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UPPER POROUS SETS WHICH ARE NOT σ-LOWER POROUS

Abstract

Let X be a nonempty, topologically complete metric space with no isolated points. We show that there exists a closed upper porous set (in a strong sense) $F \subset X$ which is not σ-lower porous (in a weak sense). More precisely, we show that there exists a closed (g_1)-shell porous set $F \subset X$ which is not σ-(g_2)-lower porous, where g_1 and g_2 are arbitrary admissible functions.

1 Introduction.

The notions of porosity and σ-porosity have been studied in many papers from different points of view. We refer the reader to [5] and [4] for motivations and applications of these notions.

In this paper, our aim is to strengthen the following well-known result:

Proposition 1.1. [4, Proposition 2.7] Let (X, ϱ) be a nonempty, topologically complete metric space with no isolated points. There exists a closed nowhere dense set $F \subset X$ which is not σ-lower porous.

As it was already mentioned in [4, Remark 2.8], it is not difficult to modify the construction used in the proof of this proposition to obtain a closed set $F \subset X$ which is upper porous and is not σ-lower porous.

Using a similar method as in the proof of Proposition 1.1 we even show a stronger result. Namely we prove that there exists a closed (g_1)-shell porous set $F \subset X$ which is not σ-(g_2)-lower porous, where g_1 and g_2 are arbitrary admissible functions.

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2 Basic Definitions and Main Result.

In this section, we present some basic definitions and formulate our main result.

In the following we suppose that (X, ϱ) is a fixed nonempty metric space. The open ball with center $x \in X$ and radius $r > 0$ will be denoted by $B(x, r)$. Further, we put $B(x, 0) := \emptyset$.

By an *admissible shell* we mean any set of the form $S(x, y, s) := \{z \in X : \varrho(x, y) - s < \varrho(x, z) < \varrho(x, y) + s\}$ where $x, y \in X$, $x \neq y$ and $0 < s < \varrho(x, y)$. Further, we put $S(x, y, 0) := \emptyset$.

We shall denote by $\mathcal{G}$ the system of all increasing continuous functions g on $[0, \infty)$ such that $g(0) = 0$ and $g(x) > x$ for $0 < x < \delta$ for some $\delta > 0$.

Definition 2.1. Let $A \subset X$, $x \in X$ and $r > 0$.

Let $\gamma(A, x, r) := \sup\{s \geq 0 : \exists\, y \in B(x, r) : B(y, s) \cap A = \emptyset\}$.

Let $\Gamma(A, x, r) := \sup\{s \geq 0 : \exists\, y \in B(x, r) : S(x, y, s) \cap A = \emptyset\}$.

(i) We say that A is *upper porous at x* if $\limsup\limits_{r \to 0_+} \frac{\gamma(A,x,r)}{r} > 0$.

(ii) We say that A is *lower porous at x* if $\liminf\limits_{r \to 0_+} \frac{\gamma(A,x,r)}{r} > 0$.

(iii) We say that A is *strongly porous at x* if $x \notin \overline{A}$ or there exists a sequence of balls $(B(y_n, s_n))_1^\infty$ such that $y_n \to x$, $B(y_n, s_n) \cap A = \emptyset$ for every $n \in \mathbb{N}$ and $\frac{s_n}{\varrho(x, y_n)} \to 1$.

(iv) We say that A is *shell porous at x* if $x \notin \overline{A}$ or $\limsup\limits_{r \to 0_+} \frac{\Gamma(A,x,r)}{r} > 0$.

Moreover, let $g \in \mathcal{G}$.

(v) We say that A is (g)-*upper porous at x* if there exists a sequence of balls $(B(y_n, s_n))_1^\infty$ such that $y_n \to x$, $B(y_n, s_n) \cap A = \emptyset$ and $g(s_n) > \varrho(x, y_n)$ for every $n \in \mathbb{N}$.

(vi) We say that A is (g)-*lower porous at x* if there exists $R > 0$ such that for every $0 < r \leq R$ there exists a ball $B(y_r, s_r)$ such that $y_r \in B(x, r)$, $B(y_r, s_r) \cap A = \emptyset$ and $g(s_r) > r$.

(vii) We say that A is (g)-*shell porous at x* if $x \notin \overline{A}$ or there exists a sequence of admissible shells $(S(x, y_n, s_n))_1^\infty$ such that $y_n \to x$, $S(x, y_n, s_n) \cap A = \emptyset$ and $g(s_n) > \varrho(x, y_n)$ for every $n \in \mathbb{N}$.

(viii) We say that A is *upper porous* (*lower porous*, *strongly porous*, *shell porous*, (g)*-upper porous*, (g)*-lower porous*, (g)*-shell porous*) if it is upper porous (lower porous, strongly porous, shell porous, (g)-upper porous, (g)-lower porous, (g)-shell porous) at each point in A.

(ix) We say that A is σ*-upper porous* (σ*-lower porous*, σ*-strongly porous*, σ*-shell porous*, σ-(g)*-upper porous*, σ-(g)*-lower porous*, σ-(g)*-shell porous*) if it is a countable union of upper porous (lower porous, strongly porous, shell porous, (g)-upper porous, (g)-lower porous, (g)-shell porous) sets.

Remark 2.2. (i) For other definitions of generalized porosities similar to the notion of (g)-upper porosity we refer the reader to [5] and [6].

(ii) The notion of (g)-lower porosity has been used in several unpublished (to my knowledge) manuscripts by D. L. Renfro, dating from around 1995-96, and it is also used at the end of his short conference note [2, Note 6].

(iii) The notion of shell porosity was introduced by R. W. Vallin in [3]. His original definition is not very suitable in general metric spaces because it does not imply upper porosity. Moreover, he admits empty shells which makes his definition not natural. Therefore we changed Vallin's definition to fix these inconveniences by considering the admissible shells only. In normed linear spaces our notion is (after some rescaling) equivalent to Vallin's original notion.

(iv) Let $g \in \mathcal{G}$ and $A \subset X$ be (g)-shell porous. Then A is (g)-upper porous.

(v) Let $g \in \mathcal{G}$ and $A \subset X$ be (g)-lower porous. Then A is (g)-upper porous.

(vi) Let $g \in \mathcal{G}$ and $A \subset X$ be (g)-upper porous. Then A is nowhere dense.

(vii) Let $g \in \mathcal{G}$, $\limsup_{t \to 0_+} \frac{g(t)}{t} < \infty$ and $A \subset X$ be (g)-upper porous ((g)-lower porous). Then A is upper porous (lower porous).

(viii) Let $g \in \mathcal{G}$, $\liminf_{t \to 0_+} \frac{g(t)}{t} = \infty$ and $A \subset X$ be upper porous (lower porous). Then A is (g)-upper porous ((g)-lower porous).

(ix) Let $g \in \mathcal{G}$, $\lim_{t \to 0_+} \frac{g(t)}{t} = 1$ and $A \subset X$ be (g)-upper porous. Then A is strongly porous.

(x) Let $g \in \mathcal{G}$, $\lim\limits_{t\to 0_+} \frac{g(t)}{t} = 1$ and $A \subset X$ be (g)-lower porous. Then A is very strongly porous (this notion was used, e.g., by P. Mattila in [1]; its definition can also be found in [5]).

Now we can formulate our main result that is an improvement of Proposition 1.1:

Theorem 2.3. *Let (X, ϱ) be a nonempty, topologically complete metric space with no isolated points. Let g_1, $g_2 \in \mathcal{G}$. There exists a closed (g_1)-shell porous set $F \subset X$ which is not σ-(g_2)-lower porous.*

The proof of this theorem will be given later in Section 4 together with its corollaries.

3 Several Lemmas.

In the proof of Theorem 2.3 we will use several lemmas. The following two lemmas give us a useful tool to show that some sets are not σ-(g)-lower porous.

Lemma 3.1. *Let $g \in \mathcal{G}$ and let A be a σ-(g)-lower porous subset of a metric space (X, ϱ). Then A can be covered by countably many closed $(2g)$-lower porous sets.*

Proof. Since A is σ-(g)-lower porous, we can write $A = \bigcup_{n\in\mathbb{N}} A_n$, where A_n is (g)-lower porous for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. For every $k \in \mathbb{N}$ put $A_{n,k} := \{x \in A_n : \forall\, 0 < r \le \frac{1}{k}\, \exists\, B(z,s) : z \in B(x,r), B(z,s) \cap A_n = \emptyset, g(s) > r\}$. Clearly, $A_n \subset \bigcup_{k\in\mathbb{N}} A_{n,k}$ and it suffices to show that $\overline{A_{n,k}}$ is $(2g)$-lower porous for every $k \in \mathbb{N}$. Fix $k \in \mathbb{N}$, $x \in \overline{A_{n,k}}$ and $0 < r \le \frac{1}{k}$. Since $x \in \overline{A_{n,k}}$, we can find $y \in A_{n,k}$ such that $\varrho(x,y) < \frac{r}{2}$. Since $y \in A_{n,k}$, there exists an open ball $B(z,s)$ such that $z \in B(y, \frac{r}{2})$, $B(z,s) \cap A_{n,k} = \emptyset$ and $g(s) > \frac{r}{2}$. Clearly, $z \in B(x,r)$ and $B(z,s) \cap \overline{A_{n,k}} = \emptyset$, because any open ball disjoint with $A_{n,k}$ is also disjoint with $\overline{A_{n,k}}$. Moreover, $g(s) > \frac{r}{2}$ and thus $(2g)(s) > r$. Thus we have proved that $\overline{A_{n,k}}$ is $(2g)$-lower porous. □

Lemma 3.2. *Let (X, ϱ) be a metric space and let F be a nonempty, topologically complete subspace of X. Let $g \in \mathcal{G}$ and let there exist a set $A \subset F$ dense in F such that F is $(2g)$-lower porous (in X) at no point $x \in A$. Then F is not a σ-(g)-lower porous subset of X.*

PROOF. Suppose on the contrary that F is σ-(g)-lower porous. Lemma 3.1 implies that $F \subset \bigcup_{n\in\mathbb{N}} F_n$, where each set F_n is closed and $(2g)$-lower porous. Since F is a topologically complete space, by Baire's Theorem there exist an open set $G \subset X$ and an index $n_0 \in \mathbb{N}$ such that $G \cap F \neq \emptyset$ and $G \cap F \subset F_{n_0}$. Since A is dense in F, there exists a point $x \in G \cap A$. Since F_{n_0} is $(2g)$-lower porous, it is $(2g)$-lower porous at all points of $G \cap F$, especially at the point x. Hence F is $(2g)$-lower porous at x, which is a contradiction with $x \in A$. □

The third lemma provides a construction of special sets that will be repeatedly used in the construction of set satisfying properties of Theorem 2.3. In its proof, we will use the following notion of ε-discreteness:

Let (X, ϱ) be a nonempty metric space and $\varepsilon > 0$. A set $A \subset X$ is *ε-discrete* if $\varrho(x, y) \geq \varepsilon$ for every two points $x, y \in A$, $x \neq y$.

Zorn's Lemma implies the existence of a *maximal ε-discrete* subset of any nonempty set in X.

Let $A, B \subset X$. We say that A is *discrete in* B if $A \subset B$ and $A' \cap B = \emptyset$ where A' denotes the set of all points of accumulation of A in X.

Lemma 3.3. *Let (X, ϱ) be a nonempty, topologically complete metric space with no isolated points. Let g, $h \in \mathcal{G}$. Then for any $z \in X$ there exists a set $M_z \subset X \setminus \{z\}$ and an open set $O_z \subset X \setminus \{z\}$ with the following properties:*

- *(a)* $(M_z)' = \{z\}$,
- *(b)* M_z *is not (h)-lower porous at* z,
- *(c)* $M_z \subset O_z$,
- *(d)* O_z *is (g)-shell porous at* z.

PROOF. Choose a point $z \in X$ arbitrarily. Since $g \in \mathcal{G}$, there exists $\delta_g > 0$ such that $g(t) > t$ for all $t \in (0, \delta_g)$. Since $h \in \mathcal{G}$, there exists $\delta_h > 0$ such that the inverse function h^{-1} is defined at least on $[0, \delta_h)$. Put $\alpha_0 := \delta_h$ and $N_0 := \emptyset$.

We will inductively construct sequences $(\alpha_n)_1^\infty$, $(y_n)_1^\infty$, $(s_n)_1^\infty$, $(N_n)_1^\infty$ and $(Q_n)_1^\infty$ such that the following conditions are satisfied for all $n \in \mathbb{N}$:

- (i) $0 < \alpha_n \leq \frac{1}{2}\alpha_{n-1}$,
- (ii) $0 < \varrho(z, y_n) < \min(\frac{\delta_g}{n}, \frac{1}{3}\operatorname{dist}(z, N_{n-1}))$,
- (iii) $0 < s_n < \varrho(z, y_n) < g(s_n)$,
- (iv) $S(z, y_n, s_n) \cap B(z, \alpha_n) = \emptyset$,

(v) N_n is discrete in X and $\operatorname{dist}(z, N_n) > 0$,

(vi) $\forall x \in B(z, \alpha_n) \setminus \{z\} : B(x, \sigma) \cap N_n = \emptyset \Rightarrow h(\sigma) < \alpha_n$,

(vii) $Q_n \subset B(z, \alpha_n) \setminus \{z\}$ is open,

(viii) $Q_n \cap B(z, \frac{2}{3} \operatorname{dist}(z, N_n)) = \emptyset$,

(ix) $N_n \subset Q_n$.

Suppose either $k = 0$, or $k \in \mathbb{N}$ and we have already constructed α_1, y_1, s_1, N_1, Q_1, ..., α_k, y_k, s_k, N_k, Q_k so that conditions (i)-(ix) are satisfied for $n = k$.

Since z is not isolated in X, there exists $y_{k+1} \in X$ such that $0 < \varrho(z, y_{k+1}) < \min(\frac{\delta_g}{k+1}, \frac{1}{3} \operatorname{dist}(z, N_k))$. Since $g(\varrho(z, y_{k+1})) > \varrho(z, y_{k+1})$, by continuity of g we can find $0 < s_{k+1} < \varrho(z, y_{k+1})$ such that $g(s_{k+1}) > \varrho(z, y_{k+1})$. Put $\alpha_{k+1} := \frac{1}{2} \min(\alpha_k, \varrho(z, y_{k+1}) - s_{k+1}) > 0$. Since $B(z, \alpha_{k+1}) \setminus \{z\}$ is nonempty, we can define N_{k+1} as a maximal $h^{-1}(\alpha_{k+1})$-discrete subset of $B(z, \alpha_{k+1}) \setminus \{z\}$. Finally, put $Q_{k+1} := B(z, \alpha_{k+1}) \setminus \overline{B(z, \frac{2}{3} \operatorname{dist}(z, N_{k+1}))}$.

It is easy to check all the conditions (i)-(iv) and (vii)-(ix) for $n = k + 1$. Condition (v) follows from $h^{-1}(\alpha_{k+1})$-discreteness of N_{k+1} and condition (vi) follows from its maximality.

Finally, we put $M_z := \bigcup_{n \in \mathbb{N}} N_n$ and $O_z := \bigcup_{n \in \mathbb{N}} Q_n$. It suffices to check properties (a)-(d).

By condition (i) we get that $\alpha_n \to 0$. Conditions (vii) and (ix) imply that $N_n \subset B(z, \alpha_n) \setminus \{z\}$ for every $n \in \mathbb{N}$. Condition (vi) and the fact that $h \in \mathcal{G}$ further assure that $N_n \neq \emptyset$ for every $n \in \mathbb{N}$. Hence $\{z\} \subset (M_z)'$. The second inclusion follows by condition (v), namely by discreteness of sets N_n in X for every $n \in \mathbb{N}$. Thus property (a) is satisfied.

Since $\alpha_n \to 0$ and by condition (vi) any open ball $B(x, \sigma)$ disjoint with M_z such that $x \in B(z, \alpha_n)$ satisfies $h(\sigma) < \alpha_n$ for every $n \in \mathbb{N}$, M_z is not (h)-lower porous at z and property (b) is satisfied.

Property (c) follows directly from condition (ix) and definitions of sets M_z and O_z.

By conditions (iv), (vii) and since $(\alpha_n)_1^\infty$ is decreasing by condition (i), we conclude that $S(z, y_n, s_n) \cap Q_k = \emptyset$ for every $n \in \mathbb{N}$ and $k \in \mathbb{N}$ such that $k \geq n$. Moreover, if $n > 1$, by conditions (ii), (iii) and (viii) we get $S(z, y_n, s_n) \subset B(z, 2\varrho(z, y_n)) \subset B(z, \frac{2}{3} \operatorname{dist}(z, N_{n-1}))$ and $Q_{n-1} \cap B(z, \frac{2}{3} \operatorname{dist}(z, N_{n-1})) = \emptyset$, so $S(z, y_n, s_n) \cap Q_{n-1} = \emptyset$ and similarly $S(z, y_n, s_n) \cap Q_k = \emptyset$ for every $k \in \mathbb{N}$, $k < n$. Hence $S(z, y_n, s_n) \cap O_z = \emptyset$ for every $n \in \mathbb{N}$. Using conditions (ii) and (iii) we easily check that $(S(z, y_n, s_n))_1^\infty$ is a sequence of admissible shells as in Definition 2.1 (vii) (where we put z instead of x and O_z instead of A). Hence O_z is (g)-shell porous at z and property (d) is also satisfied.

□

4 Proof of the Main Result and its Corollaries.

Now we can finally give the proof of our main result.

PROOF OF THEOREM 2.3. Put $g := g_1$ and $h := 2g_2$. Since $g \in \mathcal{G}$, there exists $\delta_g > 0$ such that $g(t) > t$ for all $t \in (0, \delta_g)$. Further, put $F_0 := \emptyset$. We will inductively construct closed sets $F_1 \subset F_2 \subset ...$ and open sets $G_1 \supset G_2 \supset ...$ such that the following properties (where we put $D_n := F_n \setminus F_{n-1}$) hold for each $n \in \mathbb{N}$:

(i) $(D_n)' = F_{n-1}$,

(ii) $D_n \subset G_n$,

(iii) D_n is not (h)-lower porous at x for every point $x \in D_{n-1}$ if $n > 1$,

(iv) G_n is (g)-shell porous at x for every point $x \in D_{n-1}$ if $n > 1$,

(v) for every point $x \in \overline{G_n}$ there exists an admissible shell $S(x, y_x, s_x)$ such that $0 < \varrho(x, y_x) < \frac{1}{5n}$, $S(x, y_x, s_x) \cap \overline{G_n} = \emptyset$ and $g(s_x) > \varrho(x, y_x)$.

Choose $z \in X$ and put $F_1 := \{z\}$. Since z is not isolated in X, we can find $w_z^1 \in X$ such that $0 < \varrho(z, w_z^1) < \frac{1}{5}\min(1, \delta_g)$. Thus $g(\varrho(z, w_z^1)) > \varrho(z, w_z^1)$. We can find $\delta_z^1 > 0$ such that $\varrho(z, w_z^1) + \delta_z^1 < \frac{1}{5}\min(1, \delta_g)$ and $g(\varrho(z, w_z^1)) > \varrho(z, w_z^1) + \delta_z^1$. By continuity of g we can further find $0 < s_z^1 < \varrho(z, w_z^1)$ such that $g(s_z^1) > \varrho(z, w_z^1) + \delta_z^1$. Put $\Delta_z^1 := \frac{1}{3}\min(\delta_z^1, \varrho(z, w_z^1) - s_z^1) > 0$ and $G_1 := B(z, \Delta_z^1)$. Then conditions (i), (ii) and (v) hold for $n = 1$. Indeed:

- $(D_1)' = (F_1 \setminus F_0)' = \{z\}' = \emptyset = F_0$.
- $D_1 = \{z\} \subset B(z, \Delta_z^1) = G_1$.
- Choose $x \in \overline{G_1}$. Put $y_x := w_z^1$ and $s_x := s_z^1$. Then $0 < \varrho(x, y_x) \leq \varrho(x, z) + \varrho(z, y_x) < \delta_z^1 + \varrho(z, y_x) < \frac{1}{5}$ and $g(s_x) > \varrho(z, y_x) + \delta_z^1 > \varrho(x, y_x)$. Moreover, if $t \in S(x, y_x, s_x)$, then $\varrho(z, t) \geq \varrho(x, t) - \varrho(x, z) > (\varrho(x, y_x) - s_x) - \Delta_z^1 \geq (\varrho(z, y_x) - s_x - \varrho(x, z)) - \Delta_z^1 \geq (\varrho(z, y_x) - s_x) - 2\Delta_z^1 \geq 3\Delta_z^1 - 2\Delta_z^1 = \Delta_z^1$. Hence $S(x, y_x, s_x) \cap \overline{G_1} = \emptyset$.

Further suppose that $k \geq 2$, sets F_1, G_1, ..., F_{k-1}, G_{k-1} have already been constructed and conditions (i)-(v) hold for every $n \leq k-1$.

For every point $z \in D_{k-1} = F_{k-1} \setminus F_{k-2}$ we choose another point $w_z^k \notin F_{k-1}$ such that $\varrho(z, w_z^k) < \frac{1}{5}\min(\frac{1}{k}, \delta_g, \operatorname{dist}(z, F_{k-1} \setminus \{z\}), \operatorname{dist}(z, G_{k-1}^C))$ (such a point w_z^k exists since z is neither an isolated point of X nor a point of accumulation of D_{k-1} because according to the induction hypothesis $(D_{k-1})' = F_{k-2}$ and $\operatorname{dist}(z, F_{k-2}) > 0$). Hence $g(\varrho(z, w_z^k)) > \varrho(z, w_z^k)$ and we can find $\delta_z^k > 0$ such that $\varrho(z, w_z^k) + \delta_z^k < \frac{1}{5}\min(\frac{1}{k}, \delta_g, \operatorname{dist}(z, F_{k-1} \setminus \{z\}), \operatorname{dist}(z, G_{k-1}^C))$ and $g(\varrho(z, w_z^k)) > \varrho(z, w_z^k) + \delta_z^k$. By continuity of g we can further find $0 < s_z^k < \varrho(z, w_z^k)$ such that $g(s_z^k) > \varrho(z, w_z^k) + \delta_z^k$. Put $\Delta_z^k := \frac{1}{3}\min(\delta_z^k, \varrho(z, w_z^k) - s_z^k)$ and $B_z^k := B(z, \Delta_z^k)$. Define $G_k := G_{k-1} \cap \bigcup_{z \in D_{k-1}} (B_z^k \cap O_z)$ and $F_k := F_{k-1} \cup (\bigcup_{z \in D_{k-1}} (M_z \cap B_z^k) \cap G_k)$, where M_z and O_z are sets constructed for the point z by Lemma 3.3.

Then G_k is clearly an open subset of G_{k-1}, F_k is a closed superset of F_{k-1} and $D_k = F_k \setminus F_{k-1} = \bigcup_{z \in D_{k-1}} (M_z \cap B_z^k) \cap G_k$. We will further check that conditions (i)-(v) hold for $n = k$:

- Since $(M_z \cap B_z^k \cap G_k)' = \{z\}$ for every $z \in D_{k-1}$, we conclude that $D_{k-1} \subset (D_k)'$. Therefore $(D_{k-1})' \subset (D_k)'' \subset (D_k)'$ and since by the induction hypothesis $(D_{k-1})' = F_{k-2}$, we get $F_{k-2} \subset (D_k)'$. Hence $F_{k-1} = D_{k-1} \cup F_{k-2} \subset (D_k)'$. Conversely, since M_z is discrete in $X \setminus \{z\}$ for every $z \in D_{k-1}$ and the system $\{B_z^k\}_{z \in D_{k-1}}$ is disjoint, we get $(D_k)' \subset \overline{D_{k-1}} = D_{k-1} \cup (D_{k-1})' = F_{k-1}$. So condition (i) holds.
- Condition (ii) clearly holds because $D_k = \bigcup_{z \in D_{k-1}} (M_z \cap B_z^k) \cap G_k \subset G_k$.
- Choose an arbitrary point $x \in D_{k-1}$. By Lemma 3.3 we know that M_x is not (h)-lower porous at x. Since $B_x^k \cap G_{k-1}$ is an open neighborhood of x and porosity is a local notion, even $M_x \cap B_x^k \cap G_{k-1}$ is not (h)-lower porous at x. Moreover, since $M_x \subset O_x$, we get that D_k is not (h)-lower porous at x. Thus condition (iii) is also satisfied.
- Choose an arbitrary point $x \in D_{k-1}$. By Lemma 3.3 we know that O_x is (g)-shell porous at x. Since porosity is a local notion, B_x^k is an open neighborhood of x and a direct computation shows that $B_x^k \cap B_z^k = \emptyset$ for every $z \in D_{k-1}$ such that $z \neq x$, we conclude that $\bigcup_{z \in D_{k-1}} (B_z^k \cap O_z)$ is (g)-shell porous at x. Thus G_k is (g)-shell porous at x and condition (iv) is satisfied.
- Choose $x \in \overline{G_k}$ arbitrarily. From the construction of G_k either $x \in F_{k-1}$ or there exists a point $z \in D_{k-1}$ such that $x \in \overline{B_z^k}$ where $B_z^k = B(z, \Delta_z^k)$. We will further distinguish these two cases:

In the first case, when $x \in F_{k-1}$, there exists a natural number $1 \leq i \leq k-1$ such that $x \in D_i$. Therefore, by induction hypothesis or by already proved condition (iv), we know that G_{i+1} is (g)-shell porous at x. Since $G_k \subset G_{i+1}$, we get that G_k is also (g)-shell porous at x. Thus there exists an admissible shell $S(x, y_x, s_x)$ such that $0 < \varrho(x, y_x) < \frac{1}{5k}$, $S(x, y_x, s_x) \cap G_k = \emptyset$ and $g(s_x) > \varrho(x, y_x)$. Since every admissible shell is an open set, we easily conclude that $S(x, y_x, s_x) \cap \overline{G_k} = \emptyset$.

In the second case, put $y_x := w_z^k$ and $s_x := s_z^k$. Then $0 < \varrho(x, y_x) \leq \varrho(x, z) + \varrho(z, y_x) < \delta_z^k + \varrho(z, y_x) < \frac{1}{5k}$ and $g(s_x) > \varrho(z, y_x) + \delta_z^k > \varrho(x, y_x)$. Moreover, if $t \in S(x, y_x, s_x)$, then $\varrho(z, t) \geq \varrho(x, t) - \varrho(x, z) > (\varrho(x, y_x) - s_x) - \Delta_z^k \geq (\varrho(z, y_x) - s_x - \varrho(x, z)) - \Delta_z^k \geq (\varrho(z, y_x) - s_x) - 2\Delta_z^k \geq 3\Delta_z^k - 2\Delta_z^k = \Delta_z^k$. Hence $S(x, y_x, s_x) \cap \overline{B_z^k} = \emptyset$ and consequently $S(x, y_x, s_x) \cap \overline{G_k} = \emptyset$, because a straightforward computation yields $\operatorname{dist}(x, \overline{B_{\tilde{z}}^k}) > \varrho(x, y_x) + s_x$ for every $\tilde{z} \in D_{k-1}$, $\tilde{z} \neq z$. Thus condition (v) holds as well.

Put $F := \overline{\bigcup_{k \in \mathbb{N}} F_k}$ and $A := \bigcup_{k \in \mathbb{N}} F_k$. Clearly, F is closed and A is dense in F. Since $F_1 \subset F_2 \subset ...$, we can also write $A = \bigcup_{k \in \mathbb{N}} D_k$.

Firstly, we observe that objects constructed above satisfy the following property:

$$F \subset \overline{G_n} \text{ for every } n \in \mathbb{N}. \tag{1}$$

Due to monotonicity of closure, it suffices to check that $A = \bigcup_{k \in \mathbb{N}} F_k \subset \overline{G_n}$ for every $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since $(F_k)_{k=1}^{\infty}$ increases, it suffices to prove that $\bigcup_{k=n}^{\infty} F_k = F_{n-1} \cup \bigcup_{k=n}^{\infty} D_k \subset \overline{G_n}$. Since $(G_k)_{k=1}^{\infty}$ decreases, by condition (ii) we get that $\bigcup_{k=n}^{\infty} D_k \subset \bigcup_{k=n}^{\infty} G_k = G_n$. Moreover, by condition (i), we further get that $F_{n-1} = (D_n)'$. Since $D_n \subset G_n$ by (ii), it follows that $(D_n)' \subset \overline{G_n}$. Thus $F_{n-1} \cup \bigcup_{k=n}^{\infty} D_k \subset \overline{G_n}$ and property (1) is verified.

By condition (iii), A is not (h)-lower porous at any of its points. Therefore $F = \overline{A}$ is not (h)-lower porous at any point of A. Since $h = 2g_2$ and the assumptions of Lemma 3.2 are met, F is not σ-(g_2)-lower porous.

It suffices to prove (g_1)-shell porosity of F. Choose an arbitrary point $x \in F$. Due to property (1), $x \in \overline{G_n}$ for every $n \in \mathbb{N}$. By condition (v) we get a sequence $(S(x, y_x^n, s_x^n))_1^{\infty}$ such that $y_x^n \to x$, $S(x, y_x^n, s_x^n) \cap \overline{G_n} = \emptyset$ and $g(s_x^n) > \varrho(x, y_x^n)$ for every $n \in \mathbb{N}$. Since $F \subset \overline{G_n}$ for every $n \in \mathbb{N}$ by property (1), $S(x, y_x^n, s_x^n) \cap F = \emptyset$ for every $n \in \mathbb{N}$ and hence F is (g)-shell porous at x, which finishes the proof. □

Corollary 4.1. *Let (X, ϱ) be a nonempty, topologically complete metric space with no isolated points.*

(i) Let g_1, $g_2 \in \mathcal{G}$. There exists a closed (g_1)-upper porous set $F \subset X$ which is not σ-(g_2)-lower porous.

(ii) There exists a closed strongly porous set $F \subset X$ which is not σ-lower porous.

PROOF. (i) This proposition follows directly from Theorem 2.3 and Remark 2.2 (iv).

(ii) This proposition is an immediate consequence of Corollary 4.1 (i) and Remarks 2.2 (viii), (ix). It suffices to take $g_1(t) = t + t^2$ and $g_2(t) = \sqrt{t}$ for $t \in [0, \infty)$.

□

Remark 4.2. We don't know whether there are any reasonably natural constraints that one can place on a σ-upper porosity notion (perhaps a geometric condition of some kind in the way that shell porosity is) that will force it to imply some σ-lower porosity notion.

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