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ON THE CONVERGENCE OF GENERALIZED CONTINUOUS MULTIVALUED MAPPINGS

Abstract

The main results presented in this paper concern generalized continuous multivalued mappings. An attempt has been made to formulate sufficient conditions under which convergence of nets of multivalued mappings preserves generalized continuity.

1 Introduction.

In what follows X, Y are topological spaces and $\mathcal{E}$ is a non-empty family of non-empty subsets of X. For a subset A of a topological space $Cl(A)$ denotes the closure of A and $\emptyset$, the empty set. Here $\mathbb{N}$ stands for the set of all natural numbers. A multivalued mapping is a mapping from X to $P(Y) \setminus \{\emptyset\}$ where $P(Y)$ is the power set of Y. We use capital letters F, G, H etc. to denote multivalued mappings. For a multivalued mapping $F : X \longrightarrow P(Y) \setminus \{\emptyset\}$ we write simply $F : X \longrightarrow Y$. A single-valued mapping $f : X \longrightarrow Y$ can be considered as a multivalued mapping as $x \mapsto \{f(x)\}$, $x \in X$.
For a multivalued mapping $F : X \longrightarrow Y$ and for $A \subseteq Y$, we write,

$$F^+(A) = \{x \in X : F(x) \subseteq A\}, F^-(A) = \{x \in X : F(x) \cap A \neq \emptyset\}.$$

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For a single-valued mapping $f : X \longrightarrow Y$ and for $A \subseteq Y$,

$$f^{+}(A) = f^{-}(A) = f^{-1}(A) = \{x \in X : f(x) \in A\}.$$

A multivalued mapping $F : X \longrightarrow Y$ is said to be upper (lower) $\mathcal{E}$-continuous at $x \in X$ [9] if for each open neighbourhood U of x and each open set V in Y with $F(x) \subseteq V$ $(F(x) \cap V \neq \emptyset)$ there exists $E \in \mathcal{E}$ with $E \subseteq U$ such that $E \subseteq F^{+}(V)$ $(E \subseteq F^{-}(V))$. It is called upper (lower) $\mathcal{E}$-continuous on X if it is so at every point of X.

Let $\mathcal{O} = \{E \subseteq X : E \neq \emptyset$ and open in $X\}$, $\mathcal{B}_r = \{E \subseteq X : E$ is second category with the Baire property$\}$, $\mathcal{B} = \{E \subseteq X : E$ is either non-empty open or second category with the Baire property$\}$, and $\mathcal{B}^{*} = \{E \subseteq X : E$ is not nowhere dense with the Baire property$\}$. In the case $\mathcal{E} = \mathcal{O}$,$(= \mathcal{B}_r$,$= \mathcal{B}$, $= \mathcal{B}^{*})$ we have the upper (lower) $\mathcal{E}$-continuity as the usual notion of upper (lower) quasicontinuity citebib13, (Baire continuity [10], B-continuity [10], B^{*}-continuity [3] respectively).

For a multivalued mapping $F : X \longrightarrow Y$, we write,

$$C^{+}(F) = \{x \in X : F \text{ is upper } \mathcal{E}\text{-continuous at } x\}$$

and

$$C^{-}(F) = \{x \in X : F \text{ is lower } \mathcal{E}\text{-continuous at } x\}.$$

For an open set V in a topological space (Y, τ), we write,

$$V^{+} = \{A \in P(Y) : A \subseteq V\},\ V^{-} = \{A \in P(Y) : A \cap V \neq \emptyset\}.$$

The topologies in $P(Y)$ generated by the base, $\{V^{+} : V \in \tau\}$ and subbase, $\{V^{-} : V \in \tau\}$ are respectively called the upper, lower Vietoris topologies [12]. These topologies will be denoted by τ^{+} and τ^{-} respectively.

A net $\{a_j : j \in J\}$ of elements of Y is said to be convergent to $a \in Y$ [4], if for each neighbourhood V of a, there exists $j_0 \in J$ such that $a_j \in V$ for every $j \in J$, $j \geq j_0$.

For a net $\{A_j : j \in J\}$ of subsets of Y and for $A \subseteq Y$, we write, $A \in \tau^{+} - \lim A_j$ $(A \in \tau^{-} - \lim A_j)$ if $\{A_j : j \in J\}$ converges to A in $(P(Y), \tau^{+})$ $((P(Y), \tau^{-}))$ [2].

A net $\{F_j : j \in J\}$ of multivalued mappings $F_j : X \longrightarrow Y$ is said to be τ^{+}-pointwise (τ^{-}- pointwise) convergent to a multivalued mapping $F : X \longrightarrow Y$ if for every $x \in X$, $F(x) \in \tau^{+} - \lim F_j(x)$ $(F(x) \in \tau^{-} - \lim F_j(x))$.

If Y is locally compact, then we may consider the lbc-topology [11] on the family 2^{Y} of all closed non-empty subsets of Y. The basis of the lbc-topology on 2^{Y} is the family of all sets of the form $[U_1, \ldots U_n; V_1, \ldots V_k]$ where U_i, V_j

are arbitrary open sets in Y with a compact closure and

$$[U_1, \dots U_n; V_1, \dots V_k] = \{A \in 2^Y : A \cap U_i \neq \emptyset, A \cap Cl(V_j) = \emptyset, \\ i = 1, 2, \dots n; j = 1, 2, \dots k\}.$$

It is shown in [11] that the space 2^Y with the lbc-topology is locally compact.

2 Strong Convergence of Nets.

The notion of strong convergence of multivalued mappings was introduced by Kupka and Toma in [7]. In [2], Irena Domnik considered nets of multivalued mappings upper and lower strongly convergent. A net $\{F_j : j \in J\}$ of multivalued mappings $F_j : X \longrightarrow Y$ is said to be upper (lower) strongly convergent to a multivalued mapping $F : X \longrightarrow Y$ [2] if for each open cover $\mathcal{A}$ of Y there exists $j_0 \in J$ such that for every $j \in J$, $j \geq j_0$ and for every $x \in X$,

$$F_j(x) \subseteq St(F(x), \mathcal{A}) \qquad (F(x) \subseteq St(F_j(x), \mathcal{A}))$$

where the set $St(A, \mathcal{A}) = \bigcup\{B \in \mathcal{A} : B \cap A \neq \emptyset\}$ is called the star of $A(\subseteq Y)$ with respect to a cover $\mathcal{A}$ of Y.
The upper (lower) strong convergence seems to be a generalization of the uniform convergence. For single-valued mappings, if $\{f_n : X \to (Y, d) : n \in \mathbb{N}\}$ (d a metric) upper strongly converges to f, then for a cover $\mathcal{A}$ containing all $\epsilon/4$ balls there is $m \in \mathbb{N}$ such that for any $n > m$ and any $x \in X$, $f_n(x) \in St(f(x),\mathcal{A})$, so $d(f_n(x), f(x)) < \epsilon$.
In the next theorems we formulate sufficient conditions under which the upper (lower) strong convergence preserves the upper (lower) $\mathcal{E}$-continuity. We use the general scheme of the proofs as the scheme in [2] and use the following lemma in the proof of the next theorem.

Lemma 2.1. [2] Let Y be a regular space. If A is a para-compact subset of Y and U is open in Y such that $A \subseteq U$, then there exists an open set V in Y such that $A \subseteq V \subseteq Cl(V) \subseteq U$.

Theorem 2.2. *Let Y be a regular space and $F : X \longrightarrow Y$ be a multivalued mapping with para-compact values. If a net $\{F_j : j \in J\}$ of multivalued mappings $F_j : X \longrightarrow Y$ is τ^+- pointwise and lower strongly converges to F, then*

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^+(F_j) \subseteq C^+(F).$$

PROOF. Let $x_0 \in \bigcap_{i\in J} \bigcup_{j\geq i} C^+(F_j)$, V be open in Y with $F(x_0) \subseteq V$ and let U be an open neighbourhood of x_0. By Lemma 2.1, there exists an open set W in Y such that $F(x_0) \subseteq W \subseteq Cl(W) \subseteq V$. Since $F(x_0) \in \tau^+ - \lim F_j(x_0)$ and $F(x_0) \subseteq W$, there exists $j_1 \in J$ such that $F_j(x_0) \subseteq W$ for every $j \in J$, $j \geq j_1$. Again since $\{F_j : j \in J\}$ lower strongly converges to F, corresponding to the open cover $\mathcal{A} = \{V, Y \setminus Cl(W)\}$ of Y there exists $j_2 \in J$ such that

$$F(x) \subseteq St(F_j(x), \mathcal{A}) \text{ for every } j \in J, j \geq j_2 \text{ and } x \in X.$$

Choose $j \in J\,(j \geq j_1,\ j \geq j_2)$ such that $x_0 \in C^+(F_j)$. Then there exists $E \in \mathcal{E}$ with $E \subseteq U$ such that $F_j(x) \subseteq W$ for every $x \in E$. Thus $St(F_j(x),\mathcal{A}) = V$ for every $x \in E$ and so, $F(x) \subseteq V$ for every $x \in E$. Hence $x_0 \in C^+(F)$ and consequently

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^+(F_j) \subseteq C^+(F).$$

□

Theorem 2.3. *Let Y be a regular space. If a net $\{F_j : j \in J\}$ of multivalued mappings $F_j : X \longrightarrow Y$ is τ^-- pointwise and upper strongly converges to a multivalued mapping $F : X \longrightarrow Y$, then*

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^-(F_j) \subseteq C^-(F).$$

PROOF. Let $x_0 \in \bigcap_{i\in J}\bigcup_{j\geq i} C^-(F_j)$, V be open in Y with $F(x_0) \cap V \neq \emptyset$ and let U be an open neighbourhood of x_0. Let $y_0 \in F(x_0) \cap V$. Since Y is regular, there exists an open set W in Y such that $y_0 \in W \subseteq Cl(W) \subseteq V$. Now $F(x_0) \in \tau^- - \lim F_j(x_0)$ and $F(x_0) \cap W \neq \emptyset$. So, there exists $j_1 \in J$ such that $F_j(x_0) \cap W \neq \emptyset$ for every $j \in J$, $j \geq j_1$. Again since $\{F_j : j \in J\}$ upper strongly converges to F, corresponding to the open cover $\mathcal{A} = \{V, Y \setminus Cl(W)\}$ of Y there exists $j_2 \in J$ such that

$$F_j(x) \subseteq St(F(x), \mathcal{A}) \text{ for every } j \in J,\ j \geq j_2 \text{ and } x \in X.$$

Choose $j \in J$ $(j \geq j_1,\ j \geq j_2)$ such that $x_0 \in C^-(F_j)$. Then there exists $E \in \mathcal{E}$ with $E \subseteq U$ such that $F_j(x) \cap W \neq \emptyset$ for every $x \in E$. Suppose that $F(x') \cap V = \emptyset$ for some $x' \in E$. Then $F(x') \subseteq Y \setminus V$ and so, $St(F(x'),\mathcal{A}) = Y \setminus Cl(W)$. Hence $F_j(x') \subseteq Y \setminus Cl(W)$. But $F_j(x') \cap W \neq \emptyset$. Thus we arrive at a contradiction. Hence $F(x) \cap V \neq \emptyset$ for every $x \in E$. So, $x_0 \in C^-(F)$ and consequently

$$\bigcap_{i\in J}\bigcup_{j\geq i} C^-(F_j) \subseteq C^-(F).$$

□

3 Topological Convergence of Nets.

Let Ω be the first uncountable ordinal number. For a transfinite sequence $\{A_\xi : \xi < \Omega\}$ of subsets of Y, $Li_{\xi<\Omega}A_\xi$ is defined as the set of all $y \in Y$ such that for every neighbourhood V of y there exists $\xi_0 < \Omega$ such that $A_\xi \cap V \neq \emptyset$ for every ξ, $\xi_0 < \xi < \Omega$. $Ls_{\xi<\Omega}A_\xi$ is defined as the set of all $y \in Y$ such that for every neighbourhood V of y and for every $\xi < \Omega$, there exists $\xi' < \Omega$ such that $\xi < \xi'$ and $A_{\xi'} \cap V \neq \emptyset$. $\{A_\xi : \xi < \Omega\}$ is said to be topologically convergent to $A \subseteq Y$ if $Li_{\xi<\Omega}A_\xi = Ls_{\xi<\Omega}A_\xi = A$ and in this case we write $A = Lt_{\xi<\Omega}A_\xi$ [8].

A multivalued mapping $F : X \longrightarrow Y$ is said to be a topological limit of a transfinite sequence $\{F_\xi : \xi < \Omega\}$ of multivalued mappings $F_\xi : X \longrightarrow Y$ if for every $x \in X$,

$$F(x) = Lt_{\xi<\Omega}F_\xi(x) \qquad [8]$$

and in this case we write $F = Lt_{\xi<\Omega}F_\xi$.

In this section we formulate sufficient conditions under which the topological convergence preserves the upper, lower $\mathcal{E}$-continuity. We will use the following lemmas.

Lemma 3.1. [5] Let Y be a first countable T_1-space. If a transfinite sequence $\{a_\xi : \xi < \Omega\}$ of elements of Y converges to $a \in Y$, then there exists $\xi_0 < \Omega$ such that $a_\xi = a$ for every ξ, $\xi_0 < \xi < \Omega$.

Lemma 3.2. [11] If Y is a locally compact space, then the topological convergence of nets of subsets of Y and the convergence in the space 2^Y with the lbc-topology are equivalent.

Lemma 3.3. [6] Every para-compact subset of a Hausdorff space is closed.

Theorem 3.4. *Let Y be a locally compact separable metric space and let F_ξ, $F : X \longrightarrow Y$, $\xi < \Omega$ be multivalued mappings with closed values. If $\mathcal{E}$ is countable, each F_ξ, $\xi < \Omega$ is upper (lower) $\mathcal{E}$-continuous on X and if $F = Lt_{\xi<\Omega}F_\xi$, then F is upper (lower) $\mathcal{E}$-continuous on X.*

Proof. If possible let F be not upper (lower) $\mathcal{E}$-continuous at $x_0 \in X$. Then there exists an open set V in Y with $F(x_0) \subseteq V$ $(F(x_0) \cap V \neq \emptyset)$ and there exists an open neighbourhood U of x_0 such that each $E \in \mathcal{E}$ with $E \subseteq U$ contains a point x' for which $F(x') \cap (Y \setminus V) \neq \emptyset$ $(F(x') \subseteq Y \setminus V)$. For each $E \in \mathcal{E}$ with $E \subseteq U$, choose a point $x_E \in E \subseteq U$ such that $F(x_E) \cap (Y \setminus V) \neq \emptyset$

$(F(x_E) \subseteq Y \setminus V)$ and construct the set A by choosing all such x_E's. Then for each $a \in A$,

$$F(a) \cap (Y \setminus V) \neq \emptyset \quad (F(a) \subseteq Y \setminus V). \tag{3.1}$$

Since $\mathcal{E}$ is countable, A is countable. Now $F(x) = Lt_{\xi<\Omega} F_\xi(x)$ for all $x \in X$. By Lemma 3.2, for every $x \in X$, $F(x) = \lim_{\xi<\Omega} F_\xi(x)$ in the space 2^Y with the lbc-topology. Since the space 2^Y with the lbc-topology is metrizable [1, Theorem 4], by Lemma 3.1, for each $a \in A$, there exists $\xi_a < \Omega$ such that $F_\xi(a) = F(a)$ for every ξ, $\xi_a < \xi < \Omega$. Again by Lemma 3.1, for $x_0 \in X$, there exists $\xi_0 < \Omega$ such that $F_\xi(x_0) = F(x_0)$ for every ξ, $\xi_0 < \xi < \Omega$. Choose $\xi' < \Omega$ such that $\xi_0 < \xi'$ and also $\xi_a < \xi'$ for every $a \in A$. Now $F_{\xi'}$ is upper (lower) $\mathcal{E}$-continuous at x_0 and $F_{\xi'}(x_0) \subseteq V$ $(F_{\xi'}(x_0) \cap V \neq \emptyset)$. So, there is $E \in \mathcal{E}$ with $E \subseteq U$ such that for every $e \in E$, $F_{\xi'}(e) \subseteq V$ $(F_{\xi'}(e) \cap V \neq \emptyset)$. Thus for some $a \in A$, $F_{\xi'}(a) \subseteq V$ $(F_{\xi'}(a) \cap V \neq \emptyset)$ and so, for some $a \in A$, $F(a) \subseteq V$ $(F(a) \cap V \neq \emptyset)$ which is contradictory to (3.1). So, F is upper (lower) $\mathcal{E}$-continuous on X. □

Theorem 3.5. *Let Y be a locally compact separable metric space and let F_ξ, $F : X \longrightarrow Y$, $\xi < \Omega$ be multivalued mappings with para-compact values. If for every $\xi < \Omega$, there exists $\xi' < \Omega$ such that $\xi < \xi'$ and $F_{\xi'}$, is upper $\mathcal{E}$-continuous on X and if $\{F_\xi : \xi < \Omega\}$ lower strongly converges to F so that $F = Lt_{\xi<\Omega} F_\xi$, then F is upper $\mathcal{E}$-continuous on X.*

PROOF. There is a subnet $\{F_{\xi'} : \xi' < \Omega\}$ of upper $\mathcal{E}$-continuous multivalued mappings which lower strongly converges to F and $F = Lt_{\xi'<\Omega} F_{\xi'}$. By Lemma 3.2, for every $x \in X$, $F(x) = \lim_{\xi'<\Omega} F_{\xi'}(x)$ in the space 2^Y with the lbc-topology. Since the space 2^Y with the lbc-topology is metrizable [1, Theorem 4], by Lemma 3.1, for each $x \in X$, there exists $\xi_x < \Omega$ such that $F_{\xi'}(x) = F(x)$ for every ξ', $\xi_x < \xi' < \Omega$. So $\{F_{\xi'} : \xi' < \Omega\}$ τ^+- pointwise converges to F. By Theorem 2.2,

$$\bigcap_{\xi'<\Omega} \bigcup_{\xi''\geq\xi'} C^+(F_{\xi''}) = X \subseteq C^+(F).$$

□

The proof of the next theorem is similar to that of Theorem 3.5 and so, we omit the proof.

Theorem 3.6. *Let Y be a locally compact separable metric space and let F_ξ, $F : X \longrightarrow Y$, $\xi < \Omega$ be multivalued mappings with para-compact values. If for every $\xi < \Omega$, there exists $\xi' < \Omega$ such that $\xi < \xi'$ and $F_{\xi'}$ is lower $\mathcal{E}$-continuous* *on X and if $\{F_\xi : \xi < \Omega\}$ upper strongly converges to F so that $F = Lt_{\xi<\Omega}F_\xi$, then F is lower $\mathcal{E}$-continuous on X.*

We conclude by posing the following problem: Does Theorem 3.4 hold for an arbitrary cluster system?

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