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ON THE PATH DARBOUX PROPERTY

Abstract

In this paper we introduce a notion of the path Darboux property. We examine the basic properties and investigate relationships between the path Darboux property and path continuity.

We apply standard symbols and notions. By $\mathbb{R}$, we denote the set of real numbers, $\widetilde{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, $\widetilde{\mathbb{R}}^+ = [0, \infty]$. The symbol $int(A)$ ($cl(A)$, $\ell(A)$) stands for the interior (the closure, the family of all components, respectively) of a set $A \subset \mathbb{R}$. For $x \in \mathbb{R}$ let $\mathcal{T}(x)$ denote the family of all open sets containing x.

We consider only real-valued functions defined on $\mathbb{R}$. No distinction is made between a function and its graph.

Let f be a function and $x \in \mathbb{R}$. The set of all continuity (discontinuity) points of f is denoted by C_f (D_f). By $L^+(f,x)$ ($L^-(f,x)$) we denote the set of all right-side limit numbers (all left-side limit numbers respectively) of f at x.

For a set $E \subset \mathbb{R}$ we define the set of right-side limit numbers of f at x with respect to the set E: $L^+_E(f,x)$ denotes the set of all $\alpha \in \widetilde{\mathbb{R}}$ such that there exists a sequence $(x_n)_{n=1}^{\infty}$ (where $x_n > x$ and $x_n \in E$, $n = 1, 2 \dots$) for which $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = \alpha$. Similarly $L^-_E(f,x)$ denotes the set of all $\alpha \in \widetilde{\mathbb{R}}$ such that there exists a sequence $(x_n)_{n=1}^{\infty}$ (where $x_n < x$ and $x_n \in E$, $n = 1, 2 \dots$) for which $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} f(x_n) = \alpha$. Of course

$$L^+_{\mathbb{R}}(f,x) = L^+(f,x) \text{ and } L^-_{\mathbb{R}}(f,x) = L^-(f,x).$$

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We say that a function f has the Darboux property if the image of an arbitrary connected set is connected.

In [3] Bruckner and Ceder described what it means for a real function to be Darboux at a point. We say that a function f is Darboux from the right side (from the left side) at a point x if and only if

$$\forall_{\alpha\in L^+(f,x)\setminus\{f(x)\}}\forall_{\beta\in I_{(f(x),\alpha)}}\forall_{\sigma>0}f^{-1}(\beta)\cap[x,x+\sigma)\neq\emptyset$$
$$\big(f^{-1}(\beta)\cap(x-\sigma,x]\neq\emptyset \text{ respectively}\big)$$

where (for $a,b\in\widetilde{\mathbb{R}}$) $I_{(a,b)}=(\min\{a,b\},\max\{a,b\})$.

In [4] Császár showed that a real function f has the Darboux property on interval if and only if it is Darboux at each point.

It is easy to show that if a function f is Darboux from the right side (from the left side), then $L^+(f,x)$ ($L^-(f,x)$ respectively) is a closed interval and $f(x)\in L^+(f,x)$ ($f(x)\in L^-(f,x)$ respectively).

Let $x\in\mathbb{R}$. As in [2], a path (a bilateral path) leading to x is a set E such that $x\in E$ and x is a point of accumulation (bilateral accumulation respectively) of E.

A family of (bilateral) paths at $x\in\mathbb{R}$ is a nonempty family $\mathcal{F}$ of subsets of $\mathbb{R}$ such that E is a (bilateral) path leading to x, for each $E\in\mathcal{F}$.

A system of families of (bilateral) paths on $\mathbb{R}$ is a collection $\mathcal{E}=\{\mathcal{E}(x): x\in\mathbb{R}\}$ such that $\mathcal{E}(x)$ is a family of (bilateral) paths leading to x, for $x\in\mathbb{R}$ (cf. [1]).

Let $\mathcal{E}$ be a system of families of paths on $\mathbb{R}$. We say that a function f is $\mathcal{E}$-continuous at x (or x is an $\mathcal{E}$-continuity point of f), if there exists a path $E\in\mathcal{E}(x)$ such that $f\restriction E$ is continuous at x (cf. [1]).

The set of all $\mathcal{E}$-continuity points of f will be denoted by C_f^{ε}. The function f is $\mathcal{E}$-continuous if and only if it is $\mathcal{E}$-continuous at each point.

We introduce now a notion of path Darboux property.

Definition 1. Let $\mathcal{E}$ be a system of families of paths on $\mathbb{R}$. We say that a function f is $\mathcal{E}$-Darboux from the right side at a point x if and only if there exists $\delta>0$ and $E\in\mathcal{E}(x)$ such that:

(D1) $f(x)\in L_E^+(f,x)$,

(D2) $\forall_{\alpha\in L_E^+(f,x)\setminus\{f(x)\}}\forall_{\beta\in I_{(f(x),\alpha)}\cap f([x,x+\delta))}\forall_{0<\sigma\leq\delta}\forall_{C\in\ell(E\cap[x,x+\sigma))}$
$f^{-1}(\beta)\cap C\neq\emptyset$,

(D3) $\forall_{0<\sigma\leq\delta}\forall_{C\in\ell(E\cap[x,x+\sigma))}f(C)$ is connected.

Analogously we define what it means for f to be $\mathcal{E}$-Darboux from the left side at x. We will say that a function f is $\mathcal{E}$-Darboux at a point $x \in \mathbb{R}$ if f is $\mathcal{E}$-Darboux from the right side and from the left side at x.

If f is $\mathcal{E}$-Darboux at every point, then we say that f has the $\mathcal{E}$-Darboux property. The family of all functions with the $\mathcal{E}$-Darboux property is denoted by $\mathcal{D}_{\mathcal{E}}$.

Remark 1. *Let $\mathcal{E}$ be a system of families of bilateral paths on $\mathbb{R}$ and f be a function. If there exists a path $E \in \mathcal{E}(x)$ and $\delta > 0$ such that $f \restriction (E \cap [x, x+\delta))$ ($f \restriction (E \cap (x-\delta, x])$) is continuous, then f is $\mathcal{E}$-Darboux from the right side (from the left side respectively) at x.*

Proof. Suppose that $E \in \mathcal{E}(x)$, $\delta > 0$ and $f \restriction (E \cap [x, x+\delta))$ is continuous. Because $f \restriction E$ is continuous at x, thus $L^+_E(f, x) = \{f(x)\}$. Consequently condition (D1) and (D2) hold. Let $\sigma \in (0, \delta]$ and $C \subset E \cap [x, x+\sigma)$ be a connected set. Then $f \restriction C$ is continuous. Thus $f(C)$ is connected. That is the condition (D3) is true. $\square$

Corollary 1. *If there exists a path $E \in \mathcal{E}(x)$ and a neighborhood U of x such that $f \restriction (E \cap U)$ is continuous, then f is $\mathcal{E}$-Darboux at x.*

Let $\mathcal{E}_e$ be a system of families of paths defined on $\mathbb{R}$ as

$$\mathcal{E}_e(x) = \{E \subset \mathbb{R} : \exists_{\delta>0} (x-\delta, x+\delta) \subset E\}.$$

Lemma 1. *If a function f is $\mathcal{E}_e$-Darboux from the right side (from the left side) at a point x, then f is Darboux from the right side (from the left side respectively) at x.*

Proof. Suppose that f is $\mathcal{E}_e$-Darboux from the right side at x. Let $\alpha \in L^+(f, x) \setminus \{f(x)\}$, $\beta \in I_{(f(x),\alpha)}$, $\tau > 0$. We can assume that $f(x) < \beta < \alpha$. Let $E \in \mathcal{E}_e(x)$ and $\delta \in \widetilde{\mathbb{R}}^+$ be such that conditions (D1), (D2), (D3) of Definition 1 hold. Let $0 < \sigma \le \min\{\delta, \tau\}$ be such that $[x, x+\sigma) \subset E$. Observe that $\beta \in f([x, x+\delta))$. Indeed, $C = [x, x+\sigma) \in \ell(E \cap [x, x+\sigma))$. Because $\alpha \in L^+(f, x)$ and $\alpha > \beta$, hence there exists $t \in C$ such, that $f(t) > \beta$. We have that $f(x) < \beta < f(t)$ and $f(x), f(t) \in f(C)$. From (D3) we obtain that $f(C)$ is connected, and $\beta \in f(C) \subset f([x, x+\delta))$. Condition (D2) implies that $\emptyset \ne f^{-1}(\beta) \cap C \subset f^{-1}(\beta) \cap [x, x+\tau)$. This completes the proof. $\square$

Lemma 2. *If f has the Darboux property, then it has the $\mathcal{E}_e$-Darboux property.*

PROOF. Suppose that f has the Darboux property. Let $x \in \mathbb{R}$. We show that for $E = \mathbb{R}$ and $\delta = \infty$ conditions (D1), (D2), (D3) are satisfied. In fact, because f is Darboux from the right side at x, then $f(x) \in L^+(f,x)$. This means that condition (D1) holds. Let $\alpha \in L^+(f,x) \setminus \{f(x)\}$, $\beta \in I_{(f(x),\alpha)} \cap f([x,\infty))$ and $\sigma > 0$. Observe, that $\ell(E \cap [x, x+\sigma)) = \{[x, x+\sigma)\}$ and (because f is Darboux at x) $[x, x+\sigma) \cap f^{-1}(\beta) \neq \emptyset$. Thus the condition (D2) holds. Because f has the Darboux property $f(C)$ is connected for every connected $C \subset \mathbb{R}$. Therefore condition (D3) is satisfied. So f is $\mathcal{E}_e$-Darboux from the right side at x. Analogously we can show that f is $\mathcal{E}_e$-Darboux from the left side at x. $\square$

According to Lemmas 1 and 2 we have:

Theorem 1. *The function f has the Darboux property if and only if it has the $\mathcal{E}_e$-Darboux property.*

Remark 2. *There exists a function f continuous at 0, but not $\mathcal{E}_e$-Darboux at 0.*

PROOF. Define:

$$f(x) = \begin{cases} 0 & for\ x \in (-\infty, 0], \\ \frac{1}{n} & for\ x \in (\frac{1}{n+1}, \frac{1}{n}], n = 1, 2, \ldots, \\ 1 & for\ x \in (1, \infty). \end{cases}$$

Let $E \in \mathcal{E}_e(0)$ and $\delta \in \widetilde{\mathbb{R}}^+$. Because E is a neighborhood of 0 then there exists $0 < \sigma \leq \delta$ such, that $[x, x+\sigma) \subset E$. Then $f([x, x+\sigma))$ is not connected so condition (D3) is not satisfied. $\square$

Denote by $b\mathcal{D}^0$ $[b\mathcal{D}^0_e]$ the family of all bounded functions which are Darboux [$\mathcal{E}_e$-Darboux respectively] at 0. From lemma 1 we have $b\mathcal{D}^0_e \subset b\mathcal{D}^0$. Of course if f is continuous at x, then f is Darboux at x. According to Remark 2 we obtain that $b\mathcal{D}^0 \setminus b\mathcal{D}^0_e \neq \emptyset$. Moreover we have:

Theorem 2. *The set $b\mathcal{D}^0_e$ is boundary and dense in $b\mathcal{D}^0$ (with the uniform convergence metric).*

Proof. Let $f \in b\mathcal{D}^0$ and $\epsilon > 0$. We define a functions h_1, g_1 such that $h_1(0) = g_1(0) = f(0)$, $|g_1(x) - f(x)| < \epsilon$, $|h_1(x) - f(x)| < \epsilon$ for $x \in \mathbb{R}$, g_1 is $\mathcal{E}_e$-Darboux from the right side at 0, h_1 is Darboux from the right side at 0 but is not $\mathcal{E}_e$-Darboux from the right side at 0.

Suppose that f is right-hand continuous at 0. Let $\delta > 0$ be such that $f([0,\delta)) \subset (f(0) - \frac{\epsilon}{2}, f(0) + \frac{\epsilon}{2})$ and let $t_0 \in (0,\delta)$. Let $g_1(x) = f(x)$ if $x \in (-\infty, 0] \cup [t_0, \infty)$ and g_1 be linear on $[0, t_0]$. We define h_1 as:

$$h_1(x) = \begin{cases} f(x) & \textit{if } x \in (-\infty, 0] \cup (t_0, \infty), \\ f(0) + \frac{\epsilon}{2n} & \textit{if } x \in (\frac{t_0}{n+1}, \frac{t_0}{n}], n = 1, 2, \dots \ . \end{cases}$$

Then g_1 is continuous on $[0, t_0]$, therefore 0 (according to Remark 1) g_1 is $\mathcal{E}_e$-Darboux from the right side at 0. The function h_1 is right-hand continuous at 0 but for every $0 < \sigma \leq t_0$, the set $h_1([0,\sigma))$ is not connected. Thus h_1 is Darboux from the right side at 0 but is not $\mathcal{E}_e$-Darboux from the right side at 0.

Suppose now that f is not right-hand continuous at 0. Because f is bounded and Darboux at 0 then $L^+(f,0)$ is a non-degenerate and bounded closed interval. Thus there exists $a, b \in \mathbb{R}$ such that $a < b$ and $[a,b] = L^+(f,0)$. Let $\tau > 0$ be such that $f([0,\tau)) \subset (a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2})$. Define:

$$g_1(x) = \begin{cases} b & \textit{if } \ x \in (0,\tau) \ \& \ f(x) \in (b, \infty), \\ a & \textit{if } \ x \in (0,\tau) \ \& \ f(x) \in (-\infty, a), \\ f(x) & \textit{otherwise.} \end{cases}$$

Let $c \in (a,b) \cap (b - \frac{\epsilon}{2}, b) \setminus \{f(0)\}$. There exists a sequence $(x_n)_{n=1}^{\infty} \subset (0,\tau)$ such that $x_n \searrow 0$ and $f(x_n) = c$ (for $n = 1, 2 \dots$).
Define:

$$h_1(x) = \begin{cases} b & \textit{if } \ x \in (0,\tau) \ \& \ f(x) \in (b, \infty), \\ a & \textit{if } \ x \in (0,\tau) \ \& \ f(x) \in (-\infty, a), \\ b + \frac{\epsilon}{2k} & \textit{if } \ x = x_{2k}, k = 1, 2, \dots \ , \\ f(x) & \textit{otherwise.} \end{cases}$$

Observe that:

$$L^+(g_1, 0) = [a,b]. \tag{1}$$

Let $\gamma \in (a,b)$. Then (because $\gamma \in L^+(f,0)$) there exists sequence $s_n \searrow 0$ such that $f(s_n) \in (a,b)$ (for $n = 1, 2, \dots$) and $\lim_{n\to\infty} f(s_n) = \gamma$. Then $g_1(s_n) = f(s_n)$, for $n = 1, 2, \dots$, so $\lim_{n\to\infty} g_1(s_n) = \gamma$. Hence $\gamma \in L^+(g_1, 0)$. Because $L^+(g_1, 0)$ is closed then $[a,b] \subset L^+(g_1, 0)$. But $g_1([0,\tau)) \subset [a,b]$, so $L^+(g_1, 0) \subset [a,b]$. Hence (1) is true.

Now we show that:

$$\forall_{\mu>0} \quad (a,b)\setminus\{f(0)\} \subset g_1((0,\mu)). \tag{2}$$

Let $\mu > 0$ and $\eta \in (a,b)\setminus\{f(0)\}$. Then, because f is Darboux at 0, there exists $t \in (0,\mu)$ such that $f(t) = \eta$. Because $f(t) \in (a,b)$, so $g_1(t) = f(t) = \eta$ and consequently $\eta \in g_1((0,\mu))$.

We show that for g_1, the set $E = \mathbb{R}$ and $\delta = \tau$ conditions (D1), (D2), (D3) are fulfilled. Observe first that (according to (1)) $g_1(0) = f(0) \in [a,b] = L^+(g_1,0)$. Let $\alpha \in [a,b]\setminus\{f(0)\}$ and $\beta \in I_{(f(0),\alpha)}$. We have $\beta \in (a,b)\setminus\{f(0)\}$, so (according to (2)) for every $0 < \sigma \le \tau$, $g_1^{-1}(\beta) \cap [0,\sigma) \ne \emptyset$. Thus (D2) is true. According to $g_1(0) = f(0)$ and inclusion (2) for every $0 < \sigma \le \tau$, $(a,b) \subset g_1([0,\sigma)) \subset [a,b]$, so $g_1([0,\sigma))$ is connected. This means that (D3) is true.

We show next that h_1 is Darboux from the right side at 0. Observe that $h_1(x_{2k+1}) = c$ (for $k = 1,2,\dots$), $\lim\limits_{k\to\infty} h_1(x_{2k}) = b$ and if $g_1(t) \ne c$ (for $t \in [0,\tau)$), then $h_1(t) = g_1(t)$. Consequently $L^+(h_1,0) = L^+(g_1,0) = [a,b]$. It is enough to show that for every $\nu \in (a,b)\setminus\{h_1(0)\}$ and $\lambda > 0$ there exists $u \in (0,\lambda)$ such that $h_1(u) = \nu$. Because $h_1(x_{2k+1}) = c$ (for $k = 1,2,\dots$), we can assume that $\nu \ne c$. Because f is Darboux at 0 there exists $u \in (0,\lambda)$ such that $f(u) = \nu$. Then $f(u) \in (a,b)\setminus\{c\}$ so $h_1(u) = f(u)$. Therefore $h_1(u) = \nu$.

Observe that for every $0 < \sigma \le \tau$, $h_1([0,\sigma))$ is not connected so h_1 is not $\mathcal{E}_e$-Darboux at 0.

According to the construction of g_1 and h_1 we have that $h_1(0) = g_1(0) = f(0)$, $|g_1(x) - f(x)| < \epsilon$ and $|h_1(x) - f(x)| < \epsilon$ for $x \in \mathbb{R}$.

Similarly we can define functions g_2, h_2 such that $g_2(0) = h_2(0) = f(0)$, $|g_2(x) - f(x)| < \epsilon$, $|h_2(x) - f(x)| < \epsilon$ for $x \in \mathbb{R}$ and such that g_2 is $\mathcal{E}_e$-Darboux from the left side at 0, h_2 is Darboux from the left side at 0 but is not $\mathcal{E}_e$-Darboux from the left side at 0.

Finally let g, h be such functions that $g(x) = h_1(x)$ and $h(x) = g_1(x)$ for $x \in [0,\infty)$, $g(x) = g_2(x)$ and $h(x) = h_2(x)$ for $x \in (-\infty,0)$. Then $|g(x) - f(x)| < \epsilon, |h(x) - f(x)| < \epsilon$ for $x \in \mathbb{R}$, $g \in b\mathcal{D}^0_e$ and $h \in b\mathcal{D}^0 \setminus b\mathcal{D}^0_e$, so the theorem is proved. □

Theorem 3. *There exists a system of families of bilateral paths $\mathcal{E}$ such that every $\mathcal{E}$-continuous functions has a nowhere dense set of discontinuity points and $\mathcal{C}_{\mathcal{E}} \setminus \mathcal{D}_{\mathcal{E}} \ne \emptyset \ne \mathcal{D}_{\mathcal{E}} \setminus \mathcal{C}_{\mathcal{E}}$.*

PROOF. Let (for $n = 1,2,\dots$) $C_n \subset (\frac{1}{n+1}, \frac{1}{n})$ be a Cantor set. Denote (for $n = 1,2,\dots$) by $\mathcal{I}_n$ the family of all components of the set $[\frac{1}{n+1}, \frac{1}{n}] \setminus C_n$. Let

$J_{0,n}(J_{1,n})$ (for $n = 1, 2, \dots$) be a component of $[\frac{1}{n+1}, \frac{1}{n}] \setminus C_n$ which contains a point $\frac{1}{n+1}$ $(\frac{1}{n})$. Then (for $n = 1, 2, \dots$) there exist subfamilies $\mathcal{I}_{1,n}$, $\mathcal{I}_{2,n}$ of $\mathcal{I}_n$ for which : $\mathcal{I}_{1,n} \cup \mathcal{I}_{2,n} = \mathcal{I}_n$, $\mathcal{I}_{1,n} \cap \mathcal{I}_{2,n} = \emptyset$, $C_n \subset cl(\bigcup \mathcal{I}_{1,n}) \cap cl(\bigcup \mathcal{I}_{2,n})$ and $J_{0,n}, J_{1,n} \in \mathcal{I}_{2,n}$.
Denote:

$$A = \bigcup_{n=1}^{\infty} \bigcup_{I \in \mathcal{I}_{1,n}} cl(I), \qquad C = \bigcup_{n=1}^{\infty} C_n.$$

Let $\mathcal{E}$ be a system of families of bilateral paths defined as:

$$\mathcal{E}(x) = \begin{cases} \{\mathbb{R}\} & for \quad x = 0, \\ \{A\} & for \quad x \in A, \\ \{\mathbb{R} \setminus A\} & for \quad x \in \mathbb{R} \setminus (A \cup \{0\}). \end{cases}$$

Observe that:

$$\text{if } x \notin C \text{ and } E \in \mathcal{E}(x), \text{ then } E \text{ is a neighborhood of } x \tag{3}$$

In fact let $x \in \mathbb{R} \setminus C$. Of course if $x = 0$ then (3) is true. If $x \in A \setminus C$ then $\mathcal{E}(x) = \{A\}$ and there exists an open interval $I \in \bigcup_{n=1}^{\infty} \mathcal{I}_{1,n}$ such that $x \in I$. Then $x \in I \subset A$, so A is a neighborhood of x. Let $I_1 = J_{1,1} \cup (1, \infty)$ and $I_n = J_{1,n} \cup J_{0,n-1}$ (for $n = 2, 3 \dots$). Denote $G = (-\infty, 0) \cup \bigcup_{n=1}^{\infty} I_n$. Then G is an open set and $G \subset \mathbb{R} \setminus A$. For every $x \in G$ we obtain that $\mathcal{E}(x) = \{\mathbb{R} \setminus A\}$ and $\mathbb{R} \setminus A$ is a neighborhood of x. If there exists n_0 such that $x \in (\frac{1}{n_0+1}, \frac{1}{n_0}) \setminus (C \cup A \cup G)$, then there exists an open interval $I \in \bigcup \mathcal{I}_{2,n_0}$ such that $x \in I$. Then $\mathcal{E}(x) = \{\mathbb{R} \setminus A\}$ and $I \subset \mathbb{R} \setminus A$, so $\mathbb{R} \setminus A$ is a neighborhood of x.

According to (3) we obtain that if $x \notin C$ and $E \in \mathcal{E}(x)$, then x is a point of bilateral accumulation of E. Suppose now, that $x \in C$. Then there exists n_0 such that $x \in (\frac{1}{n_0+1}, \frac{1}{n_0})$. If $x \in A$ then $\mathcal{E}(x) = \{A\}$ and x is a point of bilateral accumulation of $\bigcup \mathcal{I}_{1,n_0} \subset A$. Similarly if $x \in C \setminus A$, then $\mathcal{E}(x) = \{\mathbb{R} \setminus A\}$ and x is a point of bilateral accumulation of the set $\bigcup \mathcal{I}_{2,n_0} \subset \mathbb{R} \setminus A$. This proves that if $x \in \mathbb{R}$ and $E \in \mathcal{E}(x)$, then x is a point of bilateral accumulation of E.

Observe that C is nowhere dense. In fact let U be an open subset of $\mathbb{R}$. We can assume, that $U \cap (0, 1) \neq \emptyset$. Then there exists $n_0 \in \{1, 2, \dots\}$ such that $U \cap (\frac{1}{n_0+1}, \frac{1}{n_0}) \neq \emptyset$. The set C_{n_0} is nowhere dense so there exists an open nonempty set $V \subset U \cap (\frac{1}{n_0+1}, \frac{1}{n_0}) \setminus C_{n_0}$. Observe that $C \cap (\frac{1}{n_0+1}, \frac{1}{n_0}) = C_{n_0}$, so $V \subset U \setminus C$. According to (3) we have that if $h \in \mathcal{C}_{\mathcal{E}}$ and $x \in \mathbb{R} \setminus C$, then x is a continuity point of h. Thus $D_h \subset C$, so D_h is nowhere dense.

We show now that $\mathcal{C}_{\mathcal{E}} \setminus \mathcal{D}_{\mathcal{E}} \neq \emptyset$. In fact, let

$$f(x) = \begin{cases} 0 & if \quad x \in \mathbb{R} \setminus A, \\ \frac{1}{n} & if \quad x \in (\frac{1}{n+1}, \frac{1}{n}) \cap A, \quad n = 1, 2, 3 \ldots . \end{cases}$$

The function f is $\mathcal{E}$-continuous. Indeed, f is continuous (so $\mathcal{E}$-continuous) at 0. For every $n \in \{1, 2, \ldots\}$ the set $(\frac{1}{n+1}, \frac{1}{n}) \cap A$ is open on A, and $f \restriction A$ is constant on $(\frac{1}{n+1}, \frac{1}{n}) \cap A$. Thus (because $A \subset (0,1) \setminus \{\frac{1}{k} : k \in \mathbb{N}\}$) $f \restriction A$ is continuous. Finally observe that $f \restriction (\mathbb{R} \setminus A)$ is constant. Because for every $\sigma > 0$, $f([0, \sigma))$ is not connected then f is not $\mathcal{E}$-Darboux from the right side at 0.

Now we show, that $\mathcal{D}_{\mathcal{E}} \setminus \mathcal{C}_{\mathcal{E}} \neq \emptyset$. Let B be the set of all points of bilateral accumulation of C_1. Define:

$$g(x) = \begin{cases} 1 & if \ \ x \in B, \\ 0 & otherwise. \end{cases}$$

Observe that g is constant (with value 0) on A and on an open set $\bigcup \mathcal{I}_{2,1} \cup (\mathbb{R} \setminus [\frac{1}{2}, 1])$, thus g is $\mathcal{E}$-Darboux on $A \cup \bigcup \mathcal{I}_{2,1} \cup (\mathbb{R} \setminus [\frac{1}{2}, 1])$. It is enough to show that if $x \in C_1 \setminus A$, then g is $\mathcal{E}$-Darboux at x.

Let $x \in C_1 \setminus A$. We show that g is $\mathcal{E}$-Darboux from the right side at x. Let $E = \mathbb{R} \setminus A$, $\delta = \infty$. If $x \in B$, then $g(x) = 1$ and x is a point of bilateral accumulation of B. Similarly if $x \notin B$ then $g(x) = 0$ and (because x is a point of bilateral accumulation of $\bigcup \mathcal{I}_{2,n}$) x is a point of bilateral accumulation of $\mathbb{R} \setminus (A \cup B)$. Thus $g(x) \in L^+_E(g, x)$. Observe that $L^+_E(g, x) \subset \{0, 1\}$ and $g(\mathbb{R}) \cap (0, 1) = \emptyset$, so (D2) is true.

Let $S \subset E$ be a connected set. Because every point of B is a point of bilateral accumulation of A, we have that if $S \cap B \neq \emptyset$, then S is singleton, so $g(S)$ is connected. If $S \subset \mathbb{R} \setminus B$, then $g(S) = \{0\}$ so $g(S)$ is connected too. Consequently (D3) holds.

Analogously we can show, that for every $x \in C_1 \setminus A$, g is $\mathcal{E}$-Darboux from the left side at x. Thus $g \in \mathcal{D}_{\mathcal{E}}$.

Let $t \in B$. Then $L^+_E(g, t) = \{0, 1\}$ and the only path at t is E, so g is not $\mathcal{E}$-continuous at t (i.e. $g \notin \mathcal{C}_{\mathcal{E}}$). □

In [1] K.Banaszewski considered $\mathcal{E}$-continuity with respect to, so called, c-systems. We show now that then $\mathcal{E}$-continuity implies the $\mathcal{E}$-Darboux property.

Let $\mathcal{E}$ be a system of families of bilateral paths and $x \in \mathbb{R}$. We denote $\mathcal{E}^+(x) = \{E \cap [x, \infty); E \in \mathcal{E}(x)\}$ and $\mathcal{E}^-(x) = \{E \cap (-\infty, x]; E \in \mathcal{E}(x)\}$.

Definition 2. [1] Let $\mathcal{E}$ be a system of families of bilateral paths such that for any $x \in \mathbb{R}$, $E^+ \in \mathcal{E}^+(x)$ and $E^- \in \mathcal{E}^-(x)$ we have $E^+ \cup E^- \in \mathcal{E}(x)$.

We say that $\mathcal{E}$ is a bilateral σ-system if for every $x \in \mathbb{R}$, sequence $(x_n)_{n=1}^{\infty}$, such that $x_n \searrow x$, ($x_n \nearrow x$) and sequence of sets $(U_n)_{n=1}^{\infty}$, $(E_n)_{n=1}^{\infty}$, such that, for $n = 1, 2, \dots$, $U_n \in \mathcal{T}(x_n)$ and $E_n \in \mathcal{E}(x_n)$, there exists $E \in \mathcal{E}^+(x)$, ($E \in \mathcal{E}^-(x)$ respectively), such that $E \subset \{x\} \cup \bigcup_{n=1}^{\infty}(E_n \cap U_n)$.

We say that $\mathcal{E}$ is a c-system if it is a bilateral σ-system and for every $x \in \mathbb{R}$ and a Cantor set C such that x is a point of bilateral accumulation of C, we have $C \in \mathcal{E}(x)$.

Theorem 4. *Suppose that $\mathcal{E}$ is a c-system and $x \in \mathbb{R}$. If f is a function $\mathcal{E}$-continuous at x, then f is $\mathcal{E}$-Darboux at x.*

Proof. Let $x \in \mathbb{R}$ and f be $\mathcal{E}$-continuous at x. We show that f is $\mathcal{E}$-Darboux from the right side at x (analogously we can show, that f is $\mathcal{E}$-Darboux from the left side at x). Let $E \in \mathcal{E}(x)$ be such that $f \restriction E$ is continuous at x. We consider two cases:

1. There exists $\delta > 0$ such that $int(E \cap [x, x+\delta)) = \emptyset$.

Because $f \restriction E$ is continuous at x, so $L_E(f, x) = \{f(x)\}$. Thus (D1) and (D2) from Definition 1 are true. Observe that all components of $E \cap [x, x+\delta)$ are singletons so (D3) holds.

2. There exist sequences $(x_n)_{n=1}^{\infty}$ and $(\sigma_n)_{n=1}^{\infty}$ such that $x_n \searrow x$, $(x_n - \sigma_n, x_n + \sigma_n) \cap (x_m - \sigma_m, x_m + \sigma_m) = \emptyset$ (for $m \neq n$ $m, n \in \{1, 2 \dots\}$) and $\bigcup\limits_{n=1}^{\infty} (x_n - \sigma_n, x_n + \sigma_n) \subset E$.

Let $(C_n)_{n=1}^{\infty}$ be a sequence of a Cantor sets such that $C_n \subset (x_n - \sigma_n, x_n + \sigma_n)$ and x_n is a point of bilateral accumulation of C_n (for $n = 1, 2, \dots$). Let (for $n = 1, 2, \dots$) $C'_n = \{c \in \mathbb{R} : 2x - c \in C_n\}$. Then $C = \{x\} \cup \bigcup_{n=1}^{\infty}(C_n \cup C'_n)$ is a Cantor set and x is a point of bilateral accumulation of C. Because $\mathcal{E}$ is a c-system so $C \in \mathcal{E}(x)$. Let $\delta_1 = \infty$. Observe that $L^+_C(f, x) = \{f(x)\}$ and $int(C \cap [x, \infty)) = \emptyset$, so all components of $C \cap [x, \infty)$ are singletons. Thus for C and δ_1 conditions (D1), (D2), (D3) hold. $\square$

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