RESEARCH

Real Analysis Exchange
Vol. 35(1), 2009/2010, pp. 43–64

Ioanna Kyrezi, Mandilara 9, 71303, Iraklio, Crete, Greece. email: ioanna.kyrezi@gmail.com

MONOTONICITY PROPERTIES OF DARBOUX SUMS

Abstract

Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Dividing the interval $[a,b]$ into subintervals of equal length, we obtain partitions of $[a,b]$ for which the upper and lower Darboux sums of f constitute two sequences, which converge to the definite integral of f in $[a,b]$ from above and below respectively. We study the monotonicity properties of these sequence and we prove that their non-monotonicity is a generic (quasi-sure) property in the space $C([a,b])$.

1 Introduction.

Let $-\infty < a < b < \infty$ and let $f : [a,b] \to \mathbb{R}$ be a bounded function. By a partition P of the interval $[a,b]$ we understand a choice of points $P = \{x_0, x_1, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b.$$

We denote by $\|P\|$ the length of P, which is defined by

$$\|P\| = \max\{x_j - x_{j-1} : 1 \leq j \leq n\}.$$

Given a partition P as above we denote by E a choice of points $E = \{\xi_1, ..., \xi_n\}$ such that

$$x_{j-1} \leq \xi_j \leq x_j, \quad 1 \leq j \leq n.$$

Mathematical Reviews subject classification: Primary: 26A42
Key words: Darboux sums
Received by the editors August 5, 2008
Communicated by: Stefan Schwabik

Given a partition P and a choice of points E as above we denote by $S(f,P,E)$ the corresponding Riemann sum

$$S(f,P,E)=\sum_{j=1}^{n} f(\xi_j)(x_j-x_{j-1}).$$

Riemann's definition of the definite integral says that f is integrable in the interval $[a,b]$, if the sums $S(f,P,E)$ converge when $\|P\|\to 0$; i.e. if there is $I\in\mathbb{R}$, such that for every $\epsilon>0$, there is $\delta>0$ such that

$$|S(f,P,E)-I|<\epsilon$$

for all choices of points E, when $\|P\|<\delta$. Then I is called definite integral of f in $[a,b]$ and it is denoted by $I=\int_a^b f(t)\mathrm{d}t$. Of course if f is continuous, then f is uniformly continuous since the interval $[a,b]$ is closed and bounded and therefore it is integrable.

However, most of the elementary books of Analysis (see, for example [9]) introduce the Riemann integral using the Darboux modification of the above definition. This is an equivalent definition, which does not use arbitrary Riemann sums but the so called upper and lower Darboux sums, for which we are interested here. Namely, if P is a partition of $[a,b]$ as above, let us set

$$m_{P,j}(f)=\inf\{f(x): x_{j-1}\le x\le x_j\},\ M_{P,j}(f)=\sup\{f(x): x_{j-1}\le x\le x_j\}.$$

Then the sums

$$\underline{S}(f,P)=\sum_{j=1}^{n} m_{P,j}(x_j-x_{j-1}),\ \overline{S}(f,P)=\sum_{j=1}^{n} M_{P,j}(x_j-x_{j-1})$$

are called lower and upper Darboux sums respectively. Note that

$$\underline{S}(f,P)\le S(f,P,E)\le \overline{S}(f,P).$$

Also, if P_1,P_2 are any two partitions, then by considering a common refinement P of P_1,P_2, for example by taking $P=P_1\cup P_2$, we can easily see that

$$\underline{S}(f,P_1)\le \underline{S}(f,P)\le \overline{S}(f,P)\le \overline{S}(f,P_2).$$

The lower integral $\underline{I}$ and the upper integral $\overline{I}$ are defined by

$$\underline{I}=\sup_P \underline{S}(f,P),\ \overline{I}=\inf_P \overline{S}(f,P).$$

We say that f is Riemann integrable if $\underline{I} = \overline{I}$ and then the number $I = \underline{I} = \overline{I}$ is defined as the definite integral of f in $[a, b]$.

Let us now consider equipartitions P_n, $n \in \mathbb{N}$ of length $\frac{b-a}{n}$; that is

$$P_n = \{x_j = a + j\frac{b-a}{n}, \quad 0 \leq j \leq n\}, \; n \in \mathbb{N}$$

and let us set

$$m_{n,j} = \inf\{f(x) : x_{j-1} \leq x \leq x_j\}, \; M_{n,j} = \sup\{f(x) : x_{j-1} \leq x \leq x_j\}$$

and

$$\underline{S}_n(f) = \underline{S}_n(f, P_n) = \frac{b-a}{n} \sum_{1 \leq j \leq n} m_{n,j},$$
$$\overline{S}_n(f) = \overline{S}_n(f, P_n) = \frac{b-a}{n} \sum_{1 \leq j \leq n} M_{n,j}.$$

Then we can show that f is Riemann integrable, if and only if the limits $\lim \underline{S}_n(f)$ and $\lim \overline{S}_n(f)$ exist and

$$\lim \underline{S}_n(f) = \lim \overline{S}_n(f) = I. \tag{1.1}$$

Thus another way to define the definite integral is by using (1.1).

Let us observe that if f is continuous, then

$$m_{n,j} = \min\{f(x) : x_{j-1} \leq x \leq x_j\}, \; M_{n,j} = \max\{f(x) : x_{j-1} \leq x \leq x_j\}.$$

Thus, if we want to talk only about continuous (or piecewise continuous) functions, then the definition (1.1) is more elementary because it defines the integral as a limit of sequences and avoids the notion of infimum and supremum. This is the reason why a lot of elementary books define the integral using (1.1). Two of the most common examples that are given, in order to illustrate this definition, are the functions $f(x) = x$ and $f(x) = x^2$ in the interval $[0, 1]$. In both of these examples the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$ are increasing and decreasing respectively (see section 6). Since we obviously have $\underline{S}_n(f) \leq \underline{S}_{2n}(f)$ and $\overline{S}_n(f) \geq \overline{S}_{2n}(f)$, the reader is left with the impression that the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$ are monotone, which is not always the case.

In this paper we shall investigate the monotone behavior of the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$. We shall show that monotone behavior is an exception and not a general phenomenon.

More precisely, let us set for simplification $[a,b] = [0,1]$ and let us denote by X the space $C([0,1])$ of continuous, real-valued functions on $[0,1]$ endowed with the norm

$$\|f\|_\infty = \sup\{|f(x)|,\ x \in [0,1]\}.$$

Note that X is complete as a metric space (actually it is a Banach space). We set

$$U = \{f \in X : \underline{S}_{n+1}(f) < \underline{S}_n(f) < \overline{S}_n(f) < \overline{S}_{n+1}(f), \\ \text{for an infinite number of } n \in \mathbb{N}\}.$$

We recall that a subset of a metric space is called G_δ if it can be written as a countable intersection of open subsets.

Theorem 1. *Let X and U be as above. Then U is a dense G_δ subset of X.*

The above result shows that generically, the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$ do not have any monotonicity properties. The following result shows that if we restrict our attention to monotone functions f, then we still have the same phenomenon.

Let us denote by $X_\uparrow$ and $X_\downarrow$ the closed subsets of X containing the increasing and decreasing functions respectively and set

$$\begin{aligned} U_\uparrow = \{f \in X_\uparrow : {}& \overline{S}_n(f) < \overline{S}_{n+1}(f) \text{ for an infinite number of } n \in \mathbb{N} \text{ and} \\ & \underline{S}_{m+1}(f) < \underline{S}_m(f) \text{ for an infinite number of } m \in \mathbb{N}\}, \\ U_\downarrow = \{f \in X_\downarrow : {}& \overline{S}_n(f) < \overline{S}_{n+1}(f) \text{ for an infinite number of } n \in \mathbb{N} \text{ and} \\ & \underline{S}_{m+1}(f) < \underline{S}_m(f) \text{ for an infinite number of } m \in \mathbb{N}\}. \end{aligned}$$

Theorem 2. *Let $X_\uparrow$, $X_\downarrow$, $U_\uparrow$ and $U_\downarrow$ be as above. Then $U_\uparrow$ and $U_\downarrow$ are G_δ-dense subsets of $X_\uparrow$ and $X_\downarrow$, respectively.*

Note that in Theorem 2, the required inequalities are valid for indices which are not necessarily the same, while in Theorem 1 these inequalities hold for the same indices. So a natural question is whether it is possible to have a similar stronger version of Theorem 2. As we will show in Proposition 1 in section 6, this is not possible.

The proofs of the above results are based on Baire's category theorem (see for example [8], Chapter 5), which we have in our disposal, since X, $X_\uparrow$ and $X_\downarrow$ are complete metric spaces. For the role of Baire's theorem and the importance of generic results in Complex, Harmonic and Functional Analysis we refer to [5] and [6]. A property which holds for a dense G_δ subset is called topologically generic or quasi-sure, because, in a complete metric space, countable intersections of open dense sets are large sets. So the above results show that the non-monotonicity of the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$ is a topologically generic or quasi-sure property.

As we mentioned earlier X is a Banach space. A subset $S \subseteq X$ is called a positive cone if $af + bg \in S$ for all $f, g \in S$ and $a, b > 0$. As we can see the sets $X_\uparrow$, $X_\downarrow$ are positive cones. If $f \in U_\uparrow$ and $a > 0$, then $af \in U_\uparrow$. It is not clear though that if $f, g \in U_\uparrow$, then $f + g \in U_\uparrow$. So the question is whether the sets $U_\uparrow$ and $U_\downarrow$ contain a positive cone which is dense in $X_\uparrow$ and $X_\downarrow$ respectively. The following result gives an answer to this question.

We set

$$\begin{aligned}
\overline{U}_\uparrow &= \left\{ f \in X_\uparrow : \ \overline{S}_n(f) < \overline{S}_{n+1}(f) \text{ for an infinite number of n} \in \mathbb{N} \right\}, \\
\underline{U}_\uparrow &= \left\{ f \in X_\uparrow : \ \underline{S}_{n+1}(f) < \underline{S}_n(f) \text{ for an infinite number of n} \in \mathbb{N} \right\}, \\
\overline{U}_\downarrow &= \left\{ f \in X_\downarrow : \ \overline{S}_n(f) < \overline{S}_{n+1}(f) \text{ for an infinite number of n} \in \mathbb{N} \right\}, \\
\underline{U}_\downarrow &= \left\{ f \in X_\downarrow : \ \underline{S}_{n+1}(f) < \underline{S}_n(f) \text{ for an infinite number of n} \in \mathbb{N} \right\}.
\end{aligned}$$

Theorem 3. *The sets $\overline{U}_\uparrow$, $\underline{U}_\uparrow$, $\overline{U}_\downarrow$, $\underline{U}_\downarrow$ contain a positive cone which is dense in their corresponding space.*

In the context of Banach spaces Bayart [1] (see also [3]) introduced another type of genericity: a property on a Banach space X is said to be algebraically generic if it holds for every non-zero vector of a dense subspace of X. So the above result shows that the non-monotonicity property of the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$ is also algebraically generic, not in a Banach space but in a complete metric space which is a cone.

2 Approximation Lemmas.

In the proof of Theorems 1 and 2 we shall need the following approximation lemmas.

Lemma 1. *Let $f : [0,1] \to \mathbb{R}$ be a continuous function and let $\epsilon > 0$. Then there is $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that for every $n \geq n_0$ there is a continuous and piecewise linear function $g : [0,1] \to \mathbb{R}$ satisfying*

$$\|f - g\|_\infty < \epsilon \tag{2.1}$$

and

$$\underline{S}_{n+1}(g) < \underline{S}_n(g) < \overline{S}_n(g) < \overline{S}_{n+1}(g). \tag{2.2}$$

PROOF. Since the interval $[0,1]$ is bounded and closed the function f is uniformly continuous and therefore there is $\delta = \delta(\epsilon) > 0$ such that

$$|x-y| < \delta(\epsilon) \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{5}.$$

Let us set

$$n_0 = n_0(\epsilon) = \left[\frac{1}{\delta(\epsilon)}\right] + 3,$$

which implies that $1/n_0 < \delta$. Let us fix an integer $n \geq n_0$ and consider the partitions

$$P_n = \{x_j = \frac{j}{n},\ 0 \leq j \leq n\},\ P_{n+1} = \{y_j = \frac{j}{n+1},\ \ 0 \leq j \leq n+1\}.$$

Note that

$$x_{j-1} < y_j < x_j,\ 1 \leq j \leq n.$$

The function $g(x)$ is constructed as follows:
First we write $n = 3q + \nu$, for some $q \in \mathbb{N}$ and $\nu \in \{0,1,2\}$ and we set

$$a_m = f(x_{3m}),\ 1 \leq m \leq q.$$

We want to have $a_0 \neq a_1$, so we set

$$a_0 = \begin{cases} f(0), & \text{if } f(0) \neq a_1 (= f(x_3)), \\ f(0) - \frac{\epsilon}{10}, & \text{if } f(0) = a_1. \end{cases}$$

Next we set

$$g(x_{3m}) = a_m,\ 0 \leq m \leq q \text{ and } g(t) = a_{3q},\ x_{3q} \leq t \leq 1.$$

In each interval $I_m = [x_{3(m-1)}, x_{3m}]$, $1 \leq m \leq q$, the function $g(t)$ is defined as follows.

We choose points $z_{3(m-1)+2}, z_{3m}$, $1 \leq m \leq q$, such that

$$\begin{aligned} x_{3(m-1)} <& y_{3(m-1)+1} < x_{3(m-1)+1} \\ <& y_{3(m-1)+2} < z_{3(m-1)+2} \\ <& x_{3(m-1)+2} < y_{3m} < z_{3m} < x_{3m}. \end{aligned}$$

These points divide the interval $I_m = [x_{3(m-1)}, x_{3m}]$ into smaller sub-intervals.

We choose numbers b_m such that

$$\max(a_{m-1}, a_m) < b_m < \max(a_{m-1}, a_m) + \frac{\epsilon}{5},\ 1 \leq m \leq q.$$

The function $g(t)$ is defined as the continuous function which takes the values

$$\begin{aligned}
&g(x_{3(m-1)}) = g(y_{3(m-1)+1}) = g(x_{3(m-1)+1}) = a_{m-1}\\
&g(y_{3(m-1)+2}) = b_m,\ g(z_{3(m-1)+2}) = a_{m-1},\ g(x_{3(m-1)+2}) = b_m\\
&g(y_{3m}) = a_m,\ g(z_{3m}) = b_m,\ g(x_{3m}) = a_m
\end{aligned}$$

and which is linear in the sub-intervals that these points divide I_m.

As we can easily see, by construction we have

$$\|f-g\|_\infty < \epsilon.$$

So it remains to prove (2.2).

Let us prove first that

$$\overline{S}_n(g) < \overline{S}_{n+1}(g). \tag{2.3}$$

To this end let us set

$$\begin{aligned}
\overline{S}_n(g)|_{I_1} &= a_0(x_1-x_0)+b_1(x_2-x_1)+b_1(x_3-x_2)\\
&= a_0(y_1-x_0)+a_0(x_1-y_1)+b_1(y_2-x_1)+b_1(x_2-y_2)\\
&\quad + b_1(y_3-x_2)+b_1(x_3-y_3),\\
\overline{S}_{n+1}(g)|_{I_1} &= a_0(y_1-x_0)+b_1(y_2-y_1)+b_1(y_3-y_2)+b_1(x_3-y_3)\\
&= a_0(y_1-x_0)+b_1(x_1-y_1)+b_1(y_2-x_1)+b_1(x_2-y_2)\\
&\quad + b_1(y_3-x_2)+b_1(x_3-y_3).
\end{aligned}$$

Then we have

$$\overline{S}_n(g)|_{I_1} - \overline{S}_{n+1}(g)|_{I_1} = (b_1-a_0)(x_1-y_1) > 0. \tag{2.4}$$

Let us also set for $2 \le m \le q$,

$$
\begin{aligned}
\overline{S}_n(g)|_{I_m} &= a_{m-1}(x_{3(m-1)+1} - x_{3(m-1)}) \\
&\quad + b_m(x_{3(m-1)+2} - x_{3(m-1)+1}) + b_m(x_{3m} - x_{3(m-1)+2}) \\
&= a_{m-1}(y_{3(m-1)+1} - x_{3(m-1)}) + a_{m-1}(x_{3(m-1)+1} - y_{3(m-1)+1}) \\
&\quad + b_m(y_{3(m-1)+2} - x_{3(m-1)+1}) + b_m(x_{3(m-1)+2} - y_{3(m-1)+2}) \\
&\quad + b_m(y_{3m} - x_{3(m-1)+2}) + b_m(x_{3m} - y_{3m}), \\
\overline{S}_{n+1}(g)|_{I_m} &= b_{m-1}(y_{3(m-1)+1} - x_{3(m-1)}) \\
&\quad + b_m(y_{3(m-1)+2} - y_{3(m-1)+1}) + b_m(y_{3m} - y_{3(m-1)+2}) \\
&\quad + b_m(x_{3m} - y_{3m}) \\
&= b_{m-1}(y_{3(m-1)+1} - x_{3(m-1)}) \\
&\quad + b_m(x_{3(m-1)+1} - y_{3(m-1)+1}) + b_m(y_{3(m-1)+2} - x_{3(m-1)+1}) \\
&\quad + b_m(x_{3(m-1)+2} - y_{3(m-1)+2}) + b_m(y_{3m} - x_{3(m-1)+2}) \\
&\quad + b_m(x_{3m} - y_{3m}).
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\overline{S}_{n+1}(g)|_{I_m} - \overline{S}_n(g)|_{I_m} &= (b_{m-1} - a_{m-1})(y_{3(m-1)+1} - x_{3(m-1)}) \\
&\quad + (b_m - a_{m-1})(x_{3(m-1)+1} - y_{3(m-1)+1}) \qquad (2.5) \\
&> 0.
\end{aligned}
$$

Now we observe that

$$
\begin{aligned}
\overline{S}_n(g) &= \sum_{m=1}^{q} \overline{S}_n(g)|_{I_m} + a_q(1 - x_{3q}), \\
\overline{S}_{n+1}(g) &= \sum_{m=1}^{q} \overline{S}_{n+1}(g)|_{I_m} + b_q(y_{3q+1} - x_{3q}) + a_q(1 - y_{3q+1}).
\end{aligned}
$$

So the inequality (2.3) follows from (2.4), (2.5).

Let us now prove the inequality

$$
\underline{S}_{n+1}(g) < \underline{S}_n(g). \qquad (2.6)
$$

Arguing in a similar way, we set

$$\begin{aligned}
\underline{S}_n(g)|_{I_1} &= a_0(x_1 - x_0) + a_0(x_2 - x_1) + a_1(x_3 - x_2) \\
&= a_0(y_1 - x_0) + a_0(x_1 - y_1) + a_0(y_2 - x_1) + a_0(x_2 - y_2) \\
&\quad + a_1(y_3 - x_2) + a_1(x_3 - y_3), \\
\underline{S}_{n+1}(g)|_{I_1} &= a_0(y_1 - x_0) + a_0(y_2 - y_1) + \min(a_0, a_1)(y_3 - y_2) + a_1(x_3 - y_3) \\
&= a_0(y_1 - x_0) + a_0(x_1 - y_1) + a_0(y_2 - x_1) \\
&\quad + \min(a_0, a_1)(x_2 - y_2) + \min(a_0, a_1)(y_3 - x_2) + a_1(x_3 - y_3).
\end{aligned}$$

Then we have

$$\begin{aligned}
\underline{S}_n(g)|_{I_1} - \underline{S}_{n+1}(g)|_{I_1} &= \big(a_0 - \min(a_0, a_1)\big)(x_2 - y_2) \\
&\quad + \big(a_1 - \min(a_0, a_1)\big)(y_3 - x_2) > 0.
\end{aligned} \tag{2.7}$$

We also set for $2 \leq m \leq q$,

$$\begin{aligned}
\underline{S}_n(g)|_{I_m} &= a_{m-1}(x_{3(m-1)+1} - x_{3(m-1)}) \\
&\quad + a_{m-1}(x_{3(m-1)+2} - x_{3(m-1)+1}) + a_m(x_{3m} - x_{3(m-1)+2}) \\
&= a_{m-1}(y_{3(m-1)+1} - x_{3(m-1)}) + a_{m-1}(x_{3(m-1)+1} - y_{3(m-1)+1}) \\
&\quad + a_{m-1}(y_{3(m-1)+2} - x_{3(m-1)+1}) \\
&\quad + a_{m-1}(x_{3(m-1)+2} - y_{3(m-1)+2}) \\
&\quad + a_m(y_{3m} - x_{3(m-1)+2}) + a_m(x_{3m} - y_{3m}), \\
\underline{S}_{n+1}(g)|_{I_m} &= a_{m-1}(y_{3(m-1)+1} - x_{3(m-1)}) + a_{m-1}(y_{3(m-1)+2} - y_{3(m-1)+1}) \\
&\quad + \min(a_{m-1}, a_m)(y_{3m} - y_{3(m-1)+2}) + a_m(x_{3m} - y_{3m}) \\
&= a_{m-1}(y_{3(m-1)+1} - x_{3(m-1)}) + a_{m-1}(x_{3(m-1)+1} - y_{3(m-1)+1}) \\
&\quad + a_{m-1}(y_{3(m-1)+2} - x_{3(m-1)+1}) \\
&\quad + \min(a_{m-1}, a_m)(x_{3(m-1)+2} - y_{3(m-1)+2}) \\
&\quad + \min(a_{m-1}, a_m)(y_{3m} - x_{3(m-1)+2}) + a_m(x_{3m} - y_{3m}).
\end{aligned}$$

Then we have

$$\begin{aligned}
&\underline{S}_n(g)|_{I_m} - \underline{S}_{n+1}(g)|_{I_m} \\
&\qquad = \big(a_{m-1} - \min(a_{m-1}, a_m)\big)(x_{3(m-1)+2} - y_{3(m-1)+2}) \\
&\qquad\quad + \big(a_m - \min(a_{m-1}, a_m)\big)(y_{3m} - x_{3(m-1)+2}) \geq 0.
\end{aligned} \tag{2.8}$$

Now we observe that

$$\underline{S}_n(g) = \sum_{m=1}^{q} \underline{S}_n(g)|_{I_m} + a_q(1 - x_{3q}),$$
$$\underline{S}_{n+1}(g) = \sum_{m=1}^{q} \underline{S}_{n+1}(g)|_{I_m} + a_q(1 - x_{3q}).$$

So the inequality (2.6) follows from (2.7) and (2.8).

□

Lemma 2. *Let $f : [0,1] \to \mathbb{R}$ be a non-constant, increasing (resp. decreasing) and continuous function and let $\epsilon > 0$. Then there is $n_0 = n_0(\epsilon) \in \mathbb{N}$ such that for every $n \geq n_0$ there is an increasing (resp. decreasing) continuous and piecewise linear function $g : [0,1] \to \mathbb{R}$ satisfying*

$$\|f - g\|_\infty < \epsilon \tag{2.9}$$

and

$$\overline{S}_n(g) < \overline{S}_{n+1}(g). \tag{2.10}$$

PROOF. As in the proof of Lemma 1, we observe that f is uniformly continuous since $[0,1]$ is a bounded and closed interval and therefore there is $\delta = \delta(\epsilon) > 0$ such that

$$|x - y| < \delta(\epsilon) \Longrightarrow |f(x) - f(y)| < \frac{\epsilon}{5}.$$

Let us set

$$n_0 = n_0(\epsilon) = \left[\frac{1}{\delta(\epsilon)}\right] + 3,$$

and let us fix an integer $n \geq n_0$. Also since f is not constant, by taking a larger value for n_0 if necessary, we can assume that

$$f(\frac{1}{n}) \neq f(1),\ f(0) \neq f(\frac{n-1}{n}). \tag{2.11}$$

We consider the partitions

$$P_n = \{x_j = \frac{j}{n},\ 0 \leq j \leq n\},\ P_{n+1} = \{y_j = \frac{j}{n+1},\ 0 \leq j \leq n+1\}.$$

Note that

$$x_{j-1} < y_j < x_j,\ 1 \leq j \leq n.$$

We set

$$a_j = f(x_j),\ 0 \le j \le n.$$

The function $g(x)$ is constructed as follows:
CASE 1: f is increasing.
Then function $g(x)$ is defined as the continuous function which takes the values

$$g(y_j) = g(x_j) = a_j,\ j = 0, 1, ..., n$$

and which is linear in the intervals $[x_{j-1}, y_j]$ and $[y_j, x_j]$, $j = 1, ..., n$.

As we can easily see, by construction g is increasing and satisfies

$$\|f - g\|_\infty < \epsilon.$$

To prove (2.10), we observe that

$$\begin{aligned}
\overline{S}_n(g) &= \sum_{j=1}^{n} g(x_j)(x_j - x_{j-1}) = \frac{1}{n}\big(g(x_1) + ... + g(x_n)\big) = \frac{1}{n}(a_1 + ... + a_n),\\
\overline{S}_{n+1}(g) &= \sum_{j=1}^{n+1} g(y_j)(y_j - y_{j-1})\\
&= \frac{1}{n+1}\big(g(y_1) + ... + g(y_n) + g(y_{n+1})\big) = \frac{1}{n+1}(a_1 + ... + a_n + a_n).
\end{aligned}$$

Hence, by (2.11)

$$\begin{aligned}
\overline{S}_{n+1}(g) - \overline{S}_n(g) &= \frac{1}{n+1}(a_1 + ... + a_n + a_n) - \frac{1}{n}(a_1 + ... + a_n)\\
&= \frac{1}{n+1}a_n + (\frac{1}{n+1} - \frac{1}{n})(a_1 + ... + a_n)\\
&= \frac{1}{n+1}a_n - \frac{1}{n(n+1)}(a_1 + ... + a_n)\\
&> \frac{1}{n+1}a_n - \frac{1}{n(n+1)}na_n = 0,
\end{aligned}$$

which proves (2.10).
CASE 2: f is decreasing.
In this case, the function $g(t)$ is defined as the continuous function which takes the values

$$g(y_j) = g(x_{j-1}) = a_{j-1},\ j = 1, 1, ..., n+1$$

and which is linear in the intervals $[x_{j-1}, y_j]$ and $[y_j, x_j]$, $j = 1, ..., n$.

By construction g is decreasing and satisfies

$$\|f-g\|_\infty < \epsilon.$$

Arguing in the same way as in the previous case we have

$$\begin{aligned}
\underline{S}_n(g) &= \sum_{j=1}^{n} g(x_{j-1})(x_j - x_{j-1}) \\
&= \frac{1}{n}\big(g(x_0) + ... + g(x_{n-1})\big) = \frac{1}{n}(a_0 + ... + a_{n-1}), \\
\underline{S}_{n+1}(g) &= \sum_{j=1}^{n+1} g(y_j)(y_j - y_{j-1}) \\
&= \frac{1}{n+1}\big(g(y_0) + g(y_1) + ... + g(y_n))\big) = \frac{1}{n+1}(a_0 + a_0 + ... + a_{n-1})
\end{aligned}$$

and hence, by (2.11)

$$\begin{aligned}
\underline{S}_{n+1}(g) - \underline{S}_n(g) &= \frac{1}{n+1}(a_0 + a_0 + ... + a_{n-1}) - \frac{1}{n}(a_0 + ... + a_{n-1}) \\
&= \frac{1}{n+1}a_0 + (\frac{1}{n+1} - \frac{1}{n})(a_0 + ... + a_{n-1}) \\
&= \frac{1}{n+1}a_0 - \frac{1}{n(n+1)}(a_0 + ... + a_{n-1}) \\
&> \frac{1}{n+1}a_0 - \frac{1}{n(n+1)}na_0 = 0,
\end{aligned}$$

which proves (2.10). □

3 Proof of Theorem 1.

As mentioned in the introduction the proof of Theorem 1 is based on Baire's category theorem, and it is inspired from [7].

Let

$$V_n = \left\{g \in X : \ \underline{S}_{n+1}(g) < \underline{S}_n(g) < \overline{S}_n(g) < \overline{S}_{n+1}(g)\right\}, \ n \in \mathbb{N}$$

and

$$U_m = \bigcup_{n=m}^{+\infty} V_n, \ m \in \mathbb{N}.$$

Then the set U can be written as

$$U = \bigcap_{m \in \mathbb{N}} U_m.$$

The sets V_n, $n \in \mathbb{N}$ (hence the sets U_m, $m \in \mathbb{N}$) are open subsets of X. So U is G_δ since it is a countable intersection of open sets.

We shall prove that the sets U_m, $m \in \mathbb{N}$ are dense in X. Then, Baire's category theorem gives that U is dense in X because it is a countable intersection of open dense sets and X is complete.

Let $f \in X$ and let $\epsilon > 0$. Then, by Lemma 1 there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there is $g \in V_n$ satisfying $\|f - g\|_\infty < \epsilon$. This shows that the sets U_m, $m \in \mathbb{N}$ are dense and the theorem follows.

4 Proof of Theorem 2.

The proof of Theorem 2 is also based on Baire's category theorem. The sets $X_\uparrow$ and $X_\downarrow$ are closed subsets of X and so they are themselves complete metric spaces. We set

$$\begin{aligned}
\overline{U}_\uparrow =& \{f \in X_\uparrow : \ \overline{S}_n(f) < \overline{S}_{n+1}(f) \text{ for an infinite number of n} \in \mathbb{N}\},\\
\underline{U}_\uparrow =& \{f \in X_\uparrow : \ \underline{S}_{m+1}(f) < \underline{S}_m(f) \text{ for an infinite number of n} \in \mathbb{N}\},\\
\overline{U}_\downarrow =& \{f \in X_\downarrow : \ \overline{S}_n(f) < \overline{S}_{n+1}(f) \text{ for an infinite number of n} \in \mathbb{N}\},\\
\underline{U}_\downarrow =& \{f \in X_\downarrow : \ \underline{S}_{m+1}(f) < \underline{S}_m(f) \text{ for an infinite number of n} \in \mathbb{N}\}.
\end{aligned}$$

Then

$$U_\uparrow = \overline{U}_\uparrow \bigcap \underline{U}_\uparrow \ , \ U_\downarrow = \overline{U}_\downarrow \bigcap \underline{U}_\downarrow.$$

Since the intersection of two G_δ-dense subsets of a complete metric space is also a G_δ-dense subset, it is enough to prove that the sets $\overline{U}_\uparrow$, $\overline{U}_\downarrow$, $\underline{U}_\uparrow$, $\underline{U}_\downarrow$ are G_δ-dense. We shall give the proof only for $\overline{U}_\uparrow$. The proof for $\overline{U}_\downarrow$, $\underline{U}_\uparrow$, $\underline{U}_\downarrow$ is similar.

We set

$$V_n = \left\{g \in X_\uparrow : \ \overline{S}_n(g) < \overline{S}_{n+1}(g)\right\}, \ n \in \mathbb{N}, \ U_m = \bigcup_{n=m}^{+\infty} V_n, \ m \in \mathbb{N}.$$

Then

$$\overline{U}_\uparrow = \bigcap_{m \in \mathbb{N}} U_m.$$

Arguing as in the proof of Theorem 1, we observe that the sets V_n, $n \in \mathbb{N}$, hence the sets U_m, $m \in \mathbb{N}$, are open and therefore $\overline{U}_\uparrow$ is G_δ. In order to prove that $\overline{U}_\uparrow$ is dense, by Baire's category theorem, it is enough to prove that the sets U_m, $m \in \mathbb{N}$ are dense in $X_\uparrow$. This follows from Lemma 2. Indeed, let $f \in X_\uparrow$ and let $\epsilon > 0$. Then, by Lemma 2 there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there is $g \in V_n$ satisfying $\|f-g\|_\infty < \epsilon$, which shows that the sets U_m, $m \in \mathbb{N}$ are dense.

5 Proof of Theorem 3.

We shall give the proof only for $\overline{U}_\uparrow$. The proof for $\overline{U}_\downarrow$, $\underline{U}_\uparrow$, $\underline{U}_\downarrow$ is similar. Our method is based on [2].

Given an infinite subset M of $\mathbb{N}$, we set

$$\overline{U}_\uparrow^M = \left\{ f \in X_\uparrow : \ \overline{S}_n(f) < \overline{S}_{n+1}(f) \text{ for an infinite number of n} \in M \right\}.$$

We shall need the following lemmas.

Lemma 3. *$\overline{U}_\uparrow^M$ is a G_δ-dense subset of $X_\uparrow$.*

PROOF. Let

$$V_n = \left\{ g \in X_\uparrow : \overline{S}_n(f) < \overline{S}_{n+1}(f) \right\}, \ n \in \mathbb{N}, \ U_m = \bigcup_{n \geq m, \ n \in M} V_n, \ m \in \mathbb{N}.$$

Then

$$\overline{U}_\uparrow^M = \bigcap_{m \in \mathbb{N}} U_m.$$

The sets V_n, $n \in \mathbb{N}$ and U_m, $m \in \mathbb{N}$ are open, Also, if $f \in X_\uparrow$ and $\epsilon > 0$, then by Lemma 2, there is $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $n \in M$, there is $g \in V_n$ satisfying $\|f-g\|_\infty < \epsilon$. This shows that the sets U_m, $m \in \mathbb{N}$ are dense. So, by Baire's category theorem, $\overline{U}_\uparrow^M$ is G_δ-dense, since it can be written as a countable intersection of dense open subsets of $X_\uparrow$. □

Lemma 4. *There is a sequence g_k, $k \in \mathbb{N}$ of elements of $U_\uparrow$ and a sequence M_k, $k \in \mathbb{N}$ of subsets of $\mathbb{N}$ such that*

1. *the set $\{g_k, \ k \in \mathbb{N}\}$ is dense in $X_\uparrow$,*
2. *the sets M_k, $k \in \mathbb{N}$ have an infinite number of elements and satisfy*

$$M_k \supseteq M_{k+1}, \ k \in \mathbb{N},$$

3. we have

$$\overline{S}_n(g_k) < \overline{S}_{n+1}(g_k),\ n \in M_k,\ k \in \mathbb{N}.$$

PROOF. Since $X_\uparrow$ is a separable space we can consider a countable dense subset $\{\phi_k,\ k \in \mathbb{N}\}$ of $X_\uparrow$. The sequences g_k, $k \in \mathbb{N}$ and M_k, $k \in \mathbb{N}$ can be chosen inductively in the following way.

As g_1 we take any element of $\overline{U}_\uparrow$ satisfying

$$\|\phi_1 - g_1\|_\infty < 1$$

and we set

$$M_1 = \{n \in \mathbb{N} :\ \overline{S}_n(g_1) < \overline{S}_{n+1}(g_1)\}.$$

Such an element g_1 exists, because by Theorem 2, $\overline{U}_\uparrow$ is dense in $X_\uparrow$. Also by the way $\overline{U}_\uparrow$ was defined, the set M_1 has an infinite number of elements.

Assume now that $g_k \in \overline{U}_\uparrow$ and M_k have been chosen. Then, as g_{k+1} we take any element of $\overline{U}_\uparrow^{M_k}$ satisfying

$$\|\phi_{k+1} - g_{k+1}\|_\infty < \frac{1}{k+1}$$

and we set

$$M_{k+1} = \{n \in M_k :\ \overline{S}_n(g_{k+1}) < \overline{S}_{n+1}(g_{k+1})\}.$$

By Lemma 3 such an element g_{k+1} exists and the set M_{k+1} has an infinite number of elements.

Clearly the sequences g_k, $k \in \mathbb{N}$ and M_k, $k \in \mathbb{N}$ chosen in this way satisfy the requirements of the lemma.

□

Now we come to the proof of Theorem 3. First we recall that the positive cone generated by a set $Y \subseteq X$ is defined as the smallest positive cone that contains Y and that it is equal to the set

$$\{a_1 f_1 + a_2 f_2 + ... + a_m f_m :\ a_1, a_2, ..., a_m > 0,\ f_1, f_2, ..., f_m \in Y,\ m \in \mathbb{N}\}.$$

Let the sequences g_k, $k \in \mathbb{N}$ and M_k, $k \in \mathbb{N}$ be as in Lemma 4. Then the cone C generated the set $\{g_k,\ k \in \mathbb{N}\}$ is dense in $X_\uparrow$ and $C \subseteq \overline{U}_\uparrow$.

Indeed, the fact that C is dense in $X_\uparrow$ follows from the fact that $\{g_k,\ k \in \mathbb{N}\}$ is dense in $X_\uparrow$.

Let now $w \in C$. Then, w can be written as

$$w = a_{k_1} g_{k_1} + a_{k_2} g_{k_2} + ... + a_{k_m} g_{k_m},$$

with $k_1 < k_2 < ... < k_m$, $a_{k_1}, a_{k_2}, ..., a_{k_m} > 0$ and $m \in \mathbb{N}$.

We want to prove that $w \in \overline{U}_\uparrow$. To this end let us observe that

$$\overline{S_n}(af + bg) = a\overline{S_n}(f) + b\overline{S_n}(g),\ f, g \in X_\uparrow,\ a, b > 0,\ n \in \mathbb{N}.$$

Since

$$M_{k_1} \supseteq M_{k_2} \supseteq ... \supseteq M_{k_m}$$

we have that for all $n \in M_{k_m}$

$$\begin{aligned}\overline{S_n}(w) &= a_{k_1}\overline{S}_n(g_{k_1}) + a_{k_2}\overline{S}_n(g_{k_2}) + ... + a_{k_m}\overline{S}_n(g_{k_m})\\ &< a_{k_1}\overline{S}_{n+1}(g_{k_1}) + a_{k_2}\overline{S}_{n+1}(g_{k_2}) + ... + a_{k_m}\overline{S}_{n+1}(g_{k_m}) = \overline{S}_{n+1}(w).\end{aligned}$$

It follows that $w \in \overline{U}_\uparrow$ and this completes the proof of Theorem 3.

6 Final Comments and Examples.

As we mentioned in the introduction, it is not possible to have a stronger version of Theorem 2 in which the required inequalities are valid for the same indices. This follows from the following proposition:

Proposition 1. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous, monotone function and let us assume that for some $n \in \mathbb{N}$ we have $\overline{S}_n(f) < \overline{S}_{n+1}(f)$. Then $\underline{S}_n(f) < \underline{S}_{n+1}(f)$.*

Proof. We shall assume that f is increasing. The case when f is decreasing can be treated either by arguing in the same way, or by considering the function $-f$.

Since f is increasing we have

$$\begin{aligned}\overline{S}_n(f) =& \frac{1}{n}\big(f(\frac{1}{n}) + ... + f(\frac{n-1}{n}) + f(1)\big),\\ \underline{S}_n(f) =& \frac{1}{n}\big(f(0) + f(\frac{1}{n}) + ... + f(\frac{n-1}{n})\big).\end{aligned}$$

The assumption that $\overline{S}_n(f) < \overline{S}_{n+1}(f)$ implies

$$\frac{1}{n}\big(f(\frac{1}{n}) + ... + f(\frac{n-1}{n}) + f(1)\big) < \frac{1}{n+1}\big(f(\frac{1}{n+1}) + ... + f(\frac{n}{n+1}) + f(1)\big),$$

which in turn implies that

$$f(1) < n\big(f(\frac{1}{n+1}) + ... + f(\frac{n}{n+1})\big) - (n+1)\big(f(\frac{1}{n}) + ... + f(\frac{n-1}{n})\big). \quad (6.1)$$

If we had $\underline{S}_{n+1}(f) \leq \underline{S}_n(f)$, then we would have

$$\frac{1}{n+1}\big(f(0)+f(\frac{1}{n+1})+...+f(\frac{n}{n+1})\big) \leq \frac{1}{n}\big(f(0)+f(\frac{1}{n})+...+f(\frac{n-1}{n})\big),$$

which implies that

$$n\big(f(\frac{1}{n+1})+...+f(\frac{n}{n+1})\big)-(n+1)\big(f(\frac{1}{n})+...+f(\frac{n-1}{n})\big) \leq f(0). \quad (6.2)$$

Combining (6.1) and (6.2) we obtain that $f(1) < f(0)$, which is a contradiction as f is increasing.

□

As we mentioned in the introduction two of the most common examples that are given, in order to illustrate the definition of the definite integral, are the functions $f(x) = x$ and $g(x) = x^2$ defined in the interval $[0,1]$. Both of these functions are increasing in $[0,1]$.

Concerning the function $f(x) = x$ we have

$$\begin{aligned}\overline{S}_n(f) =& \frac{1}{n}\big(f(\frac{1}{n})+...+f(\frac{n-1}{n})+f(1)\big)\\ =& \frac{1}{n^2}(1+2+...+n) = \frac{n(n+1)}{2n^2} = \frac{1}{2}+\frac{1}{2n},\end{aligned}$$

$$\begin{aligned}\underline{S}_n(f) =& \frac{1}{n}\big(f(0)+f(\frac{1}{n})+...+f(\frac{n-1}{n})\big)\\ =& \frac{1}{n^2}(1+2+...+(n-1)) = \frac{(n-1)n}{2n^2} = \frac{1}{2}-\frac{1}{2n}.\end{aligned}$$

So the sequence $\overline{S}_n(f)$ is decreasing and the sequence $\underline{S}_n(f)$ is increasing.

Concerning the function $f(x) = x^2$ we have

$$\begin{aligned}\overline{S}_n(g) =& \frac{1}{n}\big(g(\frac{1}{n})+...+g(\frac{n-1}{n})+f(1)\big)\\ =& \frac{1}{n^3}(1+2^2+...+n^2) = \frac{n(n+1)(2n+1)}{6n^3} = \frac{1}{6}(1+\frac{1}{n})(2+\frac{1}{n}),\end{aligned}$$

$$\begin{aligned}\underline{S}_n(g) =& \frac{1}{n}\big(g(0)+g(\frac{1}{n})+...+g(\frac{n-1}{n})\big)\\ =& \frac{1}{n^2}(1+2^2+...+(n-1)^2) = \frac{(n-1)n(2n-1)}{6n^3} = \frac{1}{6}(1-\frac{1}{n})(2-\frac{1}{n}).\end{aligned}$$

So the sequence $\overline{S}_n(g)$ is decreasing and the sequence $\underline{S}_n(g)$ is increasing.

A natural question is whether there is a class of functions for which the sequences $\overline{S}_n(f)$ and $\underline{S}_n(f)$ are monotone. The following proposition gives an answer to a question posed to us by D. Betsakos and shows that such a class is the convex monotone functions.

Proposition 2. *Let $f : [0, 1] \to \mathbb{R}$ be a continuous monotone function and let us assume that it is either convex or concave. Then the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$ are monotone.*

PROOF. We shall assume that f is convex and increasing. The case when f is convex and decreasing can be treated either by arguing in the similar way and the cases when f is concave and increasing or decreasing can be treated either by arguing in the similar way, or by considering the function $-f$.

We set

$$x_k = \frac{k}{n},\ y_k = \frac{k}{n+1},\ 0 \le k \le n+1$$

Since f is increasing we have

$$\begin{aligned}\underline{S}_n(f) =& \frac{1}{n}\big(f(x_0) + f(x_1) + ... + f(x_{n-1})\big),\\ \underline{S}_{n+1}(f) =& \frac{1}{n+1}\big(f(y_0) + f(y_1) + ... + f(y_n)\big).\end{aligned}$$

We observe that

$$x_k = \frac{n-k}{n} y_k + \frac{k}{n} y_{k+1},\ 0 \le k \le n.$$

Since f is convex, this implies that

$$f(x_k) \le \frac{n-k}{n} f(y_k) + \frac{k}{n} f(y_{k+1}),\ 1 \le k \le n.$$

It follows that

$$\begin{aligned}\underline{S}_n(f) =&\frac{1}{n}\big(f(x_0)+f(x_1)+f(x_2)+...+f(x_{n-1})\big)\\ \leq&\frac{1}{n}\Big(f(y_0)+\frac{n-1}{n}f(y_1)+\frac{1}{n}f(y_2)+\frac{n-2}{n}f(y_2)+\frac{2}{n}f(y_3)\\ &\quad+...+\frac{n-(n-1)}{n}f(y_{n-1})+\frac{n-1}{n}f(y_n)\Big)\\ =&\frac{1}{n}\Big(f(y_0)+\frac{n-1}{n}f(y_1)+\frac{n-1}{n}f(y_2)+...+\frac{n-1}{n}f(y_n)\Big)\\ =&\frac{1}{n}\Big(\frac{1}{n}f(y_0)+\frac{n-1}{n}\big(f(y_0)+f(y_1)+...+f(y_n)\big)\Big)\\ =&\frac{1}{n^2}f(y_0)+\frac{n^2-1}{n^2}\frac{1}{n+1}\big(f(y_0)+f(y_1)+...+f(y_n)\big)\\ =&\frac{1}{n^2}f(y_0)+\frac{n^2-1}{n^2}\underline{S}_{n+1}(f).\end{aligned}$$

Since f is increasing we have

$$f(y_0)\leq\frac{1}{n+1}\big(f(y_0)+f(y_1)+...+f(y_n)\big)=\underline{S}_{n+1}(f).$$

Therefore

$$\underline{S}_n(f)\leq\frac{1}{n^2}S_{n+1}(f)+\frac{n^2-1}{n^2}S_{n+1}(f)=\underline{S}_{n+1}(f).$$

Let us now prove that $\overline{S}_n(f)\geq\overline{S}_{n+1}(f)$. We observe again that since f is increasing we have

$$\begin{aligned}\overline{S}_n(f)=&\frac{1}{n}\big(f(x_1)+f(x_1)+...+f(x_n)\big),\\ \overline{S}_{n+1}(f)=&\frac{1}{n+1}\big(f(y_1)+f(y_2)+...+f(y_{n+1})\big).\end{aligned}$$

We have

$$y_k=\frac{k}{n+1}x_{k-1}+\frac{n+1-k}{n+1}x_k,\ 1\leq k\leq n.$$

Since f is convex, this implies that

$$f(y_k)\leq\frac{k}{n+1}f(x_{k-1})+\frac{n+1-k}{n+1}f(x_k),\ 1\leq k\leq n.$$

It follows that

$$\begin{aligned}\overline{S}_{n+1}(f) =& \frac{1}{n+1}\big(f(y_1)+f(y_2)+f(y_3)+\ldots+f(y_n)+f(y_{n+1})\big)\\ \leq& \frac{1}{n+1}\big(\frac{1}{n+1}f(x_0)+\frac{n}{n+1}f(x_1)+\frac{2}{n+1}f(x_1)\\ &+\frac{n-1}{n+1}f(x_2)+\frac{3}{n+1}f(x_2)+\frac{n-2}{n+1}f(x_3)\\ &+\frac{4}{n+1}f(x_3)+\frac{n-3}{n+1}f(x_4)\\ &+\ldots+\frac{n}{n+1}f(x_{n-1})+\frac{1}{n+1}f(x_n)+f(x_n)\big)\\ =& \frac{1}{n+1}\Big(\frac{1}{n+1}f(x_0)+\frac{n+2}{n+1}\big(f(x_1)+f(x_2)+\ldots+f(x_n)\big)\Big)\\ =& \frac{1}{n+1}\big(\frac{1}{n+1}f(x_0)+\frac{(n+2)n}{n+1}\overline{S}_n(f)\big).\end{aligned}$$

Since f is increasing we have

$$f(x_0) \leq \frac{1}{n}\big(f(x_1)+f(x_2)+\ldots+f(x_n)\big) = \overline{S}_n(f).$$

Therefore

$$\begin{aligned}\overline{S}_{n+1}(f) \leq& \frac{1}{n+1}\big(\frac{1}{n+1}\overline{S}_n(f)+\frac{(n+2)n}{n+1}\overline{S}_n(f)\big)\\ =& \frac{1}{n+1}\frac{(n+1)^2}{n+1}\overline{S}_n(f) = \overline{S}_n(f).\end{aligned}$$

□

We point out that although the non-monotonicity of the sequences $\underline{S}_n(f)$ and $\overline{S}_n(f)$ is a generic property, it is difficult to give an explicit example of a function f satisfying Theorems 1 and 2. In the case of increasing functions, we can define such an f by a limit procedure, but still we cannot give an explicit formula.

For the reader's convenience, we give below two examples. In the first example we have $\underline{S}_4(f) < \underline{S}_3(f) < \overline{S}_3(f) < \overline{S}_4(f)$ and in the second we have $\underline{S}_{n+1}(f) < \underline{S}_n(f) < \overline{S}_n(f) < \overline{S}_{n+1}(f)$ for a given $n \in \mathbb{N}$.

The first example is the function

$$f(x) = \begin{cases} 32x^3, & \text{if } \ 0 \leq x \leq \frac{1}{2}, \\ 32(1-x)^3, & \text{if } \ \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then, we have

$$\underline{S}_2(f) = 0,\ \underline{S}_3(f) = \frac{32}{81},\ \overline{S}_3(f) = \frac{172}{81},\ \overline{S}_2(f) = 4$$

while

$$\underline{S}_4(f) = \frac{1}{4},\ \underline{S}_3(f) = \frac{32}{81},\ \overline{S}_3(f) = \frac{172}{81},\ \overline{S}_4(f) = \frac{9}{4}.$$

The second example is the following. Let $n \geq 3$ be a given integer and let $z_2 \in (\frac{2}{n+1}, \frac{2}{n})$. We consider the continuous function f which takes the values

$$f(0) = f(\frac{1}{n}) = f(z_2) = 1, \quad f(\frac{2}{n+1}) = f(\frac{2}{n}) = f(1) = 2$$

and it is linear in the intervals $[0, \frac{1}{n}]$, $[\frac{1}{n}, \frac{2}{n+1}]$, $[\frac{2}{n+1}, z_2]$, $[z_2, \frac{2}{n}]$, $[\frac{2}{n}, 1]$. Then

$$\underline{S}_{n+1}(f) = 2 - \frac{3}{n+1},\ \underline{S}_n(f) = 2 - \frac{2}{n},\ \overline{S}_n(f) = 2 - \frac{1}{n},\ \overline{S}_{n+1}(f) = 2 - \frac{1}{n+1}$$

and since $n \geq 3$, we have

$$\underline{S}_{n+1}(f) < \underline{S}_n(f) < \overline{S}_n(f) < \overline{S}_{n+1}(f).$$

Acknowledgement. I would like to thank Vasilis Nestoridis for suggesting this problem and for his help and encouragement. I would also like to thank Manolis Katsoprinakis for several helpful discussions.

References

[1] F. Bayart, *Topological and algebraic genericity of divergence and universibility*, Studia Math., **167** (2005), 161–181.

[2] F. Bayart, K.-G. Grosse-Erdmann, V. Nestoridis, and C. Papadimitropoulos, *Abstract theory of universal series and applications*, Proc. London Math. Soc., To appear.

[3] L. Bernal-Gonzalez, *Densely hereditarily hypercyclic sequences and large hypercyclic manifolds*, Proc. Amer. Math. Soc., **127** (1999), 3279–3285.

[4] G. Costakis, V. Vlachou, *Identical approximative sequence for various notions of universality*, J. Approx. Theory, **132** (2005), 15–24.

[5] K.-G. Grosse-Erdmann, *Universal families and hypercyclic operators*, Bull. Amer. Math. Soc., **36** (1999), 345–381.

[6] J.P. Kahane, *Baire's Category Theorem and trigonometric series*, J. d'Analyse Math., **80** (2000), 143–182.

[7] V. Nestoridis, *Universal Taylor series*, Ann. Inst. Fourier, **46** (1996), 1293–1306.

[8] W. Rudin, *Real and Complex Analysis*, Second edition, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., New York-Düsseldorf-Johannesburg, 1974.

[9] M. Spivak, *Calculus*, Second edition, Publish or Perish, Berkeley, California, 1980.