refer to C as the parent set of the pIFS. Recall that the set of non-empty compact subsets of C forms a complete metric space under the Hausdorff metric d. Call this space $\mathcal{K}_C$.

We denote $T^j = \bigcup_{t=1}^{p} \mathcal{S}_t^j$ and $\mathcal{T}^j = T^1 \circ T^2 \circ \ldots \circ T^j$ (where $(S \circ T)(x) = S(T(x))$). Define the attractor of $\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j\}$, denoted $\mathcal{A}(\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j)$, to be the unique non-empty compact set E so as $E = \lim_{j\to\infty} \mathcal{T}^j(E)$ (in $\mathcal{K}_C$). In order to answer existence and well-definedness questions we will prove an analogue of the fundamental theorem of IFSs (Theorem 2.3). First we must establish some inequalities.

Lemma 2.1. *Let A and B be compact. Let T_1 and T_2, be Lipschitz transformations on $A \cup B$, so $||T_1 - T_2||_\infty, d(A, B) < \infty$. Then*

$$d\big(T_1(A), T_2(B)\big) \leq d(A, B) \max\big\{\operatorname{lip}(T_1), \operatorname{lip}(T_2)\big\} + ||T_1 - T_2||_\infty.$$

PROOF. We prove a slightly stronger result. Since $d(\ ,\ )$ is a metric,

$$d\big(T_1(A), T_2(B)\big) \leq d\big(T_1(A), T_1(B)\big) + d\big(T_1(B), T_2(B)\big).$$

Also, note that

$$d\big(T_1(A), T_1(B)\big) \leq d(A, B)\ \mathrm{lip}(T_1)$$
$$d\big(T_1(B), T_2(B)\big) \leq \sup_{x \in B} ||T_1(x) - T_2(x)||$$

and the result follows. □

Lemma 2.2. *Let $\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j$ be a perturbation of the IFS $\{S_t\}_{t=1}^p$. Then*

(i) $d(\mathcal{T}^j(A), \mathcal{T}^j(B)) \leq d(A, B)\, \chi^j$

(ii) $$d\big(\mathcal{T}^j(A), \mathcal{T}^k(B)\big) \leq d(\mathcal{T}^{j-v}(A), \mathcal{T}^{k-v}B)\, \chi^v + \sum_{l=0}^{v-1} \Big(\Big[\max_{t=1,\ldots,p} \big(||\mathcal{S}_t^{j-l} - \mathcal{S}_t^{k-l}||_\infty \big) \Big] \chi^l \Big)$$

where χ is such that $\sup_{t,j} \big(\mathrm{lip}(S_t^{(j)}) \big) \leq \chi < 1$.

PROOF. Note that $\mathcal{T}^j$, $\mathcal{S}_t^j$ and $\mathcal{S}_t$ all map elements of $\mathcal{K}_C$ to elements of $\mathcal{K}_C$. Note also that (i) follows from (ii), so we prove (ii). Let $A, B \in \mathcal{K}_C$. For any $j, k > 1$,

$$\begin{aligned} d(\mathcal{T}^j(A), \mathcal{T}^k(B)) &= d\Big(\bigcup_{t=1}^p \mathcal{S}_t^j(\mathcal{T}^{j-1}(A)), \bigcup_{t=1}^p \mathcal{S}_t^k(\mathcal{T}^{k-1}(B) \Big) \\ &\leq \max_t \Big[d\big(S_t^j(\mathcal{T}^{j-1}(A)), \mathcal{S}_t^k(\mathcal{T}^{k-1}(B)) \big) \Big] \end{aligned}$$

Applying Lemma 2.1 we get

$$\begin{aligned} &\max_t \big[d\big(\mathcal{S}_t^j(\mathcal{T}^{j-1}(A)), \mathcal{S}_t^k(\mathcal{T}^{k-1}(B)) \big) \big] \\ &\leq d\big(\mathcal{T}^{j-1}(A), \mathcal{T}^{k-1}(B)\big) \max_{t=1,\ldots,p\ m=j,k} \big(|\, \mathrm{lip}(\mathcal{S}_t^m)| \big) + \max_{t=1,\ldots,p} \big(||\mathcal{S}_t^j - \mathcal{S}_t^k||_\infty \big). \end{aligned}$$

We note $\mathcal{S}_t^j \to \mathcal{S}_t$ implies $\mathrm{lip}\, \mathcal{S}_t^j \to \mathrm{lip}\, \mathcal{S}_t$, and also (because of the restricted domain), $\mathrm{lip}\, \mathcal{S}_t^{(j)} \leq \mathrm{lip}\, S_t^{(j)} < 1$. t takes values from the finite set $\{1, \ldots, p\}$, hence there is some χ such that $\sup_{t,j} \big(\mathrm{lip}\, \mathcal{S}_t^{(j)} \big) < \chi < 1$. So,

$$\begin{aligned} &d(\mathcal{T}^j(A), \mathcal{T}^k(B)) \\ &\qquad \leq \chi\, d\big(\mathcal{T}^{j-1}(A), \mathcal{T}^{k-1}(B)\big) + \max_{t=1,\ldots,p} \big(||S_t^j - S_t^k||_\infty \big). \end{aligned}$$

Repeated application v times gives, for $j, k > v$,

$$\begin{aligned} &d\big(\mathcal{T}^j(A), \mathcal{T}^k(B)\big) \\ &\qquad \leq d(\mathcal{T}^{j-v}(A), \mathcal{T}^{k-v}B)\,\chi^v + \sum_{l=0}^{v-1}\Big(\Big[\max_{t=1,\ldots,p}\big(||\mathcal{S}_t^{j-l} - \mathcal{S}_t^{k-l}||_\infty\big)\Big]\chi^l\Big) \end{aligned}$$

which is the result. □

Theorem 2.3. *Let $\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j$ be a perturbation of the IFS $\{S_t\}_{t=1}^p$. Let C be a parent set of $\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j$. Then there exists a non-empty compact set $E \subseteq C$ such that*

(I) $\lim_{j\to\infty} \mathcal{T}^j(E) = E$ (under the Hausdorff metric).

(II) If $E' \subseteq C$ is any non-empty compact set then $\lim_{j\to\infty} \mathcal{T}^j(E') = E$.

(III) If C' is some other parent set and $E_{C'}$ is the unique non empty compact set arising out of using C' as the domain for the restricted $S_t^{(j)}$, then $E = E_{C'}$.

Proof. Recall $\mathcal{K}_C$ is the metric space of non-empty compact subsets of C. We prove first that for some $E' \in \mathcal{K}_C$, $\mathcal{T}^j(E')$ is a d-Cauchy sequence (and hence is convergent). We then show that the limit $E \in \mathcal{K}_C$ is independent of E'. Finally we show E does not depend on C, in the sense of (III) above.

Let $E' \in \mathcal{K}_C$, and let $\epsilon > 0$. Recall that by Lemma 2.2 there exists χ such that $\sup_{t,l}\big(\mathrm{lip}(S_t^l)\big) < \chi < 1$. $\mathcal{S}_t^j \to \mathcal{S}_t$ and so $\mathcal{S}_t^j$ is Cauchy, $t \in \{1, \ldots, p\}$ so there exists M_1 such that

$$\sup_{t,\ j>k>M_1}\big(||\mathcal{S}_t^j - \mathcal{S}_t^k||_\infty\big) < \frac{\epsilon}{2}(1-\chi).$$

For any $A \in \mathcal{K}$, for all j, k, $\mathcal{T}^j(A), \mathcal{T}^k(A) \subseteq C$, and so $d(\mathcal{T}^j(A), \mathcal{T}^k(A)) \leq |C|$. Let M_2 be sufficiently large so as for $v \geq M_2$,

$$\chi^v < \frac{\epsilon}{2|C|}$$

Now, for $j > k > M = M_1 + M_2$, we have (by Lemma 2.2(ii))

$$d(\mathcal{T}^j(E'), \mathcal{T}^k(E') \le d\big(\mathcal{T}^{j-M_2}(E'), \mathcal{T}^{k-M_2}(E')\big)\,\chi^{M_2} + \sum_{l=0}^{M_2-1}\Big(\chi^l \max_{t=1,\dots,p}\big(||\mathcal{S}_t^{j-l} - \mathcal{S}_t^{k-l}||_\infty\big)\Big).$$

In the sum on right, $j-l, k-l > M - (M_2 - 1) > M_1$, so

$$\max_{t=1,\dots,p}\big(||\mathcal{S}_t^{j-l} - \mathcal{S}_t^{k-l}||\big) < \frac{\epsilon}{2}(1-\chi).$$

Also $d\big(\mathcal{T}^{j-M_2}(E'), \mathcal{T}^{k-M_2}(E')\big) < |C|$ and $\chi^{M_2} < \frac{\epsilon}{2}|C|^{-1}$. So

$$d\big(\mathcal{T}^j(E'), \mathcal{T}^k(E')\big) \le |C||C|^{-1}\,\frac{\epsilon}{2} + \frac{\epsilon}{2}(1-\chi)\sum_{l=0}^{M_2-1}\chi^l \le \frac{\epsilon}{2} + \frac{\epsilon}{2}(1-\chi)\frac{1}{1-\chi} = \epsilon.$$

This proves that $\mathcal{T}^j(E')$ is Cauchy. Hence there exists a non-empty compact set $E \in \mathcal{K}_C$ such that $\mathcal{T}^j(E') \to E$ under d.
In order to prove (I) and (II) we must show that E did not depend on $E' \in \mathcal{K}_C$.

Let $A, B \in \mathcal{K}_C$. Then, by Lemma 2.2(i), $d(\mathcal{T}^j(A), \mathcal{T}^j(B)) \le d(A,B)\,\chi^j$. Clearly the RHS of this tends to zero as $j \to \infty$. Let E_0 be the limit of $\mathcal{T}^j(A)$; so that $d(\mathcal{T}^j(A), E_0) \to 0$. Noting that

$$d(\mathcal{T}^j(B), E_0) \le d(\mathcal{T}^j(B), \mathcal{T}^j(A)) + d(\mathcal{T}^j(A), E_0)$$

we have $\mathcal{T}^j(B) \to E_0$. Hence any two sets $A, B \in \mathcal{K}_C$ have the same limiting set under the application of the pIFS; the limit is therefore dependent only on the pIFS itself.

We still need to prove (III); that E does not depend on C. Let C' be some other set such that $S_t^j(C'), S_t(C') \subseteq C'$, and (by (I) and (II)) let $E_{C'} \subseteq C'$ denote the unique non-empty compact set such that $\lim_{j\to\infty} \mathcal{T}^j(E_{C'}) = E_{C'}$.

Let $\delta = d(C, C')$. Since C and C' are compact, $[C \cup C']_\delta = \{x \in \mathbb{R}^d : \operatorname{dist}(x, C \cup C') \le \delta\}$ is compact. Let $S_t^j(x) \in S_t^j\big([C \cup C']_\delta\big)$. Then there exists $y \in C$ s.t $\operatorname{dist}(y,x) < \delta$. $S_t^j(y) \in C$, and $d\big(S_t^j(y), S_t^j(x)\big) \le \chi \operatorname{dist}(y,x)$. Hence $S_t^j(y) \in S_t^j\big([C \cup C']_\delta\big)$.

Let $\widetilde{\mathcal{S}}_t^j$ be the restriction of S_t^j to $[C \cup C']_\delta$, $\widetilde{T}^j = \bigcup_{t=1}^{p} \widetilde{\mathcal{S}}_t^j$ and $\widetilde{\mathcal{T}}_t^j = \widetilde{T}^1 \circ \ldots \circ \widetilde{T}^j$. By (I) and (II) there is a unique non-empty compact set $E_{[C\cup C']_\delta}$ such that for any non-empty compact set $E'' \subseteq [C \cup C']_\delta$, $\lim_{j\to\infty} \widetilde{\mathcal{T}}^j(E'') = E_{[C\cup C']_\delta}$. Of course, this implies that for any non-empty compact set $E' \subseteq C$, $\lim_{j\to\infty} \widetilde{\mathcal{T}}^j(E') = E_{[C\cup C']_\delta}$. Hence $E = E_{C'} = E_{[C\cup C']_\delta}$. □

This result is, of course, proof that the attractor of a pIFS exists, is unique, and does not depend on the choice of parent set. It generalises the fundamental theorem of IFSs. We now restrict our attention to a more specialised class of pIFSs.

3 Setup and Conditions.

Ultimately we are interested in the measure of the attractors of pIFSs. The reader will have noticed that our definition of a pIFS from an IFS was quite general. We now restrict ourselves to more familiar ground. We will require the functions in the IFSs making up our pIFS and it's limiting IFS to be the composition of a translation and a linear similarity.

Fix $p > 1$, and let c_t, c_t^j be vectors for $t = 1, \ldots, p$. Let A_t^j, A_t be linear transformations on $\mathbb{R}^d$. We define S_t^j and S_t by $S_t^{(j)}(x) = c_t^{(j)} + A_t^{(j)}(x)$. These are the transformations in our IFS and pIFS.

Will we delay a formal statement of our conditions on the $A_t^{(j)}$s and $c_t^{(j)}$s until after we have set up more notation. First, we give the natural conditions under which $\{\{S_t^j\}_{t=1}^p\}_j$ is a pIFS. Our conditions in 3.4 will trivially imply the conditions for the following results.

Lemma 3.1. *Suppose there exist some constants Ω and $\lambda < 1$ such that $||c_t^{(j)}|| < \Omega$ and $||A_t^{(j)}x|| \leq \lambda||x||$. Then for large r the set $\overline{B}_r(0) = \{x : ||x|| \leq r\}$ is a non-empty compact set so as $S_t^j(\overline{B}_r(0)), S_t(\overline{B}_r(0)) \subseteq \overline{B}_r(0)$.*

Proof. This follows from noting $||c_t^{(j)} + A_t^{(j)}x|| \leq \Omega + \lambda||x||$ and taking $r \geq \frac{\Omega}{1-\lambda}$. □

In other words, $\{\{S_t^j\}_{t=1}^p\}_j$ has arbitrarily large sets as potential parent sets.

Bearing this in mind, we *restrict* S_t^j, S_t *to* $C = \overline{B}_r(0)$ and work with the metric spaces $(\mathcal{K}_C, d)$ and $(\{f : C \to C : f \text{ is bounded}\}, || \, ||_\infty)$. Defining the transformations and then immediately restricting their domain has avoided a lot of what is now superfluous notation (the distinction between S and $\mathcal{S}$). We restrict our $A_t^{(j)}$ similarly.

The natural conditions under which $\{\{S_t^j\}_{t=1}^p\}_j$ is a pIFS are obvious.

Lemma 3.2. *If $c_t^j \to c_t$ in $|| \, ||$, $A_t^j \to A_t$ in $|| \, ||_\infty$, and $\{\{S_t^j\}_{t=1}^p\}_j$ has a parent set, $\{\{\mathcal{S}_t^j\}_{t=1}^p\}_j$ is a pIFS of $\{S_t\}_{t=1}^p$.*

PROOF. We leave the proof (that defined on C, $S_t^j \to S_t$) to the reader. □

Theorem 2.3 applies and the attractor of the pIFS $\{\{S_t^j\}_{t=1}^p\}_j\}$ is well defined.

We write $\mathbf{i} = \{i_1, i_2, \ldots\}$ and $\mathbf{j} = \{j_1, j_2, \ldots\}$ where i_k, j_k are drawn from $\{1, \ldots, p\}$.
Let

$$k_{\mathbf{i},\mathbf{j}} = \sup\{n : \forall l \leq n, i_l = j_l\}$$

so as ($k_{\mathbf{i},\mathbf{j}} = \infty \iff \mathbf{i} = \mathbf{j}$). Define

$$\begin{aligned}
x_{i_1,\ldots,i_n} &= c_{i_1} + \ldots + A_{i_1} \ldots A_{i_{n-1}} c_{i_n} \\
x'_{i_1,\ldots,i_n} &= c^1_{i_1} + \ldots + A^1_{i_1} \ldots A^{n-1}_{i_{n-1}} c^n_{i_n} \\
x^n_{i_1,\ldots,i_m} &= c^1_{i_1} + \ldots + A^{n-1}_{i_{n-1}} c^n_{i_n} + A^1_{i_1} \ldots A^n_{i_n} (c_{i_{n+1}} + \ldots + A_{i_{n+1}} \ldots A_{i_{m-1}} c_{i_m})
\end{aligned}$$

Also,

$$\begin{aligned}
x_{\mathbf{i}} &= \lim_{n\to\infty} x_{i_1,\ldots,i_n} \\
x'_{\mathbf{i}} &= \lim_{n\to\infty} x'_{i_1,\ldots,i_n} \\
x^n_{\mathbf{i}} &= \lim_{m\to\infty} x^n_{i_1,\ldots,i_m}.
\end{aligned}$$

Our conditions in 3.4 will imply that all these sums converge. Define

$$\begin{aligned}
F_{\{A_t\},\{c_t\}_{t=1}^p} &= \{x_{\mathbf{i}} : \mathbf{i} \in \{1, 2, \ldots, p\}^{\mathbb{N}}\} \\
E_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p} &= \{x'_{\mathbf{i}} : \mathbf{i} \in \{1, 2, \ldots, p\}^{\mathbb{N}}\} \\
E^n_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p} &= \{x^n_{\mathbf{i}} : \mathbf{i} \in \{1, 2, \ldots, p\}^{\mathbb{N}}\}.
\end{aligned}$$

Call these F-sets, E-sets, and E^n-sets respectively. Note the E^n-sets depend on A_t and c_t as well as their perturbed equivalents; since these may be obtained as limits from their perturbed equivalents we do not notate this.

$\mathbf{i}$ should be thought of as the 'address' of the point $x^\star_{\mathbf{i}}$ and the expressions above define the way in which this address is interpreted. To understand its geometric meaning, we give the following result.

Theorem 3.3. *Suppose all the series $x_{\mathbf{i}}$, $x'_{\mathbf{i}}$ and $x^n_{\mathbf{i}}$ converge in C. Then (i) $\mathcal{A}(\{S_t\}_{t=1}^p) = F_{\{A_t\},\{c_t\}_{t=1}^p}$ and (ii) $\mathcal{A}(\{\{S_t^j\}_{t=1}^p\}_j\}) = E_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p}$. If we define a pIFS $\{\{R_t^j\}_{t=1}^p\}_j$ by $R_t^j = S_t^j$ for $j \leq n$ and $R_t^j = S_t$ for $j > n$ we then have (iii) $\mathcal{A}(\{\{R_t^j\}_{t=1}^p\}_j) = E^n_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p}$.*

Proof. Note that all these assertions are special cases of *(ii)*. It is easy for us to calculate

$$T^1 \circ T^2 \circ \ldots \circ T^n(\{0\}) = \{c^1_{i_1} + \ldots + A^1_{i_1} \ldots A^{n-1}_{i_{n-1}} c^n_{i_n} \ : \{i_1, \ldots, i_n\} \in \{1, \ldots, p\}^n\}$$

and Theorem 2.3 gives that these sets converge to $\mathcal{A}(\{\mathcal{S}^j_t\}^p_{t=1}\}_j\})$. It is a simple matter to show they also converge to $E_{\{\{A^j_t\}_j, \{c^j_t\}_j\}^p_{t=1}}$. □

Conditions 3.4.
Our notation is now set up and we state our conditions.

(1) $A^{(j)}_t$ *is linear and there exists* $a^{(j)}_t$ *such that for all* $x \in \mathbb{R}^d$,

$$||A^{(j)}_t x|| = a^{(j)}_t ||x||.$$

Moreover, there exists $\lambda < 1$ *and* $\eta > 0$ *such that for all* j, t, $\eta \leq a^{(j)}_t \leq \lambda$.

(2) $c^j_t \to c_t$ *in* $\mathbb{R}^d$.

(3) $A^j_t \to A_t$ *in the* $|| \, ||_\infty$ *norm on functions on the parent set* C.

(4) There is some $\gamma > 0$ *such that for all* $1 \leq t \leq p$, $\frac{a^j_t}{a_t} \in [\gamma, \infty)$, *and*

$$\sum_{j=1}^{\infty} |a^j_t - a_t| < \infty.$$

(5) $\mathbf{i} = \mathbf{j} \iff x'_{\mathbf{i}} = x'_{\mathbf{j}}$, *and for all* n, $\mathbf{i} = \mathbf{j} \iff x^n_{\mathbf{i}} = x^n_{\mathbf{j}}$.

Linearity and *(1)* imply that our $S^{(j)}_t$ are similitudes. *(1)* implies $A^j_t(0) = 0$. Note if $a_t \neq 0$ the existence of η and λ is implied by *(2)*. *(2)* and *(3)* are the natural conditions to make our setup a pIFS. Conditions *(1)-(3)* are more than sufficient to imply all the preceding results hold.

As we will see, *(4)* is not only the strength of convergence we require to approximate our pIFS attractor, but a condition to make it's measure positive and finite. The existence of γ is clearly implied by *(1)* but it will be convenient for us to have it included as part of *(4)* as well.

We record a consequence of *(5)*, in particular a consequence of $\mathbf{i} = \mathbf{j} \iff x^n_{\mathbf{i}} = x^n_{\mathbf{j}}$.

We say an IFS $\{S_t\}^p_{t=1}$ satisfies the *strong separation condition* (SSC) if there exists $\alpha > 0$ and a compact set K such that $d(S_{t_1}(K), S_{t_2}(K)) > \alpha$ when $t_1 \neq t_2$. Our conditions above, in particular the effect of *(5)*, are more than enough for the following lemma.

Lemma 3.5. *Assume (1)-(3). Suppose for some n, $\mathbf{i} = \mathbf{j} \iff x^n_{\mathbf{i}} = x^n_{\mathbf{j}}$. Then the SSC holds for $\{S_t\}_{t=1}^p$. In particular, there exists some $\alpha > 0$ such that for all $\mathbf{i} \neq \mathbf{j}$,*

$$||(c_{i_{k_{\mathbf{i},\mathbf{j}}+1}} - c_{j_{k_{\mathbf{i},\mathbf{j}}+1}}) + (A_{i_{k_{\mathbf{i},\mathbf{j}}+1}} c_{i_{k+2}} - A_{j_{k_{\mathbf{i},\mathbf{j}}+1}} c_{j_{k_{\mathbf{i},\mathbf{j}}+2}}) + \ldots|| \geq \alpha. \qquad (2)$$

PROOF. For $\mathbf{i}$ and $\mathbf{j}$ with $i_1 \neq j_1$, $||x^n_{1,\ldots,1,i_1,i_2,\ldots} - x^n_{1,\ldots,1,j_1,j_2,\ldots}|| > 0$ (where the index begins with n 1's), so by *(1)*, $||x_{\mathbf{i}} - x_{\mathbf{j}}|| > 0$. Now consider the action of $\{S_t\}_{t=1}^p$ on its own attractor $F_{\{A_t\},\{c_t\}_{t=1}^p}$. For $t_1 \neq t_2$, $S_{t_1}(F_{\{A_t\},\{c_t\}_{t=1}^p})$ and $S_{t_2}(F_{\{A_t\},\{c_t\}_{t=1}^p})$ are two non-intersecting closed sets, and hence have a positive distance between them. There are only finitely many $S_t(F_{\{A_t\},\{c_t\}_{t=1}^p})$, hence there is a positive distance between any pair of them. That is to say, the SSC holds with $G = F_{\{A_t\},\{c_t\}_{t=1}^p}$.

By the SSC, there exists some $\alpha > 0$ such that for all $\mathbf{i}$ and $\mathbf{j}$ with $i_1 \neq i_2$,

$$||x_{\mathbf{i}} - x_{\mathbf{j}}|| = ||(c_{i_1} - c_{j_1}) + (A_{i_1} c_{i_2} - A_{j_1} c_{j_2}) + \ldots|| \geq \alpha.$$

For *any* $\mathbf{i}$ and $\mathbf{j}$ we then consider the addresses

$$\mathbf{i}' = \{i_{k_{\mathbf{i},\mathbf{j}}+1}, i_{k_{\mathbf{i},\mathbf{j}}+2}, \ldots\} \quad \text{and} \quad \mathbf{j}' = \{j_{k_{\mathbf{i},\mathbf{j}}+1}, j_{k_{\mathbf{i},\mathbf{j}}+2}, \ldots\}$$

which by definition of $k_{\mathbf{i},\mathbf{j}}$ satisfy $i'_1 = i_{k_{\mathbf{i},\mathbf{j}}+1} \neq j'_1 = j_{k_{\mathbf{i},\mathbf{j}}+1}$. Thus

$$||(c_{i_{k_{\mathbf{i},\mathbf{j}}+1}} - c_{j_{k_{\mathbf{i},\mathbf{j}}+1}}) + (A_{i_{k_{\mathbf{i},\mathbf{j}}+1}} c_{i_{k+2}} - A_{j_{k_{\mathbf{i},\mathbf{j}}+1}} c_{j_{k_{\mathbf{i},\mathbf{j}}+2}}) + \ldots|| \geq \alpha$$

which is the result. □

4 pIFSs as Limits of ppIFSs.

This section constitutes a proof that in the setup defined in the previous section, assuming conditions *(1)-(5)*,

$$\mathcal{H}^s(E_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p}) = \lim_{n\to\infty} \mathcal{H}^s(E^n_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p}).$$

We outline our approach. It is clear that the sets $E^n_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p}$ converge in some sense to $E_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p}$. We define a sequence of transformations $\phi^n : E^n_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p} \to E_{\{\{A^j_t\}_j,\{c^j_t\}_j\}_{t=1}^p}$ by $\phi^n(x^n_{\mathbf{i}}) = x'_{\mathbf{i}}$. We would like to say that each ϕ^n is bilipschitz with bilipschitz bounds tending to 1. The result would then follow using the standard result concerning $\mathcal{H}$ and bilipschitz functions (Proposition 2.2 in [1]). We cannot quite achieve this; it turns out our ϕ^n are only bilipschitz when considering points sufficiently close together. This is a minor inconvenience which we can work around.

Bearing in mind that we hope our ϕ^n are bilipschitz, the quantity we should consider is the following. We note that, providing $n < k_{\mathbf{i},\mathbf{j}}$ and writing $k = k_{\mathbf{i},\mathbf{j}}$ in the long formula,

$$\begin{aligned}
&\frac{||x'_{\mathbf{i}} - x'_{\mathbf{j}}||}{||x^n_{\mathbf{i}} - x^n_{\mathbf{j}}||} \\
&= \frac{||A^1_{i_1} \dots A^n_{i_n} A^{n+1}_{i_{n+1}} \dots A^k_{i_k} \left[(c^{k+1}_{i_{k+1}} - c^{k+1}_{j_{k+1}}) + (A^{k+1}_{i_{k+1}} c^{k+2}_{i_{k+2}} - A^{k+1}_{j_{k+1}} c^{k+2}_{j_{k+2}}) + \dots \right]||}{||A^1_{i_1} \dots A^n_{i_n} A_{i_{n+1}} \dots A_{i_k} \left[(c_{i_{k+1}} - c_{j_{k+1}}) + (A_{i_{k+1}} c_{i_{k+2}} - A_{j_{k+1}} c_{j_{k+2}}) + \dots \right]||} \\
&= \Big(\prod_{l=n}^{k_{\mathbf{i},\mathbf{j}}} \frac{a^l_{i_l}}{a_{i_l}} \Big) \frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}
\end{aligned}$$

where the terms in square brackets in the long formula are

$$A^{\mathbf{i},\mathbf{j}}_k = \sum_{l=k+1}^{\infty} \Big[\Big(\prod_{m=k+1}^{l-1} A^m_{i_m} \Big) c^l_{i_l} - \Big(\prod_{m=k+1}^{l-1} A^m_{j_m} \Big) c^l_{j_l} \Big]$$

$$B^{\mathbf{i},\mathbf{j}}_k = \sum_{l=k+1}^{\infty} \Big[\Big(\prod_{m=k+1}^{l-1} A_{i_m} \Big) c_{i_l} - \Big(\prod_{m=k+1}^{l-1} A_{j_m} \Big) c_{j_l} \Big]$$

(abusing notation slightly in as much as when Π has it's argument as functions $A^\star$, it signifies function composition rather than multiplication). We used *(1)* to make terms with indexes $< n$ cancel on top and bottom. Note that the formula 2 in Lemma 3.5 can now be rewritten as

$$||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}|| \geq \alpha.$$

We will show in Lemma 4.1 that we can choose δ_n such that $||x^n_{\mathbf{i}} - x^n_{\mathbf{j}}|| < \delta_n$ implies $k_{\mathbf{i},\mathbf{j}} > n$. Then we show that $\prod_{l=n}^{k_{\mathbf{i},\mathbf{j}}} \frac{a^l_{i_l}}{a_{i_l}}$ tends uniformly to 1 as $n \to \infty$ with $k_{\mathbf{i},\mathbf{j}} > n$ (uniformly over the $\mathbf{i}$'s and $\mathbf{j}$'s). Thirdly, we show we may choose K such that for all $\mathbf{i}, \mathbf{j}$ with $k_{\mathbf{i},\mathbf{j}} > K$, $\frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}$ is arbitrarily close to one. Then, given $\epsilon > 0$ we can obtain δ_n and the uniform convergence will get us the result.

Lemma 4.1. *For all n there exists δ_n such that $||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n|| < \delta_n$ implies $k_{\mathbf{i},\mathbf{j}} > n$.*

PROOF. By *(5)*, $\bigcup_{i_1,\dots,i_n} S_{i_1}^1 \dots S_{i_n}^n(F_{\{A_t\},\{c_t\}_{t=1}^p})$ is a finite disjoint union of closed sets. Hence there is some $\zeta > 0$ such that

$$\operatorname{dist}\left(S_{i_1}^1 \dots S_{i_n}^n(F_{\{A_t\},\{c_t\}_{t=1}^p}), S_{j_1}^1 \dots S_{j_n}^n(F_{\{A_t\},\{c_t\}_{t=1}^p})\right) > \zeta$$

for all $\{i_1, \dots, i_n\} \neq \{j_1, \dots, j_n\}$.

Let $\delta_n = \zeta$. Then if $||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n|| < \zeta$ we must have $\{i_1, \dots, i_n\} = \{j_1, \dots, j_n\}$, so $k_{\mathbf{i},\mathbf{j}} > n$. □

Lemma 4.2. *For all $\epsilon > 0$ there exists N such that for all $k > n > N$,*

$$1 - \epsilon < \prod_{l=n}^{k} \frac{a_{i_l}^l}{a_{i_l}} < 1 + \epsilon$$

for all $\mathbf{i}, \mathbf{j}$.

PROOF. Let us write $b_{i_l}^l = \frac{a_{i_l}^l}{a_{i_l}}$. In view of *(1)*, this allows us to rewrite *(4)* as: For all $1 \leq t \leq p$,

$$\sum_{j=1}^{\infty} |1 - b_t^j| < \infty.$$

Hence $\sum_{j=n}^{k} \max_t |1 - b_t^j| \to 0$ as $n, k \to \infty$.

$$\prod_{l=n}^{k} b_{i_l}^l = \exp \sum_{l=n}^{k} \log b_{i_l}^l \leq \exp \sum_{l=n}^{k} b_{i_l}^l - 1 \leq \exp \sum_{l=n}^{k} \max_t |1 - b_t^l|.$$

Hence there is some N_1 independent of $\mathbf{i}$ such that $k > n > N_1$ implies $\prod_{l=n}^{k} b_{i_l}^l < 1 + \epsilon$. The lower limit requires more care.

$$\begin{aligned} \prod_{l=n}^{k} b_{i_l}^l &= \exp \sum_{l=n}^{k} \log b_{i_l}^l \\ &\geq \exp \sum_{l=n}^{k} \chi_{\{b_{i_l}^l < 1\}} \log b_{i_l}^l \end{aligned}$$

where χ is the indicator function, because log is positive on $[1, \infty)$. This leaves us to deal with only $b^l_{i_l} \in [\gamma, a]$. There exists $W > 0$ such that for all $x \in [\gamma, 1]$, $\log x \geq W(x-1)$. (We cannot find W such that this holds on $[\gamma, \infty)$ as well). Thus

$$\begin{aligned}\prod_{l=n}^{k} b^l_{i_l} &\geq \exp \sum_{l=n}^{k} \chi_{\{b^l_{i_l}<1\}} W(b^l_{i_l} - 1) \\ &\geq \exp\Big(- W \sum_{l=n}^{k} \chi_{\{b^l_{i_l}<1\}} 1 - b^l_{i_l}\Big) \\ &\geq \exp\Big(- W \sum_{l=n}^{k} \chi_{\{b^l_{i_l}<1\}} \max |1 - b^l_{i_l}|\Big).\end{aligned}$$

As before, there is some N_2 independent of $\mathbf{i}$ such that $n > N_2$ implies $1 - \epsilon < \prod_{l=n}^{k} b^l_{i_l}$. Taking $N = \max\{N_1, N_2\}$ gives the result. □

This completes the first section of the proof. We show

$$\frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||} \to 1$$

in two steps, first that $||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}} - B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}|| \to 0$ and then we use that

$$\left|1 - \frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}\right| \leq \frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}} - B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}.$$

Lemma 4.3. *Let* $\Lambda^{\mathbf{i},\mathbf{j}}_k = A^{\mathbf{i},\mathbf{j}}_k - B^{\mathbf{i},\mathbf{j}}_k$. *Then for all* $\epsilon > 0$ *there exists* K*, for all* $\mathbf{i} = \{i_1, i_2, \ldots\}$ *and* $\mathbf{j} = \{j_1, j_2, \ldots\}$ *for which* $k_{\mathbf{i},\mathbf{j}} > K$*,*

$$||\Lambda^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}|| < \epsilon.$$

Proof. By *(2)* there exists φ such that for all j and t, $||c^j_t||, ||c_t|| \leq \varphi$. Let $\epsilon > 0$. By *(1)* (using geometric series), there exists q such that $\sum_{l=q}^{\infty} 4\lambda^l < \frac{\epsilon}{3\varphi}$.

Writing $k = k_{\mathbf{i},\mathbf{j}}$,

$$
\begin{aligned}
\Lambda^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}} &= \overbrace{(c^{k+1}_{i_{k+1}} - c_{i_{k+1}}) - (c^{k+1}_{j_{k+1}} - c_{j_{k+1}})}^{Y_0} \\
&+ \overbrace{(A^{k+1}_{i_{k+1}} c^{k+2}_{i_{k+2}} - A_{i_{k+1}} c_{i_{k+2}}) - (A^{k+1}_{j_{k+1}} c^{k+2}_{j_{k+2}} - A_{j_{k+1}} c_{j_{k+2}})}^{Y_1} \\
&+ \overbrace{(A^{k+1}_{i_{k+1}} A^{k+2}_{i_{k+2}} c^{k+3}_{i_{k+3}} - A_{i_{k+1}} A_{i_{k+2}} c_{i_{k+3}}) - (A^{k+1}_{j_{k_{\mathbf{i},\mathbf{j}}+1}} A^{k+1}_{j_{k+1}} c^{k+3}_{j_{k+3}} - A_{j_{k+1}} A_{j_{k+2}} c_{j_{k+3}})}^{Y_2} \\
&+ \dots
\end{aligned}
$$

Define Y_l as shown (the index l corresponds to the number of A transformations in the term Y_l). Rearrangement is justified as the series converge geometrically.

By *(1)*, $||Y_l|| \leq 4\varphi\lambda^l$ and so, $\sum_{l=q}^{\infty} ||Y_l|| < \frac{\epsilon}{3}$. By *(2)* there exists K_1 so as $\sup_{t,j>K_1} ||c^j_t - c_t|| < \frac{\epsilon}{6}$. By *(2)* and *(3)* there exists K_2 such that both

$$
\sup_{t,\, j>K_2} ||A^j_t - A_t||_\infty < \frac{\epsilon}{6(q-1)(q-2)} \quad \text{and} \quad \sup_{t,\, j>K_2} ||c^j_t - c_t|| < \frac{\epsilon}{12(q-1)}.
$$

We note, writing $k = k_{\mathbf{i},\mathbf{j}}$,

$$
\begin{aligned}
&\big|\big|(A^{k+1}_{i_{k+1}} \dots A^{k+l}_{i_{k+l}} c^{k+l+1}_{i_{k+l+1}}) - (A_{i_{k+1}} \dots A_{i_{k+l}} c_{i_{k+l+1}})\big|\big| \\
&= \big|\big|(A^{k+1}_{i_{k+1}} \dots A^{k+l}_{i_{k+l}} c^{k+l+1}_{i_{k+l+1}} - A_{i_{k+1}} A^{k+2}_{i_{k+2}} \dots A^{k+l}_{i_{k+l}} c^{k+l+1}_{i_{k+l+1}}) \\
&\quad + (A_{i_{k+1}} A^{k+2}_{i_{k+2}} A^{k+3}_{i_{k+3}} \dots A^{k+l}_{i_{k+l}} c^{k+l+1}_{i_{k+l+1}} - A_{i_{k+1}} A_{i_{k+2}} A^{k+3}_{i_{k+3}} \dots A^{k+l}_{j_{k+l}} c^{k+l+1}_{j_{k+l+1}}) + \dots \\
&\quad \dots + (A_{i_{k+1}} \dots A_{i_{k+l}} c^{k+l+1}_{i_{k+l+1}} - A_{i_{k+1}} \dots A_{i_{k+l}} c_{i_{k+l+1}})\big|\big| \\
&\leq ||A^{k+1}_{i_{k+1}} - A_{i_{k+1}}||_\infty + \dots + ||A^{k+l}_{i_{k+l}} - A_{i_{k+l}}||_\infty + ||c^{k}_{i_{k+l+1}} - c_{i_{k+l+1}}||
\end{aligned}
$$

using *(1)* to give us $||A_t x|| \leq \lambda ||x|| < ||x||$ and $||(A^j_t - A_t)(x)|| \leq ||A^j_t - A_t||_\infty$. For $k = k_{\mathbf{i},\mathbf{j}} > K_2$,

$$
\begin{aligned}
&\big|\big|(A^{k+1}_{i_{k+1}} \dots A^{k+l}_{i_{k+l}} c^{k+l+1}_{i_{k+l+1}}) - (A_{i_{k+1}} \dots A_{i_{k+l}} c_{i_{k+l+1}})\big|\big| \\
&\qquad\qquad \leq l \sup_{t,\, j>K_2} ||A^j_t - A_t||_\infty + \sup_{t,\, j>K_2} ||c^j_t - c_t||
\end{aligned}
$$

A corresponding inequality holds for terms relating to $\mathbf{j}$. Hence

$$\begin{aligned}\sum_{l=1}^{q-1}||Y_l|| &\leq \sum_{l=1}^{q-1} 2\Big(l \sup_{t,\, j>K_2}||A_t^j - A_t||_\infty + \sup_{t,\, j>K_2}||c_t^j - c_t||\Big)\\ &= 2\sup_{t,\, j>K_2}||A_t^j - A_t||_\infty\Big(\sum_{l=1}^{q-1} l\Big) + 2(q-1)\sup_{t,\, j>K_2}||c_t^j - c_t||\\ &= (q-1)(q-2)\sup_{t,\, j>K_2}||A_t^j - A_t||_\infty + 2(q-1)\sup_{t,\, j>K_2}||c_t^j - c_t||\\ &< \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.\end{aligned}$$

So for all $\mathbf{i}$, $\mathbf{j}$ with $k_{\mathbf{i},\mathbf{j}} > K = \max\{K_1, K_2\}$,

$$||\Lambda^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}|| = \Big|\Big|\sum_{l=0}^{\infty} Y_l\Big|\Big| \leq 2\sup_{t,j>K_1}||c_t^j - c_t|| + \sum_{l=1}^{q-1}||Y_l|| + \sum_{l=q}^{\infty}||Y_l|| < \epsilon,$$

as required. □

Lemma 4.4. *For all $\epsilon > 0$ there exists K, for all $\mathbf{i} = \{i_1, i_2, \ldots\}$ and $\mathbf{j} = \{j_1, j_2, \ldots\}$ for which $k_{\mathbf{i},\mathbf{j}} > K$,*

$$1 - \epsilon < \frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||} < 1 + \epsilon$$

Proof. Let $\epsilon > 0$. Note $\left|1 - \frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}\right| \leq \frac{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}} - A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||} = \frac{||\Lambda^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}$. Lemma 3.5 gives $||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}|| \geq \alpha > 0$ and by Lemma 4.3 there exists K, such for all $\mathbf{i}$ and $\mathbf{j}$ with $k_{\mathbf{i},\mathbf{j}} > K$, $||\Lambda^{\mathbf{i},\mathbf{j}}_k|| < \frac{\epsilon}{\alpha}$, which implies $\left|1 - \frac{||A^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}{||B^{\mathbf{i},\mathbf{j}}_{k_{\mathbf{i},\mathbf{j}}}||}\right| < \epsilon$. □

This completes the second part of the proof. It only remains to fit everything together.

Theorem 4.5. *For all $\epsilon > 0$ there exists N such that for all $n > N$ there is some δ_n such that for all $\mathbf{i}$ and $\mathbf{j}$ with $||x^n_{\mathbf{i}} - x^n_{\mathbf{j}}|| < \delta_n$,*

$$||x^n_{\mathbf{i}} - x^n_{\mathbf{j}}||(1-\epsilon) \leq ||\phi^n(x^n_{\mathbf{i}}) - \phi^n(x^n_{\mathbf{j}})|| \leq (1+\epsilon)||x^n_{\mathbf{i}} - x^n_{\mathbf{j}}||.$$

PROOF. Let $\epsilon > 0$. By Lemma 4.2 there exists M such that for all $k \geq n > M$, for all $\mathbf{i}$ and $\mathbf{j}$,

$$1 - \frac{\epsilon}{3} \leq \prod_{l=n}^{n} \frac{a_{i_l}^l}{a_{i_l}} \leq 1 + \frac{\epsilon}{3}.$$

By Lemma 4.4 there exists K such that for all $\mathbf{i}, \mathbf{j}$ with $k_{\mathbf{i},\mathbf{j}} > K$,

$$1 - \frac{\epsilon}{3} \leq \frac{||A_k^{\mathbf{i},\mathbf{j}}||}{||B_k^{\mathbf{i},\mathbf{j}}||} \leq 1 + \frac{\epsilon}{3}.$$

Let $N = \max\{K, M\}$ and consider $n > N$. By Lemma 4.1 there exists δ_n such that $||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n|| < \delta_n$ implies $k_{\mathbf{ij}} > n$.

If $\mathbf{i} = \mathbf{j}$ then the conclusion is trivial. If not, $x_{\mathbf{i}}^n \neq x_{\mathbf{j}}^n$ by *(5)* and

$$\frac{||x_{\mathbf{i}}' - x_{\mathbf{j}}'||}{||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n||} = \Big(\prod_{l=n}^{k_{\mathbf{i},\mathbf{j}}} \frac{a_{i_l}^l}{a_{i_l}} \Big) \frac{||A_{k_{\mathbf{i},\mathbf{j}}}^{\mathbf{i},\mathbf{j}}||}{||B_{k_{\mathbf{i},\mathbf{j}}}^{\mathbf{i},\mathbf{j}}||}.$$

Then

$$(1 - \epsilon/3)^2 \leq \frac{||\phi(x_{\mathbf{i}}^n) - \phi(x_{\mathbf{j}}^n)||}{||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n||} \leq (1 + \epsilon/3)^2,$$

and (for small ϵ),

$$(1 - \epsilon) \leq \frac{||\phi(x_{\mathbf{i}}^n) - \phi(x_{\mathbf{j}}^n)||}{||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n||} \leq (1 + \epsilon).$$

Multiplying through by $||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n||$ gives the result. □

We have now established a strong sense in which ϕ^n approximates the identity function. However, our ϕ^n are not truly bilipschitz! We only have

$$||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n||(1 - \epsilon) \leq ||\phi^n(x_{\mathbf{i}}^n) - \phi^n(x_{\mathbf{j}}^n)|| \leq (1 + \epsilon)||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n||$$

for points close together, in fact those for which $||x_{\mathbf{i}}^n - x_{\mathbf{j}}^n|| < \delta_n$. The standard result about Hausdorff measure and bilipschitz transformations (2.4(b) in [1]) does not apply. However the following is true:

Let $F \subseteq \mathcal{R}^d$ and $f : F \to f(F)$ be a bijection. Suppose there is some $\delta > 0$ and $\lambda, \lambda' > 0$ such that $||x - y|| \leq \delta$ implies $\lambda' ||x - y|| \leq ||f(x) - f(y)|| \leq \lambda ||x - y||$. Then $(\lambda')^s \mathcal{H}^s(F) \leq \mathcal{H}^s(f(F)) \leq (\lambda)^s \mathcal{H}^s(F)$.

The proof of this is essentially the same as if f were bilipschitz: Let $\{U_i\}$ be a cover of F; then $\{f(F \cap U_i)\}$ is a cover of $f(F)$. For all $0 < \sigma < \delta$, take $|U_i| < \sigma \leq \delta$ then $|f(F \cap U_i)| < \lambda\sigma$ and from the definition of Hausdorff measure, $\mathcal{H}^s(f(F)) \leq \lambda^s \mathcal{H}^s(F)$. The other inequality follows from the same argument applied to f^{-1} and $f(F)$.

It is this result which we now use.

Corollary 4.6. $\mathcal{H}^s(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}) = \lim_{n\to\infty} \left(\mathcal{H}^s(E^n_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p})\right)$.

Proof. By *(5)*, ϕ^n is a bijection. Let $\epsilon > 0$. The result above gives

$$(1-\epsilon)^s \mathcal{H}^s(E^n_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}) \leq \mathcal{H}^s(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p})$$

$$\mathcal{H}^s(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}) \leq (1+\epsilon)^s \mathcal{H}^s(E^n_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p})$$

for all $n > N$ for some N. Since $(1 \pm \epsilon)^s \to 1$ as $\epsilon \to 0$, the stated result follows. □

5 Applying 4.6.

In this section we calculate $\mathcal{H}^s(E^n_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p})$ in terms of $\mathcal{H}^s(F_{\{A_t\},\{c_t\}_{t=1}^p})$ and hence show that

$$0 < \mathcal{H}^s(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}) < \infty$$

for $s = \dim \mathcal{H}(F_{\{A_t\},\{c_t\}_{t=1}^p})$.

The following result has been lurking ever since we set up our notation.

Lemma 5.1. *For all n,*

$$E^n_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p} = \bigcup_{\substack{i_1,\ldots,i_n \\ \in\{1,\ldots p\}}} S^1_{i_1} \circ \ldots \circ S^n_{i_n}(F_{\{A_t\},\{c_t\}_{t=1}^p}).$$

This union is disjoint.

Proof. The expression follows from our definition of E^n-sets and F-sets. *(5)* gives $\mathbf{i} = \mathbf{j} \iff x^n_{\mathbf{i}} = x^n_{\mathbf{j}}$, hence the sets $x'_{i_1,\ldots,i_n} + A^1_{i_1} \ldots A^n_{i_n} F_{\{A_t\},\{c_t\}_{t=1}^p}$ must be disjoint. □

Corollary 5.2. *It holds that*

$$\mathcal{H}^s(E^n_{\{\{A^j_t\}_j,\{c^j_t\}_j\}^p_{t=1}}) = \mathcal{H}^s(F_{\{A_t\},\{c_t\}^p_{t=1}}) \sum_{\substack{i_1,\ldots,i_n\\ \in\{1,\ldots p\}}} (a^1_{i_1}\ldots a^n_{i_n})^s$$
$$= \mathcal{H}^s(F_{\{A_t\},\{c_t\}^p_{t=1}}) \prod_{j=1}^n \sum_{t=1}^p (a^j_t)^s.$$

PROOF. Note that

$$S^1_{i_1}\circ\ldots\circ S^n_{i_n}(F_{\{A_t\},\{c_t\}^p_{t=1}}) = x'_{i_1,\ldots,i_n} + A^1_{i_1}\ldots A^n_{i_n} F_{\{A_t\},\{c_t\}^p_{t=1}}.$$

Taking the measure on both sides in (3) and using disjointness we have

$$\mathcal{H}^s(E^n_{\{\{A^j_t\}_j,\{c^j_t\}_j\}^p_{t=1}}) = \sum_{\substack{i_1,\ldots,i_n\\ \in\{1,\ldots p\}}} \mathcal{H}^s\big(x'_{i_1,\ldots,i_n} + A^1_{i_1}\ldots A^n_{i_n} F_{\{A_t\},\{c_t\}^p_{t=1}}\big)$$
$$= \sum_{\substack{i_1,\ldots,i_n\\ \in\{1,\ldots p\}}} \mathcal{H}^s\big(A^1_{i_1}\ldots A^n_{i_n} F_{\{A_t\},\{c_t\}^p_{t=1}}\big)$$
$$= \sum_{\substack{i_1,\ldots,i_n\\ \in\{1,\ldots p\}}} (a^1_{i_1}\ldots a^n_{i_n})^s\, \mathcal{H}^s\big((F_{\{A_t\},\{c_t\}^p_{t=1}}\big)$$

because the A^j_t are similitudes.

The second form of the result follows from noting that

$$\Big(\sum_{\substack{i_1,\ldots,i_n\\ \in\{1,\ldots p\}}} (a^1_{i_1}\ldots a^n_{i_n})^s\Big)\Big(\sum_{t=1}^p (a^{n+1}_t)^s\Big) = \sum_{\substack{i_1,\ldots,i_{n+1}\\ \in\{1,\ldots p\}}} (a^1_{i_1}\ldots a^{n+1}_{i_{n+1}})^s.$$

□

It is now clear that the E^n-sets and the F-set have the same dimension. We show this is also common to the E-set.

Lemma 5.3. *Let* $s > 0$ *and* $0 < \eta < \lambda < 1$. *Then there exists* $\delta > 0$ *such that for some* $M \geq 0$ *and for all* $x \in [\eta, \lambda]$, $|\epsilon| < \delta$, $x^s - |\epsilon| M \leq (x+\epsilon)^s \leq x^s + |\epsilon| M$.

PROOF. This is an obvious consequence of the mean value theorem. □

Theorem 5.4. *Let* $s = \dim_{\mathcal{H}}(F_{\{A_t\},\{c_t\}_{t=1}^p})$ *be the unique positive real satisfying* $\sum_{t=1}^{p} |a_t|^s = 1$*. Then* $0 < \prod_{j=1}^{\infty} \sum_{t=1}^{p} (a_t^j)^s < \infty$*, so*

$$0 < \mathcal{H}^s(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}) < \infty$$

and $\dim_{\mathcal{H}}(E_{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}) = s$*.*

Proof. By Lemma 5.3, let $\delta > 0$ be small enough, and $M \geq 0$ be large enough, so as for all $x \in (\eta, \lambda)$ and $|\epsilon| < \delta$,

$$x^s - |\epsilon|M \leq (x+\epsilon)^s \leq x^s + |\epsilon|M.$$

By *(4)*, there exists J for all $j \geq J$, $\max_t |a_t^j - a_t| < \delta$. So for all $j \geq J$, with $\epsilon = a_t^j - a_t$,

$$(a_t)^s - |\epsilon|M \leq \big(a_t + \epsilon\big)^s \leq (a_t)^s + |\epsilon|M,$$

$$(a_t)^s - |\epsilon|M \leq (a_t^j)^s \leq (a_t)^s + |\epsilon|M.$$

Noting that $|\epsilon| \leq \max_t |a_t^j - a_t|$ and summing over t we obtain

$$\sum_{t=1}^{p} \left((a_t)^s - M\Big(\max_t |a_t^j - a_t|\Big)\right) \leq \sum_{t=1}^{p} (a_j^t)^s \leq \sum_{t=1}^{p} \left((a_t)^s + M\Big(\max_t |a_t^j - a_t|\Big)\right).$$

We shall want the left hand side of this inequality again. Let us refer to it as $(\star)$. Continuing as we were,

$$1 - \Big(\max_t |a_t^j - a_t|\Big)pM \leq \sum_{t=1}^{p} (a_j^t)^s \leq 1 + \Big(\max_t |a_t^j - a_t|\Big)pM$$

$$\Big|\sum_{t=1}^{p} (a_t^j)^s - 1\Big| \leq pM \max_t |a_t^j - a_t|.$$

This tells us that

$$\sum_{j=J}^{\infty} \Big|\sum_{t=1}^{p} (a_t^j)^s - 1\Big| \leq pM \sum_{j=J}^{\infty} \max_t |a_t^j - a_t|.$$

By *(2)*, the right hand side converges absolutely to something finite, so

$$\sum_{j=1}^{\infty} \Big|\sum_{t=1}^{p} (a_j^t)^s - 1\Big| < \infty.$$

Hence $\prod_{j=1}^{\infty}\sum_{t=1}^{p}(a_t^j)^s$ converges and is finite.

Now the lower bound. We work in similar fashion to the lower bound calculation in 4.2.

$$\begin{aligned}\prod_{j=J}^{N}\sum_{t=1}^{p}(a_t^j)^s &= \exp\sum_{j=J}^{N}\Big(\log\sum_{t=1}^{p}(a_t^j)^s\Big)\\ &\geq \exp\sum_{j=J}^{N}\Big(\chi_{\left[\sum_{t=1}^{p}(a_t^j)^s\leq 1\right]}\log\sum_{t=1}^{p}(a_t^j)^s\Big).\end{aligned}$$

Note that $\sum_{t=1}^{p}(a_t^j)^s \geq \eta^s$. If $\eta^s \geq 1$ we already have our lower bound, $e^0 = 1$. If not, there exists some large W such that $x \in [\eta^s, 1]$ implies $W(x-1) \leq \log x$. Thus

$$\begin{aligned}\prod_{j=J}^{N}\sum_{t=1}^{p}|a_t^j|^s &\geq \exp\sum_{j=J}^{N}\Big(\chi_{\left[\sum_{t=1}^{p}(a_t^j)^s\leq 1\right]}W\big(\sum_{t=1}^{p}(a_t^j)^s-1\big)\Big)\\ &= \exp\sum_{j=J}^{N}\Big(\chi_{\left[\sum_{t=1}^{p}(a_t^j)^s\leq 1\right]}W\sum_{t=1}^{p}\big((a_t^j)^s-(a_t)^s\big)\Big)\\ &\geq \exp\Big[-pWM\sum_{j=J}^{\infty}\max_t|a_t^j-a_t|\Big]\end{aligned}$$

where we have used $(\star)$ to get the last inequality. By *(4)* the right hand side of this is > 0, so $\prod_{j=1}^{\infty}\sum_{t=1}^{p}(a_t^j)^s > 0$. The result is now a consequence of Lemma 5.2 □

We are now in a position to give the result we have been aiming for.

Theorem 5.5.

$$\mathcal{H}^s\Big(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}\Big) = \mathcal{H}^s\Big(F_{\{A_t\},\{c_t\}_{t=1}^p}\Big)\times\prod_{j=1}^{\infty}\sum_{t=1}^{p}(a_t^j)^s$$

and $0 < \mathcal{H}^s\big(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}\big) < \infty$ *so as* $\dim_{\mathcal{H}}(E_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}) = s$ *where* s *is the unique solution to* $\sum_{t=1}^{p}(a_t)^s = 1$ *(i.e.* $s = \dim_{\mathcal{H}}\big(F_{\{A_t\},\{c_t\}_{t=1}^p}\big)$*).*

Proof. This now follows from Corollary 4.6, Corollary 5.2 and Theorem 5.4. □

6 Checking *(5)*.

When the pIFS is known it should not be a problem to determine if *(1)-(4)* are satisfied. As yet we have no real means of verifying *(5)*. In this section we need assume only *(1)-(3)*. Consider the following.

Let $\{\{S_t^j\}_{t=1}^p\}_j$ be a pIFS of $\{S_t\}_{t=1}^p$. We define a strong separation condition for pIFSs as *SSC(pIFS): There exists a non-empty set $\mathcal{G}$ such that:*

(i) $\mathcal{A}(\{S_t\}_{t=1}^p) \subseteq \mathcal{G}$.

(ii) There exists M such that for each $m > M$ there exists ς such that for any $i_1, \ldots, i_m, j_1, \ldots, j_m$, $\operatorname{dist}\left(S_{i_1}^1 \ldots S_{i_m}^m(\mathcal{G}), S_{j_1}^1 \ldots S_{j_m}^m(\mathcal{G})\right) > \varsigma$.

Recall that $E^m_{\{\{A_t^j\}_j,\{c_t^j\}_j\}_{t=1}^p}$ is composed of p^m scaled copies of $F_{\{A_t\},\{c_t\}_{t=1}^p}$. This condition states that each of these is contained in a scaled copy of $\mathcal{G}$, disjoint from other scaled copies of $\mathcal{G}$. If we had the SSC on the limiting IFS, $\mathbf{i} \neq \mathbf{j} \Rightarrow x_{\mathbf{i}} \neq x_{\mathbf{j}}$ and hence $\mathbf{i} \neq \mathbf{j}$ iff $x_{\mathbf{i}}^m \neq x_{\mathbf{j}}^m$ (so *(5)* would hold).

Our SSC for pIFSs deliberately avoids saying anything about separation conditions for the limiting IFS. Even if all the IFSs in our pIFS satisfy the SSC it is not implied that the limiting IFS also does.

We conclude with a simple numerical condition which combined with the SSC on the limiting IFS implies *(5)*. The case we wish to isolate is when

(i) $\mathcal{G}$ is the closed ball about 0 whose radius is

$$G = \sup\{||y|| \,:\, y \in F_{\{A_t\},\{c_t\}_{t=1}^p}\}.$$

(ii) For all $m \geq M$, $i_1, \ldots, i_m$ and $t \in \{1, \ldots, p\}$, $S_{i_1}^1 \ldots S_{i_m}^m S_t^{m+1}(\mathcal{G}) \subseteq S_{i_1}^1 \ldots S_{i_m}^m(\mathcal{G})$.

With a little extra effort it should be possible to modify the argument below to work for more complicated $\mathcal{G}$. The condition which will make (i) and (ii) occur is:

(5a) For all n, $\displaystyle \max_t \frac{||c_t^n||}{1-a_t^n} \leq G < \min_{s\neq t} \frac{||c_s^n - c_t^n||}{a_s^n + a_t^n}$.

See 6.2 for a geometrical explanation of the inequalities. We first show how to calculate G. Note that for $t = 1, \ldots, p$,

$$\sup\{||A_t x|| \,:\, ||x|| \leq 1\} \leq a_t < 1.$$

It is a standard result that (for linear transformations between Banach spaces) this implies $I - A_t$ is invertible, where I denotes the identity transformation. Hence S_t has a unique fixed point

$$y_t = (I - A)^{-1} c_t.$$

Lemma 6.1. $\mathrm{conv}(F_{\{A_t\},\{c_t\}_{t=1}^p}) = \mathrm{conv}\{y_1, \ldots, y_p\}$ *where* y_t *is the unique fixed point of* S_t. *Hence*

$$G = \max\{(I - A)^{-1}(c_t) \, : \, t = 1, \ldots, p\}.$$

PROOF. Define $Y = \mathrm{conv}\{y_1, \ldots, y_p\}$. Hence

$$Y = \{\alpha_1 y_1 + \ldots + \alpha_p y_p \, : \, \sum_{j=1}^{p} \alpha_j = 1, \, 0 \leq \alpha_j \leq 1\}.$$

In this form it is easy to prove that $\bigcup_{t=1}^{p} S_t(Y) \subseteq Y$. Let $\mathcal{T} = \bigcup_t S_t$, the application of $\{S_t\}_{t=1}^p$. By the fundamental theorem of IFSs (see [1]), $R^j = \overbrace{\mathcal{T} \circ \ldots \circ \mathcal{T}}^{j \text{ times}}(Y)$ is a decreasing sequence of sets converging to $F_{\{A_t\},\{c_t\}_{t=1}^p}$. As $y_t \in F_{\{A_t\},\{c_t\}_{t=1}^p}$, so $\mathrm{conv}\{y_1, \ldots, y_p\} \subseteq \mathrm{conv}(F_{\{A_t\},\{c_t\}_{t=1}^p})$. Hence,

$$F_{\{A_t\},\{c_t\}_{t=1}^p} \subseteq \mathrm{conv}\{y_1, \ldots, y_p\} \subseteq \mathrm{conv}(F_{\{A_t\},\{c_t\}_{t=1}^p}).$$

Taking convex hulls gives the result. □

We conclude with proof that *(5a)* functions as planned.

Proposition 6.2. *In the presence of the SSC on* $\{S_t\}_{t=1}^p$, *condition (5a) implies that (5) holds with* $\mathcal{G} = \overline{B}_G(0)$.

PROOF. We use induction on the elements in the sequence of prefractals beginning from $\{0\}$. First note the consequences of either side of the inequality in *(5a)*. From $\max_t \frac{||c_t^n||}{1 - a_t^n} \leq G$: Then for all $i_1, i_2, \ldots, i_n$ we have $\frac{||c_{i_n}^n||}{1 - a_{i_n}^n} \leq G$ which implies

$$||c_{i_n}^n|| \, (a_{i_1}^1 \ldots a_{i_{n-1}}^{n-1}) + (a_{i_1}^1 \ldots a_{i_n}^n) G \leq (a_{i_1}^1 \ldots a_{i_{n-1}}^{n-1}) G.$$

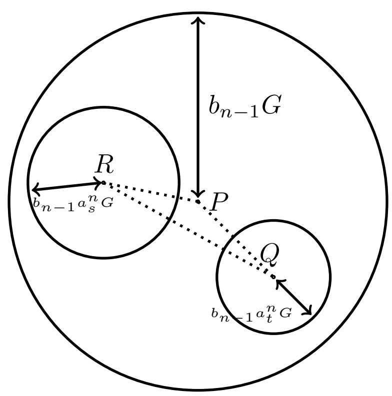
Geometrically, this means that for each $i_1, \ldots, i_{n-1}$, for each t, a closed ball of radius $(a^1_{i_1} \ldots a^{n-1}_{i_{n-1}})G$ about $c^1_{i_1} + \ldots + a^1_{i_1} \ldots a^{n-2}_{i_{n-2}} c^{n-1}_{i_{n-1}}$ contains all t closed balls of radius $(a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} a^n_t)G$ about $c^1_{i_1} + \ldots + a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} c^n_t$. We will refer to this implication as (†).

From $G < \min_{s \neq t} \dfrac{||c^n_s - c^n_t||}{a^n_s + a^n_t}$: Then for all $i_1, i_2, \ldots, 1_{n-1}$ and $s \neq t$ we have that

$$||(c^1_{i_1} + \ldots + a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} c^n_s) - (c^1_{i_1} + \ldots + a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} c^n_t)|| \\ > \big((a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} a^n_t) + (a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} a^n_s)\big)G.$$

Geometrically, this implies that for all $i_1, \ldots, i_{n-1}$ and $s \neq t$, the ball of radius $(a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} a^n_t)G$ about $c^1_{i_1} + \ldots + a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} c^n_s$ and the ball of radius $(a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} a^n_s)G$ about $c^1_{i_1} + \ldots + a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} c^n_t$ are positively separated ($\Rightarrow$ do not intersect). We refer to this implication as ($\star$).

Here is a pictorial demonstration of what (†) and ($\star$) mean.



$$P = x'_{i_1, \ldots i_{n-1}},\ Q = x'_{i_1, \ldots, i_{n-1}, t},\ R = x'_{i_1, \ldots, i_{n-1}, s},\ b_n = \prod_{k=1}^{n} a^k_{i_k}$$
$$\text{recall } x'_{i_1, \ldots, i_n} = c^1_{i_1} + \ldots + a^1_{i_1} \ldots a^{n-1}_{i_{n-1}} c^n_{i_n}$$

(†) states that the inner circles must be contained in the outer circle. ($\star$) states that the inner circles must be positively separated. It is now easy to see how the induction should work.

By Corollary 5.1,

$$E^n_{\{\{A^j_t\}_j, \{c^j_t\}_j\}^p_{t=1}} = \bigcup_{\substack{i_1, \ldots, i_n \\ \in \{1, \ldots p\}}} \Big(x'_{i_1, \ldots, i_m} + A^1_{i_1} \ldots A^n_{i_n} F_{\{a_t\}, \{c_t\}^p_{t=1}} \Big).$$

For each n, enumerate the sets

$$\left(c^1_{i_1} + A^1_{i_1} c^2_{i_2} + \ldots + A^1_{i_1} \ldots A^{n-1}_{i_{n-1}} c^n_{i_n}\right) + A^1_{i_1} \ldots a^n_{i_n} \overline{B}_G(0)$$

as $C^n_{i_1,\ldots,i_n}$. We can now restate (†) and ($\star$) as follows:

(†): For all n, for all $i_1, \ldots i_n$, for all t, $C^{n+1}_{i_1,\ldots,i_n,t} \subseteq C^n_{i_1,\ldots,i_n}$,

($\star$): For all n, $i_1, \ldots i_{n-1}$, when $s \neq t$, $C_{i_1,\ldots,i_{n-1},t} \cap C_{i_1,\ldots,i_{n-1},s} = \emptyset$.

When $n = 1$, by ($\star$) we have all the C^1_l are disjoint. Then by (†) we have that for each i_1, for each l, $C^2_{i_1,l} \subseteq C^1_{i_1}$. By ($\star$), all the $C^2_{i_1,l}$ are disjoint, which gives that for all i_1, i_2, j_1, j_2, if either $i_1 \neq j_1$ or $i_2 \neq j_2$, we have $C^2_{i_1,i_2} \cap C^2_{j_1,j_2} = \emptyset$. Continuing in this fashion (we leave the statement of a formal induction for the reader) gives that if there exists some $l < n$ such that $i_l \neq j_l$, $C^n_{i_1,\ldots i_n} \cap C^n_{j_1,\ldots,j_n} = \emptyset$.

Inside each $C^n_{i_1,\ldots,i_n}$ sits a scaled copy of $F_{\{a_t\},\{c_t\}^p_{t=1}}$. Due to the SSC any two points in this scaled copy have a positive distance between them. This proves that $\mathbf{i} = \mathbf{j} \iff x^n_{\mathbf{i}} = x^n_{\mathbf{j}}$.

To show that $\mathbf{i} = \mathbf{j} \iff x'_{\mathbf{i}} = x'_{\mathbf{j}}$, let $\mathbf{i} \neq \mathbf{j}$ and consider $E^{k_{\mathbf{i},\mathbf{j}}}_{\{\{A^j_t\}_j,\{c^j_t\}_j\}^p_{t=1}}$. From the above we know this is split into $p^{k_{\mathbf{i},\mathbf{j}}}$ positively separated (scaled and translated) copies of $\overline{B}_G(0)$. Let B_1 and B_2 be copies of $\overline{B}_G(0)$ containing $x^{k_{\mathbf{i},\mathbf{j}}}_{\mathbf{i}}$ and $x^{k_{\mathbf{i},\mathbf{j}}}_{\mathbf{j}}$ respectively. We note that $x^n_{\mathbf{i}} \to x'_{\mathbf{i}}$ as $n \to \infty$ and similarly for $\mathbf{j}$. The sequence $x^n_{\mathbf{i}}$ is contained in B_1 and similarly $x^n_{\mathbf{j}}$ in B_2 for $n > k_{\mathbf{i},\mathbf{j}}$, so $x'_{\mathbf{i}} \neq x'_{\mathbf{j}}$. The reverse implication is trivial and hence $\mathbf{i} = \mathbf{j} \iff x'_{\mathbf{i}} = x'_{\mathbf{j}}$.

Therefore, *(5)* holds. □

7 Comments.

There are a some comments I would like to make.

7.1 On Weakening Condition *(5)*.

It is my belief that the exact formula of 5.5 will fail if one were to weaken *(5)*. A natural question is to ask whether the OSC on all the IFSs $\{S^{(j)}_t\}^p_{t=1}$ would be sufficient in place of *(5)*. We discuss the OSC in the following form: *There exists a convex open set R such that $\cup_t S^{(j)}_t(R) \subseteq R$ with the union on the left being disjoint.*

Note that if the limiting IFS satisfies the SSC then so, in the tail, does the pIFS. Hence the question is really whether or not we can use the OSC on $\{S_t\}^p_{t=1}$.

The major use of the SSC in the proofs given was in lemma 4.4 as part of obtaining bilipschitz bounds on ϕ^n. This is no longer possible, at least not in same fashion if we do not have the SSC. In fact, unless our pIFS is exactly it's limiting IFS the functions ϕ^n are not bilipschitz (for *any* bilipschitz bounds). Worse still, pairs of points at which ϕ^n fails to be bilipschitz can be shown to be essentially dense in the E^n-set.

To understand what has gone wrong here we should step back from trying to use the method of section 3. Essentially the result is about a sequence of sets which converges pointwise - we want to know if this convergence preserves Hausdorff measure (and dimension).

Suppose we had a way of consistently covering our sets in the sequence (with δ-covers as $\delta \to 0$) in a more efficient manner than is possible for the limit set. Without wishing to quantify the word 'more', from the definition of $\mathcal{H}^s$ it is unlikely that measure would be preserved. Can this happen in our scenario?

Consider a three part Cantor set on $[0, 1]$, with two first level images touching and one standing free. In the second stage of its construction we observe an interval of length 3/16. If the two first level images had not touched this would not exist! Clearly we can cover this more efficiently than if there was a gap.

Consider a pIFS of this Cantor set where the contraction ratios are kept the same but the central interval $S_2^j[0, 1]$ (as imaged by the IFSs in our pIFS) is moved closer to $S_1^j[0, 1]$ as $j \to \infty$. The limiting IFS has them touching. If the result of Theorem 5.5 were to hold here, the IFS and pIFS would have attractors of equal measure.

However, as we observed, it looks like we might be able to cover the sequence of E^n-sets more efficiently than we can cover the E-set they are supposed to converge to. Results in [3] suggest (but I have not been able to obtain exact enough figures) that the covering will be significantly more effective and Theorem 5.5 will fail. The issue is that too much mass could concentrate about the joining points. If Theorem 5.5 held in cases like these a new method of proof would be needed.

7.2 A Stronger Result?

Theorem 1 in [2] states the following result: Let $0 < a_1^j + a_2^j < 1$, and let $E = \cap_j E^j$ where E^j is defined by:

- $E^0 = [0, 1]$.
- E^j is obtained from E^{j-1} be deleting all but the left most a_1^j and the right most a_2^j portion of each of it's constituent intervals.

Let $(a_1^j)^{s^j} + (a_2^j)^{s^j} = 1$. Then if the limit $s^j \to s$ exists,

$$\mathcal{H}^s(E) = \liminf_{l\to\infty} \prod_{j=1}^{l} \big((a_1^j)^s + (a_2^j)^s\big).$$

Our result would apply to this situation when $a_t^j \to a_t$ for some $a_1, a_2 > 0$ with $\sum |a_t^j - a_t| < \infty$. In this case, the unperturbed attractor has measure 1 (see [5]).

In one dimension, restricted to two part IFSs, this is a more general result than ours. Of course one might hope this more general result extended into higher dimensions as well. I do not expect this will work. The lim inf in the result relies on the natural covering (the sequence of prefractals) being as efficient a cover as can be found (in terms of $\sum_i |U_i|^s$ for small δ-covers). In one dimension considering only two part Cantor sets with the SSC something like this is always the case. In more complex scenarios the natural covering is often beaten by something more exotic.

Acknowledgment. The idea for this paper came out of the St. Andrews University Mathematics Summer School. Professor Ken Falconer supervised me and a fellow Oxford undergraduate, Bernhard Elsner, and we came up with the outlines of the method in Section 4, focused on the two part perturbed Cantor set. Whilst the generalisations and other results are entirely my own, I would not have been able to write this paper without their input. I am very grateful to Dr Ben Hambly (also of St Anne's College) for his advice on presentation, style and LaTeX. Finally, my thanks to the anonymous referee for the extent and clarity of their comments.

References

[1] Kenneth Falconer, *Fractal Geometry*, Mathematical Foundations and Applications, Second edition, John Wiley & Sons, Hoboken, NJ, 2003.

[2] Sandra Meinershagen, *The Hausdorff measure and packing measure on a perturbed Cantor set*, Real Anal. Exchange, **27** (2001-02), 177–190.

[3] Elizabeth Ayer and Robert S. Strichartz, *Exact Hausdorff measure and intervals of maximum density for Cantor sets*, Trans. Am. Math. Soc., **351** (1999), 3725–3741.

[4] In Soo Baek, *Dimensions of weakly convergent deranged Cantor sets*, Real Anal. Exchange, **23** (1997-98), 689–696.

[5] Jacques Marion, *Measures de Hausdorff d'un fractal à similitude interne*, Ann. Sc. Math. Québec, **10** (1986), 51–84.