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DIVERGENCE IN MEASURE OF REARRANGED MULTIPLE ORTHOGONAL FOURIER SERIES

Abstract

Let $\{\varphi_n(x),\, n = 1, 2, \dots\}$ be an arbitrary complete orthonormal system (ONS) on the interval $I := [0, 1)$ that consists of a.e. bounded functions. Then there exists a rearrangement $\{\varphi_{\sigma_1(n)},\, n = 1, 2, \dots\}$ of the system $\{\varphi_n(x),\, n = 1, 2, \dots\}$ that has the following property: for arbitrary nonnegative, continuous and nondecreasing on $[0, \infty)$ function $\phi(u)$ such that $u\phi(u)$ is a convex function on $[0, \infty)$ and $\phi(u) = o(\ln u)$, $u \to \infty$, there exists a function $f \in L(I^2)$ such that $\int_{I^2} |f(x, y)|\, \phi(|f(x, y)|)\, dx\, dy < \infty$ and the sequence of the square partial sums of the Fourier series of f with respect to the double system $\{\varphi_{\sigma_1(m)}(x)\varphi_{\sigma_1(n)}(y),\, m, n \in \mathbb{N}\}$ on I^2 is essentially unbounded in measure on I^2.

1 Introduction.

In the theory of orthogonal series A. Olevskiĭ's fundamental method for investigating arbitrary complete ONS and bases in function spaces, based on some special properties of the Haar system, is well known ([4]-[6]). In particular, the following theorem holds ([4], p.60, see also [3], p. 294).

Theorem 1 (A. Olevskiĭ). *For any complete ONS $\{\varphi_l(x),\, l \in \mathbb{N}\}$ on I there exists a Haar-type system $\{\tilde{\chi}_l(x),\, l = 0, 1, 2, \dots\}$ and polynomials with respect to the system $\{\varphi_l(x),\, l \in \mathbb{N}\}$ with*

$$Q_m(x) = \sum_{l=k(m)+1}^{k(m+1)} a_l \varphi_l(x),\ \text{for } m \in \mathbb{N},\ 0 = k(0) < k(1) < k(2) < \dots, \tag{1}$$

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such that

$$\tilde{\chi}_m(x) = Q_m(x) + \gamma_m(x), \text{ for } m \in \mathbb{N} \tag{2}$$

$$\|\gamma_m\|_{L^2}^2 \leq 32^{-m-1}, \text{ for } m \in \mathbb{N}. \tag{3}$$

In the present paper we use A. Olevskiĭ's method to study convergence in measure of the Fourier series with respect to rearranged multiple complete ONS.

We start with the following definition. Let (X, Σ, ν) be σ-finite measurable space, $E \in \Sigma$ and $\nu(E) > 0$. Let also a sequence of measurable real-valued functions $\{f_n(x)\}_{n=1}^{\infty}$ be defined and a.e. finite on E. Then we say that the sequence $\{f_n(x)\}_{n=1}^{\infty}$ is essentially divergent in measure on E if for every $E_1 \subset E$, $E_1 \in \Sigma$ and $\nu(E_1) > 0$, the sequence is divergent in measure (that is, does not converge in measure to an a.e. finite and measurable function) on E_1. Let μ_N, $N \in \mathbb{N}$, denote Lebesgue measure in the Euclidean space R^N. If F is a Lebesgue measurable set in R^2, with $0 < \mu_2 F < \infty$, then let $L^0(F)$ denote the set of all Lebesgue measurable functions on $F \subset R^2$ that are finite a.e. on F.

A sequence $\{f_n(x, y),\ n \in \mathbb{N}\}$ of functions from $L^0(F)$ is called bounded in measure on F if for any $\epsilon > 0$ there is a constant $R_1 > 0$ such that $\mu_2\{(x, y) \in F : |f_n(x, y)| \geq R_1\} \leq \epsilon$ for any $n \in \mathbb{N}$. A sequence $\{f_n(x, y),$ $n \in \mathbb{N}\}$ of functions from $L^0(F)$ is called essentially unbounded in measure on F if for any Lebesgue measurable set $E \subset F$, $\mu_2 E > 0$, the sequence is not bounded in measure on E. It is clear that any sequence of measurable a.e. finite functions that is essentially unbounded in measure on a measurable set E is essentially divergent in measure on the same set.

We shall denote the set of all non-negative integers by $\mathbb{Z}_0$.

By a dyadic interval in $I := [0, 1)$ we shall mean an interval of the form

$$\Delta_n^{(k)} := [k2^{-n}, (k+1)2^{-n}), \text{ with } (0 \leq k < 2^n, n, k \in Z_0). \tag{4}$$

The Haar system $\{\chi_l(x),\ l \in \mathbb{Z}_0\}$ is defined as follows. Set $\chi_0(x) := 1$. For $n, k \in Z_0$ with $0 \leq k < 2^n$ define $\chi_l(x)$ on I by

$$\chi_{2^n+k}(x) := \chi_n^{(k)}(x) := \begin{cases} 2^{\frac{n}{2}}, & \text{if } x \in \Delta_{n+1}^{(2k)} \\ -2^{\frac{n}{2}}, & \text{if } x \in \Delta_{n+1}^{(2k+1)} \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Let $E_n^{(k)}$ $(0 \leq k < 2^n,\ n, k \in Z_0)$ be a family of measurable sets, where $E_n^{(k)} \subset I$, $\mu_1\{E_n^{(k)}\} = \frac{1}{2^n}$, $E_n^{(k)} \cap E_n^{(l)} = \emptyset$ if $k \neq l$, and $E_n^{(k)} = E_{n+1}^{(2k)} \cup E_{n+1}^{(2k+1)}$.

A Haar-type system $\{\tilde{\chi}_l(x),\, l \in \mathbb{Z}_0\}$ on I is defined as follows. Set $\tilde{\chi}_0(x) := 1$. For $n, k \in Z_0$ with $0 \le k < 2^n$ define $\tilde{\chi}_l(x)$ on I by

$$\tilde{\chi}_{2^n+k}(x) := \tilde{\chi}_n^{(k)}(x) := \begin{cases} 2^{\frac{n}{2}}, & \text{if } x \in E_{n+1}^{(2k)} \\ -2^{\frac{n}{2}}, & \text{if } x \in E_{n+1}^{(2k+1)} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\{\varphi_n(x),\, n \in \mathbb{N}\}$ be an arbitrary ONS on the interval I. Fourier coefficients of a function $f \in L(I)$ with respect to the system are denoted by $b_n^{(\varphi)}(f)$ and the partial sums by $S_N^{(\varphi)}(f, x)$. Fourier coefficients of a function $h \in L(I^2)$ in the system $\{\varphi_n(x)\varphi_m(y),\, n, m \in \mathbb{Z}_0\}$, $(x, y) \in I^2$, are denoted by $b_{n,m}^{(\varphi)}(h)$ and the rectangular partial sums by $S_{N,M}^{(\varphi)}(h, x, y)$. The partial sums $S_{N,N}^{(\varphi)}(h, x, y)$ are called square partial sums.

Let (X, ρ) be a metric space. It is said that a double sequence $\{x_{m,n}\}$ of elements of X converges by rectangles to an element $a \in X$ if for any number $\epsilon > 0$ there exists a number N_ϵ such that $\rho(x_{m,n}, a) < \epsilon$ whenever $m > N_\epsilon$ and $n > N_\epsilon$.

It is well-known that (see, for example [3], p. 71) the Fourier-Haar series of any function $f \in L(I)$ is unconditionally convergent (that is, it converges for every ordering of the terms) in measure on I. In integral classes of functions wider than $LLn^+L(I^2)$ there are no product bases that are unconditional with respect to the convergence in measure by rectangles [7].

For any $f \in LLn^+L(I^2)$ the series $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \epsilon_{i,j} a_{i,j}^{(\chi)}(f)\chi_i(x)\chi_j(y)$ converges by rectangles for any $\epsilon_{i,j} = \epsilon_i^{'}\epsilon_j^{''}$ with $\epsilon_i^{'}, \epsilon_j^{''} = +1$ or -1 in $L^p(I^2)$ metric for every $p \in (0, 1)$ (see the Remark 1 after Theorem 1 in [7]). It is known also that the double Fourier-Haar series of any Lebesgue integrable function on I^2 is convergent in the metric $L(I^2)$ by rectangles. In this paper we prove first the following theorem.

Theorem 2. *There exists a rearrangement* $\{\psi_n(x) := \chi_{\sigma(n)}(x),\, n \in \mathbb{Z}_0\}$ *of the one-dimensional Haar system that has the following property: for arbitrary nonnegative, continuous and nondecreasing on* $[0, \infty)$ *function* $\phi(u)$ *such that* $u\phi(u)$ *is a convex function on* $[0, \infty)$ *and*

$$\phi(u) = o(\ln u),\; u \to \infty, \tag{6}$$

there exists a function $g \in L(I^2)$ *such that* $\int_{I^2} \mid g(x,y) \mid \phi(\mid g(x,y) \mid)\, dx\, dy < \infty$ *and the sequence of the square partial sums of the Fourier series of* g *with respect to the double system* $\{\chi_{\sigma(m)}(x)\chi_{\sigma(n)}(y),\, m, n \in \mathbb{Z}_0\}$ *on* I^2 *is essentially unbounded in measure on* I^2 *.*

Using the A.Olevskiĭ's method we generalize this theorem in the general case, namely we prove that:

Theorem 3. *Let $\{\varphi_n(x),\, n \in \mathbb{N}\}$ be an arbitrary complete ONS on the interval I that consists of a.e. bounded functions. Then there exists a rearrangement $\{\varphi_{\sigma_1(n)},\, n \in \mathbb{N}\}$ of the system $\{\varphi_n(x),\, n \in \mathbb{N}\}$ that has the following property: for arbitrary nonnegative, continuous and nondecreasing on $[0,\infty)$ function $\phi(u)$ such that $u\phi(u)$ is a convex function on $[0,\infty)$ and $\phi(u) = o(\ln u)$, $u \to \infty$, there exists a function $f \in L(I^2)$ such that $\int_{I^2} \mid f(x,y) \mid \phi(\mid f(x,y) \mid)\, dx\, dy < \infty$ and the sequence of the square partial sums of the Fourier series of f with respect to the double system $\{\varphi_{\sigma_1(m)}(x)\varphi_{\sigma_1(n)}(y),\, m,n \in \mathbb{N}\}$ on I^2 is essentially unbounded in measure on I^2.*

Taking in account Tkebuchava's [7] positive result mentioned above, one can see easily that Theorems 2 and 3 are sharp: the condition (6) cannot be replaced by $\phi(u) = O(\ln u)$.

2 The Proof of Theorem 2.

For a number $h \in (0,1)$, I_h denotes the interval $[0, 1-h]$. For each pair of numbers $(\theta,\eta) \in I_h^2$ and a number $h \in (0,1)$, introduce the function of two variables (x,y) defined on I^2 by

$$\delta_{\theta,\eta,h}(x,y) := \begin{cases} h^{-2}, & \text{if } (x,y) \in [\theta,\theta+h]\times[\eta,\eta+h]; \\ 0, & \text{otherwise on } I^2. \end{cases} \tag{7}$$

Let $(x,y) \in I^2$ and $\Delta_0^{(i_0)}, \Delta_1^{(i_1)}, \ldots, \Delta_l^{(i_l)}, \ldots$ be the sequence of all dyadic intervals that contain x and let $\Delta_0^{(j_0)}, \Delta_1^{(j_1)}, \ldots, \Delta_l^{(j_l)}, \ldots$ be the sequence of all dyadic intervals that contain y.

We show that if $0 \le k \le N$, N is an arbitrary positive integer and

$$\theta \in \Delta_{2k}^{(i_{2k})} \setminus \Delta_{2k+2}^{(i_{2k+2})}, \tag{8}$$

then

$$2^{2k+1} \ge \Big| \sum_{p=0}^{N} \sum_{l=0}^{2^{2p}-1} \chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta) \Big| \ge 2^{2k-1}. \tag{9}$$

Indeed, when $p > k$, then (see (4), (5), (8)) $\theta \notin \Delta_{2p}^{(i_{2p})}$ and therefore

$\chi_{2^{2p}+i_{2p}}(\theta)=0$ and when $p\le k$, then $\theta\in\Delta_{2p}^{(i_{2p})}$ and therefore $|\ \chi_{2^{2p}+i_{2p}}(\theta)\ |=\sqrt{2^{2p}}$. Thus

$$\Big|\sum_{p=0}^{N}\sum_{l=0}^{2^{2p}-1}\chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta)\Big|=\Big|\sum_{p=0}^{k}\chi_{2^{2p}+i_{2p}}(x)\chi_{2^{2p}+i_{2p}}(\theta)\Big|$$
$$\ge|\chi_{2^{2k}+i_{2k}}(x)\chi_{2^{2k}+i_{2k}}(\theta)|-\Big|\sum_{p=0}^{k-1}\chi_{2^{2p}+i_{2p}}(x)\chi_{2^{2p}+i_{2p}}(\theta)\Big|$$
$$\ge 2^{2k}-\sum_{p=0}^{k-1}2^{2p}\ge 2^{2k-1}.$$

On the other hand,

$$\Big|\sum_{p=0}^{N}\sum_{l=0}^{2^{2p}-1}\chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta)\Big|=\Big|\sum_{p=0}^{k}\chi_{2^{2p}+i_{2p}}(x)\chi_{2^{2p}+i_{2p}}(\theta)\Big|$$
$$\le\sum_{p=0}^{k}2^{2p}\le 2^{2k+1}.$$

Consequently the estimate (9) is proved.

We introduce ordered sets. Let

$$D_j:=\{2^j,2^j+1,\ \ldots,\ 2^{j+1}-1\},\ j\in\mathbb{Z}_0 \tag{10}$$

be the j^{th} block where the natural order is preserved. We introduce ordered packets of blocks according to the following list

$$B_i:=\{D_{20i-20},\ D_{20i-18},\ \ldots,\ D_{20i-2}\},\ i\in\mathbb{N}. \tag{11}$$

Now we define the rearrangement σ of Z_0 according to the following list

$$0,B_1,D_1,B_2,D_3,B_3,D_5,\ldots,B_i,D_{2i-1},\ldots. \tag{12}$$

Let n be an arbitrary positive integer. Introduce numbers

$$p_n:=1+\sum_{i=1}^{n}(card(B_i)+card(D_{2i-1})),\ n\in\mathbb{N}, \tag{13}$$

where $card(F)$ denotes the number of elements of a finite set F.

Set

$$\psi_j(x) := \chi_{\sigma(j)}(x),\ j \in \mathbb{Z}_0. \tag{14}$$

Now we have (see (4), (5), (10)-(14)) for all $(x,\theta) \in I^2$ and $n \in \mathbb{N}$,

$$\begin{aligned}\Big|\sum_{j=0}^{p_n} \psi_j(x)\psi_j(\theta)\Big| &\geq \Big|\sum_{p=0}^{10n-1}\sum_{l=0}^{2^{2p}-1} \chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta)\Big| - 1 \\ &\qquad - \Big|\sum_{p=0}^{n-1}\sum_{m=0}^{2^{2p+1}-1} \chi_{2^{2p+1}+m}(x)\chi_{2^{2p+1}+m}(\theta)\Big| \\ &\geq \Big|\sum_{p=0}^{10n-1}\sum_{l=0}^{2^{2p}-1} \chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta)\Big| - 1 - 2^{2n}.\end{aligned} \tag{15}$$

Introduce the set $A_n := A_n(x,y)$ to be

$$\bigcup_{\frac{49N_n}{100} \leq k \leq \frac{50N_n}{100}} ((\Delta_{2k}^{(i_{2k})} \backslash \Delta_{2k+2}^{(i_{2k+2})}) \times (\Delta_{2N_n-2k}^{(j_{2N_n-2k})} \setminus \Delta_{2N_n-2k+2}^{(j_{2N_n-2k+2})})), \tag{16}$$

$$\text{where } N_n := 10n-1. \tag{17}$$

We have from (4)

$$\mu_2 A_n \geq \sum_{\frac{49N_n}{100} \leq k \leq \frac{50N_n}{100}} 2^{-2k-2} 2^{2N_n+2k-2} \geq \frac{10n-1}{100}\frac{1}{2^{20n+2}}. \tag{18}$$

Let $(\theta,\eta) \in A_n$. Then there exists a positive integer k, $\frac{49N_n}{100} \leq k \leq \frac{50N_n}{100}$, such that (see (9), (16)) $2^{2\frac{50N_n}{100}+1} \geq 2^{2k+1} \geq |\sum_{p=0}^{N_n}\sum_{l=0}^{2^{2p}-1} \chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta)| \geq 2^{2k-1}$ and $2^{2\frac{51N_n}{100}+1} \geq 2^{2N_n-2k+1} \geq |\sum_{p=0}^{N_n}\sum_{l=0}^{2^{2p}-1} \chi_{2^{2p}+l}(y)\chi_{2^{2p}+l}(\eta)| \geq$

2^{2N_n-2k-1}. Then, in light of (15) and (17), we conclude that for $n>2$,

$$\begin{aligned}
&\Big|\sum_{j=0}^{p_n}\psi_j(x)\psi_j(\theta)\Big|\Big|\sum_{j=0}^{p_n}\psi_j(y)\psi_j(\eta)\Big| \\
&\quad\geq \Big|\sum_{p=0}^{10n-1}\sum_{l=0}^{2^{2p}-1}\chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta)\Big|\Big|\sum_{p=0}^{10n-1}\sum_{l=0}^{2^{2p}-1}\chi_{2^{2p}+l}(y)\chi_{2^{2p}+l}(\eta)\Big| \\
&\qquad\qquad - 2^{2n+1}\Big|\sum_{p=0}^{10n-1}\sum_{l=0}^{2^{2p}-1}\chi_{2^{2p}+l}(x)\chi_{2^{2p}+l}(\theta)\Big| \qquad (19) \\
&\qquad\qquad - 2^{2n+1}\Big|\sum_{p=0}^{10n-1}\sum_{l=0}^{2^{2p}-1}\chi_{2^{2p}+l}(y)\chi_{2^{2p}+l}(\eta)\Big| + 2^{4n+2} \\
&\quad\geq \frac{1}{4}2^{20n-2} - 2^{17n+2} \geq \frac{2^{20n-2}}{8}.
\end{aligned}$$

Thus (see (18)) for all $(x,y)\in I^2$, we have

$$\mu_2\{(\theta,\eta)\in I^2 : \text{inequality (19) holds}\} \geq \frac{10n-1}{2^{20n+2}100}. \tag{20}$$

Introduce the functions for $n\in\mathbb{N}$ and $(x,y,\theta,\eta)\in I^4$

$$K_n(x,y,\theta,\eta) := \sum_{i=0}^{p_n}\psi_i(x)\psi_i(\theta)\sum_{j=0}^{p_n}\psi_j(y)\psi_j(\eta). \tag{21}$$

Introduce the set

$$\Theta_n := \bigcup_{i=1}^{2^{20n}}\bigcup_{j=1}^{2^{20n}}[\frac{i-1}{2^{20n}},\frac{i}{2^{20n}}-\frac{1}{2^{100n}})\times[\frac{j-1}{2^{20n}},\frac{j}{2^{20n}}-\frac{1}{2^{100n}}). \tag{22}$$

It is clear that

$$\mu_2\Theta_n \geq 1-\frac{2}{2^{80n}}. \tag{23}$$

Let a set $E\subset I^2$, $\mu_2E>0$, be arbitrary. We introduce numbers

$$h_n := 2^{-200n}, \tag{24}$$

$$\xi_n := \xi_n(E) = \frac{9}{10}\frac{(10n-1)\mu_2E}{2400}. \tag{25}$$

From (10)-(13) it follows that

$$\max_{0\le j\le p_n} \sigma(j) \le 2^{20n-1}. \tag{26}$$

Now let $(x,y,\theta,\eta) \in I^2 \times \Theta_n$. Then we have (see (4), (5), (7), (22))

$$\sum_{i=0}^{p_n}\sum_{j=0}^{p_n} a^{\psi}_{i,j}((\delta_{\theta,\eta,h_n}))\psi_i(x)\psi_j(y) = \sum_{i=0}^{p_n}\psi_i(x)\psi_i(\theta)\sum_{j=0}^{p_n}\psi_j(y)\psi_j(\eta). \tag{27}$$

It is clear that (see (20), (21), (23))

$$\mu_2\{(\theta,\eta)\in\Theta_n : |K_n(x,y,\theta,\eta)| \ge \frac{1}{8}2^{20n-2}\} \ge \frac{9}{10}\frac{10n-1}{2^{20n+2}100} \tag{28}$$

for $(x,y) \in I^2$. We shall show that for an arbitrary given set $E \subset I^2$, $\mu_2 E > 0$, and for each integer $n > r_0$ (r_0 is a positive constant depending on E) there exist (all depending on the set E) a positive integer $q(n)$ and the following finite sequences: a sequence of disjoint measurable sets $\{B_i^{(n)}\}_{i=1}^{q(n)}$, $B_i^{(n)} \subset E$, $i = 1,2,\ldots,q(n)$; a sequence of pairs of numbers $\{(\theta_i^{(n)},\eta_i^{(n)})\}_{i=1}^{q(n)}$, $(\theta_i^{(n)},\eta_i^{(n)}) \in \Theta_n$, $i = 1,2,\ldots,q(n)$, such that (see (24), (25))

$$|S^{\psi}_{p_n,p_n}(\delta_{\theta_i^{(n)},\eta_i^{(n)},h_n},x,y)| \ge \frac{2^{20n-2}}{8},\ \forall (x,y)\in B_i^{(n)},\ i=1,2,\ldots,q(n) \tag{29}$$

$$\mu_2\{\cup_{i=1}^{q(n)} B_i^{(n)}\} \ge \frac{\mu_2 E}{6} > 0, \tag{30}$$

$$\mu_2\{B_i^{(n)}\} \ge \frac{\xi_n}{2^{20n}},\ \forall i=1,2,\ldots,q(n). \tag{31}$$

Set

$$A_1^{(n)} := \{(x,y,\theta,\eta)\in E\times\Theta_n : |K_n(x,y,\theta,\eta)| \ge \frac{1}{8}2^{20n-2}\} \tag{32}$$

We have for all $(x,y)\in E$ (see (28)), $\int_{\Theta_n}\chi_{A_1^{(n)}}(x,y,\theta,\eta)\,d\theta\,d\eta \ge \frac{9}{10}\frac{10n-1}{2^{20n+2}100}$, where $\chi_{A_1^{(n)}}(x,y,\theta,\eta)$ is the characteristic function of the set $A_1^{(n)}$. Using Fubini's theorem we conclude that there exists a pair of numbers $(\theta_1^{(n)},\eta_1^{(n)}) \in \Theta_n$ such that $\mu_2\{(x,y)\in E : (x,y,\theta_1^{(n)},\eta_1^{(n)}) \in A_1^{(n)}\} \ge \frac{9}{10}\frac{\mu_2 E}{6}\frac{10n-1}{2^{20n+2}100}$.

Now we let $B_1^{(n)} := \{(x,y)\in E : (x,y,\theta_1^{(n)},\eta_1^{(n)}) \in A_1^{(n)}\}$. From (21), (25), (31), (29), (27), and (32), we see that the first step in the construction is complete. We now assume that the p-th step of the construction is complete. If it happens that $\mu_2\{\cup_{i=1}^{p} B_i^{(n)}\} \ge \frac{\mu_2 E}{6} > 0$, then the construction is complete.

Suppose, on the contrary, that $\mu_2\{\cup_{i=1}^p B_i^{(n)}\} < \frac{\mu_2 E}{6}$. Define $A_{p+1}^{(n)}$ to be

$$\{(x,y,\theta,\eta) \in (E \setminus \cup_{i=1}^p B_i^{(n)}) \times \Theta_n : |K_n(x,y,\theta,\eta)| \geq \frac{2^{20n-2}}{8}\}. \tag{33}$$

Using Fubini's theorem we conclude that there exists a pair of numbers $(\theta_{p+1}^{(n)}, \eta_{p+1}^{(n)}) \in \Theta_n$ such that $\mu_2\{(x,y) \in E \setminus \cup_{i=1}^p B_i^{(n)} : (x,y,\theta_{p+1}^{(n)}, \eta_{p+1}^{(n)}) \in A_{p+1}^{(n)}\} \geq \frac{9}{10}\frac{\mu_2 E}{6}\frac{10n-1}{2^{20n+2}100}$. Now we set $B_{p+1}^{(n)} :=\{(x,y) \in E \setminus \cup_{i=1}^p B_i^{(n)} : (x,y,\theta_{p+1}^{(n)}, \eta_{p+1}^{(n)}) \in A_{p+1}^{(n)}\}$. From (21), (25), (31), (29), (27), and (33), we see that the $(p+1)^{st}$ step in the construction is complete.

It follows now from the construction (see (25), (31)) that after the p^{th} step we have $\mu_2(\cup_{i=1}^p B_i^{(n)}) = \sum_{i=1}^p \mu_2 B_i^{(n)} \geq \frac{9p}{10}\frac{\mu_2 E}{6}\frac{10n-1}{2^{20n+2}100}$ and consequently, this inequality cannot hold for sufficiently large numbers p. We can conclude now that the construction terminates at some finite step $q(n)$.

Define (see (7)) $f_i^{(n)}(x,y)$ to be

$$\delta_{\theta_i^{(n)},\eta_i^{(n)},h_n}(x,y) = \begin{cases} h_n^{-2}, & (x,y) \in [\theta_i^{(n)}, \theta_i^{(n)} + h_n] \times [\eta_i^{(n)}, \eta_i^{(n)} + h_n]; \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

Introduce the functions

$$\Phi_n^{(t)}(x,y) := \Phi_n^{(t)}(E;x,y) = \sum_{i=1}^{q(n)} \frac{\xi_n}{2^{20n}} f_i^{(n)}(x,y) r_i(t),\ (x,y,t) \in I^3 \tag{35}$$

for $n > r_0$, where $\{r_i(t),\ i \in \mathbb{N}\}$ is the Rademacher system. Consider the set

$$H_n = \bigcup_{i=1}^{q(n)} B_i^{(n)}. \tag{36}$$

Let (x,y) be any point from H_n. Then (see (29), (34)) for some positive integer $i_0 = i_0(x,y)$, $1 \leq i_0 \leq q(n)$, we have

$$|S_{p_n,p_n}^{\psi}(f_{i_0}^{(n)}, x, y)| \geq \frac{1}{8} 2^{20n-2}. \tag{37}$$

Clearly (see (35), (34)), $\Phi_n^{(t)}(x,y) \in L(I^2)$ for each fixed $t \in [0,1)$. Further it follows from (35) that for any $t \in [0,1)$

$$\begin{aligned} S_{p_n,p_n}^{\psi}(\Phi_n^{(t)}, x, y) = r_{i_0}(t) \frac{\xi_n}{2^{20n}} S_{p_n,p_n}^{\psi}(f_{i_0}^{(n)}, x, y) \\ + \sum_{i \neq i_0} r_i(t) S_{p_n,p_n}^{\psi}(f_i^{(n)}, x, y) \frac{\xi_n}{2^{20n}}. \end{aligned} \tag{38}$$

The following easily verifiable fact is well known (see for example [1], p.10): Let $\sum_{i=1}^{m}\omega_i r_i(t)$ be an arbitrary polynomial with real coefficients in the Rademacher system and i_0 a fixed integer, $1 \le i_0 \le m$. Then $\mu_1\{t \in [0,1) : \omega_{i_0} r_{i_0}(t) \sum_{i\neq i_0}\omega_i r_i(t) \ge 0\} \ge \frac{1}{2}$.

Introduce the set

$$Q := \{(x,y,t) \in H_n \times [0,1) : |S^{\psi}_{p_n,p_n}(\Phi_n^{(t)},x,y)| \ge \frac{1}{32}\xi_n\}. \tag{39}$$

According to (37)-(39) we conclude that for all $(x,y) \in H_n$ we have the inequality $\int_0^1 \chi_Q(x,y,t)\,dt \ge \frac{1}{2}$, where $\chi_Q(x,y,t)$ is the characteristic function of Q. Consequently (see (30), (36), (39)), there exists a number $t_0 \in [0,1)$ such that $\mu_2\{(x,y) \in H_n :| S^{\psi}_{p_n,p_n}(\Phi_n^{(t_0)},x,y) |\ge \frac{1}{32}\xi_n\} \ge \frac{\mu_2 E}{12}$. We observe that (see (31), (30), (34), and (35))

$$\int_0^1\int_0^1 |\Phi_n^{(t_0)}(x,y)|\,dx\,dy \le \sum_{i=1}^{q(n)}\frac{\xi_n}{2^{20n}} \le \sum_{i=1}^{q(n)}\mu_2\{B_i^{(n)}\} \le 1. \tag{40}$$

Introduce the notations

$$G_n(x,y) := G_n(E;x,y) := \Phi_n^{(t_0)}(x,y),\ (x,y) \in I^2 \tag{41}$$

$$\psi(u) := u\phi(u),\ \ u \in [0,\infty). \tag{42}$$

Taking account of an assumption on ψ we see that if a number $C \in [0,1)$, then for any $x > 0$,

$$\psi(Cx) \le C\psi(x),\ x > 0. \tag{43}$$

We note that (see (34), (25), and (6)) there exists a sequence of positive integers $\{\epsilon_n := \epsilon_n(E)\ n > r_0\}$ such that $\lim_{n\to\infty}\epsilon_n = 0$ and for all integers $n > r_0$,

$$\max_{1\le i\le q(n)}\int_{I^2} |f_i^{(n)}(x,y)|\phi(|f_i^{(n)}(x,y)|)\,dx\,dy \le \epsilon_n\xi_n. \tag{44}$$

The function ψ in (42) is a non-decreasing convex function. From (43), (41), and (35) we have

$$\int_0^1\int_0^1 \psi(|G_n(x,y)|)\,dx\,dy \le \int_0^1\int_0^1 \frac{\xi_n}{2^{20n}}\sum_{i=1}^{q(n)}\psi(|f_i^{(n)}(x,y)|)\,dx\,dy \tag{45}$$
$$\le \xi_n\epsilon_n.$$

Let S_n denote a finite one-dimensional sequence of all intervals $\Delta_k^{(i,j)} := \Delta_k^{(i)} \times \Delta_k^{(j)}$, where $i, j, = 0, 1, 2, \dots, 2^k - 1$, $k = 0, 1, 2, \dots, n$. According to the following scheme $S_0, S_1, S_2, \dots, S_k, \dots$, we obtain a sequence of sets

$$E_1, E_2, \dots, E_k, \dots, \tag{46}$$

that has the following properties:

i. For each positive integer k there exists a triple of non negative integers (n, i, j), where $0 \leq i, j \leq 2^n - 1$, such that $E_k = \Delta_n^{(i,j)}$ and
ii. For each triple of non negative integers (n, i, j), where $i, j = 0, 1, 2, \dots, 2^n - 1$, there exists an increasing sequence of positive integers $\{l_p = l_p(n, i, j)\}_{p=1}^{\infty}$ such that for every $p \in \mathbb{N}$,

$$E_{l_p} = \Delta_n^{(i,j)}. \tag{47}$$

Now for the sequence of sets in (46) we will construct by induction an increasing sequence of positive integers $\{n_j\}_{j=1}^{\infty}$, sequence of positive numbers $\{\delta_j\}_{j=2}^{\infty}$ such that for all $j \in \mathbb{N}$ we have

$$\frac{a_1}{\nu_1} \leq \frac{1}{2} \text{ and } \epsilon_{n_1}(E_1) \leq \frac{1}{4}, \tag{48}$$

$$\frac{a_{j+1}}{\nu_{j+1}} \leq \frac{1}{2}\frac{a_j}{\nu_j}, \tag{49}$$

$$\epsilon_{n_{j+1}}(E_{j+1}) \leq \frac{1}{4}\epsilon_{n_j}(E_j), \tag{50}$$

$$n_{j+1} > n_j, \tag{51}$$

$$\mu_2\{(x, y) \in E_j : |S^{\psi}_{p_{n_j}, p_{n_j}}(\Psi_j, x, y)| \geq \frac{1}{32}\nu_j\} \geq \frac{\mu_2 E_j}{12} > 0, \tag{52}$$

$$2\frac{a_{j+1}}{\nu_{j+1}} 2^{40 n_j} p_{n_j}^2 < \frac{a_j}{96} \text{ and} \tag{53}$$

$$\mu_2\{(x, y) \in I^2 : |S^{\psi}_{p_{n_{j+1}}, p_{n_{j+1}}}(\alpha_j, x, y)| \geq \delta_{j+1}\} \leq \frac{1}{36}\mu_2 E_{j+1}, \tag{54}$$

$$\text{where } \alpha_j(x.y) := \sum_{i=1}^{j} \frac{a_i}{\nu_i}\Psi_i(x, y) \tag{55}$$

$$\text{and } \frac{a_{j+1}}{96} > \max(j + 1, \delta_{j+1}), \tag{56}$$

where

$$a_j := \min\Big(\frac{1}{\sqrt{\epsilon_{n_j}(E_j)}}, \sqrt{\xi n_j(E_j)}\Big), \tag{57}$$

$$\nu_j := \xi_{n_j}(E_j), \text{ and} \tag{58}$$

$$\Psi_j(x,y) := G_{n_j}(E_j; x, y). \tag{59}$$

The constructions of the integer n_1 and the number δ_2 are contained by the description of the general $(k+1)^{st}$ step of the induction. Let the numbers $\{n_j\}_{j=1}^k$, $\{\delta_j\}_{j=2}^k$ be already defined so that they satisfy (48)-(59). According to the inequalities of Chebyshev and Parseval we obtain for all positive numbers δ and for all positive integers n, $\mu_2\{(x,y) \in I^2 : |S_{n,n}^{\psi}(\alpha_k, x, y)| \geq \delta\} \leq \frac{\|\alpha_k\|^2_{L^2(I^2)}}{\delta^2}$ and consequently one can choose a positive number δ_{k+1} such that for all positive integers n we have (see (57), (59), (34), (35), and (41)), $\mu_2\{(x,y) \in I^2 : |S_{n,n}^{\psi}(\alpha_k, x, y)| \geq \delta_{k+1}\} \leq \frac{1}{36}\mu_2 E_{k+1}$. Now we can obtain an integer n_{k+1} large enough so that the relations (48)-(51) and (53)-(56) are satisfied for $j = k$ and the relations (52) and (57)-(59) are satisfied for $j = k+1$. The construction of sequences $\{n_j\}_{j=1}^\infty$, $\{\delta_j\}_{j=2}^\infty$ is now completed.

Introduce the functions defined on I^2 by

$$g(x,y) := \sum_{i=1}^{\infty} \frac{a_i}{\nu_i} \Psi_i(x,y), \tag{60}$$

$$\beta_k(x,y) := \sum_{i=k+1}^{\infty} \frac{a_i}{\nu_i} \Psi_i(x,y). \tag{61}$$

It is obvious that (see (48), (49), (40), (41), (59), (60) and (61)) for any $k \in \mathbb{N}$,

$$\int_0^1 \int_0^1 |\beta_k(x,y)|\, dx\, dy \leq \sum_{i=k+1}^{\infty} \frac{a_i}{\nu_i} \leq 2\frac{a_{k+1}}{\nu_{k+1}} \text{ and} \tag{62}$$

$$\int_0^1 \int_0^1 |g(x,y)|\, dx\, dy \leq \sum_{i=1}^{\infty} \frac{a_i}{\nu_i} < 1.$$

Now let $E_0 \subset I^2$ be an arbitrary Lebesgue measurable set, $\mu_2 E_0 > 0$. It is clear that there exist a triple of non negative integers (n_0, i_0, j_0), where $0 \leq i_0, j_0 \leq 2^n - 1$, and an increasing sequence of positive integers $\{k_q\}_{q=1}^\infty$ such that (see (47))

$$\mu_2\{E_0 \cap \Delta_{n_0}^{(i_0,j_0)}\} \geq \frac{215}{216}\mu_2 \Delta_{n_0}^{(i_0,j_0)} \text{ and} \tag{63}$$

$$E_{k_q} = \Delta_{n_0}^{(i_0,j_0)} \tag{64}$$

for all $q \in \mathbb{N}$. From (55), (60), and (61) we have for all $q = 2, 3, \ldots$, $g(x,y) = \alpha_{k_q-1}(x,y) + \frac{a_{k_q}}{\nu_{k_q}} \Psi_{k_q}(x,y) + \beta_{k_q}(x,y)$. Set for all $q \in \mathbb{N}$

$$r_q := p_{n_{k_q}}. \tag{65}$$

Then it is obvious that for all $q \in \mathbb{N}$, $\mu_2\{(x,y) \in E_{k_q} : |S^{\psi}_{r_q,r_q}(\frac{a_{k_q}}{\nu_{k_q}} \Psi_{k_q}, x, y)| \geq \frac{1}{32} a_{k_q}\} \leq \mu_2\{(x,y) \in E_{k_q} : |S^{\psi}_{r_q,r_q}(\alpha_{k_q-1}, x, y)| \geq \frac{1}{96} a_{k_q}\} + \mu_2\{(x,y) \in E_{k_q} : |S^{\psi}_{r_q,r_q}(\beta_{k_q}, x, y)| \geq \frac{1}{96} a_{k_q}\} + \mu_2\{(x,y) \in E_{k_q} : |S^{\psi}_{r_q,r_q}(g, x, y)| \geq \frac{1}{96} a_{k_q}\}$. Using (4), (5), (53), (26), (62), and (65) we obtain that for all $(x,y) \in I^2$ and any $q \in \mathbb{N}$, $|S^{\psi}_{r_q,r_q}(\beta_{k_q}, x, y)| \leq \frac{2a_{k_q+1}}{\nu_{k_q+1}} 2^{40 n_{k_q}} r_q^2 < \frac{a_{k_q}}{96}$. Consequently (see (52), (54), (56), (64), and (65)) we conclude that for any $q \in \mathbb{N}$, $\mu_2\{(x,y) \in \Delta^{(i_0,j_0)}_{n_0} : |S^{\psi}_{r_q,r_q}(g, x, y)| \geq \frac{1}{96} a_{k_q}\} \geq \frac{1}{36} \mu_2 \Delta^{(i_0,j_0)}_{n_0}$ and consequently (see ((63)) for any $q \in \mathbb{N}$, $\mu_2\{(x,y) \in E_0 \cap \Delta^{(i_0,j_0)}_{n_0} : |S^{\psi}_{r_q,r_q}(g, x, y)| \geq \frac{1}{96} a_{k_q}\} \geq \frac{5}{216} \mu_2 \Delta^{(i_0,j_0)}_{n_0}$. Obviously (see (51) and (56)) the sequence of the square partial sums of Fourier series of g is not bounded in measure on E_0.

From (55) and (60) we obtain that $\psi(|g(x,y)|) = \lim_{k\to\infty} \psi(|\alpha_k(x,y)|)$ a.e. We note that for a.e. (x,y) and $k \in \mathbb{N}$ (see (55), (57), (58), and (43)) $\psi(|\alpha_k(x,y)|) \leq \psi(\sum_{i=1}^k \frac{a_i}{\nu_i} |\Psi_i(x,y)|) \leq \sum_{i=1}^k \frac{a_i}{\nu_i} \psi(|\Psi_i(x,y)|)$. We see that the sequence of functions $\{\psi(\sum_{i=1}^k \frac{a_i}{\nu_i} |\Psi_i(x,y)|, k \in \mathbb{N}\}$ is increasing and for all $k \in \mathbb{N}$ (see (57)-(59), (50), (41), (44), and (45)),

$$\int_0^1 \int_0^1 \psi(\sum_{i=1}^k \frac{a_i}{\nu_i} |\Psi_i(x,y)|)\, dx\, dy \leq \sum_{i=1}^k \nu_i \epsilon_{n_i}(E_i) \frac{a_i}{\nu_i} \leq \sum_{i=1}^k \sqrt{\epsilon_{n_i}(E_i)} \leq 1.$$

It follows now that the limit of the sequence $\{\psi(\sum_{i=1}^k \frac{a_i}{\nu_i} |\Psi_i(x,y)|), k \in \mathbb{N}\}$ is integrable on I^2 and this limit is a majorant of the sequence $\{\psi(|\alpha_k(x,y)|), k \in \mathbb{N}\}$. Consequently, the limit of the latter, that is the function $\psi(|g(x,y)|)$ is also integrable on I^2. Theorem 2 is proven.

3 The Proof of Theorem 3.

We consider the rearrangement $\{\sigma(j)\}$ and the function g from Theorem 2. It is clear that $g(x,y) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} b^{(\chi)}_{i,j}(g) \chi_i(x) \chi_j(y)$ where $b^{(\chi)}_{i,j}(g)$ are the Fourier coefficients of a function g in the double Haar system and the series converges in the metric $L(I^2)$ by rectangles.

Let $\{\tilde{\chi}_l(x), l \in \mathbb{Z}_0\}$ be the Haar-type system that corresponds to the given complete ONS $\{\varphi_l(x), l \in \mathbb{Z}_0\}$ in Theorem 1. Then according to the properties of the Haar type system ([4], pp. 60-62) we can find a function $f(x,y)$ that has the same distribution as $g(x,y)$ and is such that

$$f(x,y)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{i,j}^{(\chi)}(g)\tilde{\chi}_i(x)\tilde{\chi}_j(y) \tag{66}$$

in the metric $L(I^2)$ by rectangles.

We introduce ordered blocks of integers (see (1))

$$H_j:=\{k(\sigma(j))+1,k(\sigma(j))+2,\dots,k(\sigma(j)+1)\},\quad j\in\mathbb{Z}_0, \tag{67}$$

where the natural order is preserved. We define the rearrangement $\{\sigma_1(j), j\in\mathbb{N}\}$ of the set of all positive integers, according to the following list of the blocks

$$H_0,H_1,\dots,H_j,H_{j+1},\dots. \tag{68}$$

Set

$$\Phi_n(x):=\varphi_{\sigma_1(n)}(x),\ n\in\mathbb{N}, \tag{69}$$

$$q(r):=\sum_{p=0}^{r}card(H_p),\ r\in\mathbb{Z}_0,\ q(-1):=0. \tag{70}$$

G.A. Karagulyan proved that ([2], p. 47.) if $h(x,y)\in L(I^2)$, then all the series (see (1), (2)) $\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}b_{l,s}^{(\chi)}(h)Q_l(x)Q_s(y)$, $\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}b_{l,s}^{(\chi)}(h)\tilde{\chi}_l(x)\gamma_s(y)$ and $\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}b_{l,s}^{(\chi)}(h)\gamma_l(x)\tilde{\chi}_s(y)$, and $\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}b_{l,s}^{(\chi)}(h)\gamma_l(x)\gamma_s(y)$ are convergent in the metric of $L(I^2)$ by rectangles, while the last three series are convergent by rectangles a.e. on I^2.

Let $q(p_1-1)<n\le q(p_1)$ and $q(p_2-1)<m\le q(p_2)$ with $p_1,p_2\in\mathbb{N}$. Then we have (see (1), (2), (66)-(69))

$$c_{n,m}^{(\Phi)}(f)=b_{\sigma(p_1),\sigma(p_2)}^{(\chi)}(g)d_{\sigma_1(n)}d_{\sigma_1(m)}+\sum_{j=0}^{\infty}b_{\sigma(p_1),j}^{(\chi)}(g)d_{\sigma_1(n)}c_m^{(j)} \tag{71}$$

$$+\sum_{i=0}^{\infty}b_{i,\sigma(p_2)}^{(\chi)}(g)d_{\sigma_1(m)}c_n^{(i)}+\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}b_{i,j}^{(\chi)}(g)c_n^{(i)}c_m^{(j)},$$

where $c_n^{(i)} = \int_0^1 \gamma_i(x)\Phi_n(x)\,dx$. From (1), (2), (66)-(71) we have

$$\begin{aligned}
S^{\Phi}_{q(N),q(N)}(f;x,y) &= \sum_{p_1=0}^{N}\sum_{p_2=0}^{N}\sum_{(n=q(p_1-1)+1)}^{q(p_1)}\sum_{m=q(p_2-1)+1}^{q(p_2)} c_{n,m}^{(\Phi)}(f)\Phi_n(x)\Phi_m(y)\\
&= \sum_{p_1=0}^{N}\sum_{p_2=0}^{N} b^{(\chi)}_{\sigma(p_1),\sigma(p_2)}(g)Q_{\sigma(p_1)}(x)Q_{\sigma(p_2)}(y)\\
&\qquad + \sum_{j=0}^{\infty}\sum_{p_1=0}^{N} b^{(\chi)}_{\sigma(p_1),j}(g)Q_{\sigma(p_1)}(x)S^{\Phi}_{q(N)}(\gamma_j,y) \qquad (72)\\
&\qquad + \sum_{i=0}^{\infty}\sum_{p_2=0}^{N} b^{(\chi)}_{i,\sigma(p_2)}(g)Q_{\sigma(p_2)}(y)S^{\Phi}_{q(N)}(\gamma_i,x)\\
&\qquad + \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} b^{(\chi)}_{i,j}(g)S^{\Phi}_{q(N)}(\gamma_i,x)S^{\Phi}_{q(N)}(\gamma_j,y)\\
&:= J^{(1)}(N,x,y) + J^{(2)}(N,x,y)\\
&\qquad + J^{(3)}(N,x,y) + J^{(4)}(N,x,y).
\end{aligned}$$

We have according to (2)

$$\begin{aligned}
J^{(1)}(N,x,y) &= \sum_{p_1=0}^{N}\sum_{p_2=0}^{N} b^{(\chi)}_{\sigma(p_1),\sigma(p_2)}(g)\tilde{\chi}_{\sigma(p_1)}(x)\tilde{\chi}_{\sigma(p_2)}(y)\\
&\qquad - \sum_{p_1=0}^{N}\sum_{p_2=0}^{N} b^{(\chi)}_{\sigma(p_1),\sigma(p_2)}(g)\gamma_{\sigma(p_1)}(x)\tilde{\chi}_{\sigma(p_2)}(y)\\
&\qquad - \sum_{p_1=0}^{N}\sum_{p_2=0}^{N} b^{(\chi)}_{\sigma(p_1),\sigma(p_2)}(g)\tilde{\chi}_{\sigma(p_1)}(x)\gamma_{\sigma(p_2)}(y) \qquad (73)\\
&\qquad + \sum_{p_1=0}^{N}\sum_{p_2=0}^{N} b^{(\chi)}_{\sigma(p_1),\sigma(p_2)}(g)\gamma_{\sigma(p_1)}(x)\gamma_{\sigma(p_2)}(y)\\
&:= I^{(1)}(N,x,y) - I^{(2)}(N,x,y) - I^{(3)}(N,x,y) + I^{(4)}(N,x,y).
\end{aligned}$$

Set for $l \in \mathbb{N}$

$$F_l^{(1)}(t) := \int_0^1 g(s,t)\chi_l(s)\,ds. \qquad (74)$$

Then we have for each $N \in \mathbb{N}$, $I^{(2)}(N,x,y) = \sum_{p_2=0}^{N} b_{\sigma(p_2)}^{(\chi)} \big(\sum_{p_1=0}^{N} F_{\sigma(p_1)}^{(1)} \gamma_{\sigma(p_1)}(x) \big) \tilde{\chi}_{\sigma(p_2)}(y)$. It is known that (see [4], pp. 60-62) for any Haar type system $\{\tilde{\chi}_l(x)$ with $l \in \mathbb{N}\}$, there exists a Lebesgue measurable function $u(x) : I \to I$ such that $\tilde{\chi}_l(x) = \chi_l(u(x))$ a.e. for all $l \in \mathbb{N}$ and for every Lebesgue measurable set $G \subset I$ the set $u^{-1}(G)$ is Lebesgue measurable and $\mu_1\{u^{-1}(G)\} = \mu_1 G$. Besides, for each n there exists a measure preserving mapping $u_n(x) : I \to I$ that is one-to-one a.e. and is such that $\tilde{\chi}_l(x) = \chi_l(u_n(x))$ a.e. for all $l = 0, 1, \ldots, n$.

According to the weak (1,1) type property of rearrangements of the one-dimensional Haar system (see [3], p. 71), we have for any $x \in I$, $\mu_1\{y \in I : |I^{(2)}(N,x,y)| \geq R\} \leq \frac{C}{R} \| \sum_{p_1=0}^{N} F_{\sigma(p_1)}^{(1)} \gamma_{\sigma(p_1)}(x) \|_{L(I)}$, where C is an absolute constant.

On the other hand we have (see (74), (5), (3)) $\| \sum_{p_1=0}^{N} F_{\sigma(p_1)}^{(1)} \gamma_{\sigma(p_1)} \|_{L(I^2)} \leq \sum_{p_1=0}^{N} \|F_{\sigma(p_1)}^{(1)}\|_{L(I)} \|\gamma_{\sigma(p_1)}\|_{L^2(I)} \leq \sum_{p_1=0}^{\infty} \frac{1}{32^{0,5\sigma(p_1)+0,5}} \sqrt{\sigma(p_1)+1} \|g\|_{L(I)} := C_1 < \infty$, where the constant C_1 does not depend on N. Thus we obtain for every $N \in \mathbb{N}$ and $R > 0$, $\mu_2\{(x,y) \in I^2 : |I^{(2)}(N,x,y)| \geq R\} \leq \frac{C_2}{R}$, where the constant C_2 does not depend on N. Therefore the sequence $\{I^{(2)}(N,x,y), N \in \mathbb{Z}_0\}$ is bounded in measure on I^2. The sequence $\{I^{(3)}(N,x,y), N \in \mathbb{Z}_0\}$ is bounded in measure on I^2 for similar reasons.

From (5), (3), and (73) we obtain

$$\begin{aligned} \|I^{(4)}(N,x,y)\|_{L(I^2)} &\leq \sum_{p_1=0}^{N} \sum_{p_2=0}^{N} |b_{\sigma(p_1),\sigma(p_2)}^{(\chi)}(g)| \|\gamma_{\sigma(p_1)}(x)\|_{L^2(I)} \|\gamma_{\sigma(p_2)}(y)\|_{L^2(I)} \\ &\leq \sum_{p_1=0}^{\infty} \sum_{p_2=0}^{\infty} \frac{\sqrt{\sigma(p_1)+1}}{32^{0,5\sigma(p_1)+0,5}} \frac{\sqrt{\sigma(p_2)+1}}{32^{0,5\sigma(p_2)+0,5}} \|g\|_{L(I)} \\ &:= C_3 < \infty. \end{aligned}$$

Consequently the sequence $\{I^{(4)}(N,x,y)\}$ is bounded in measure on I^2. We note that (2) and (72) imply

$$\begin{aligned} J^{(2)}(N,x,y) = &\sum_{j=0}^{\infty} \sum_{p_1=0}^{N} b_{\sigma(p_1),j}^{(\chi)}(g) \tilde{\chi}_{\sigma(p_1)}(x) S_{q(N)}^{\Phi}(\gamma_j, y) \\ &- \sum_{j=0}^{\infty} \sum_{p_1=0}^{N} b_{\sigma(p_1),j}^{(\chi)}(g) \gamma_{\sigma(p_1)}(x) S_{q(N)}^{\Phi}(\gamma_j, y) \\ := &\, J^{(2,1)}(N,x,y) - J^{(2,2)}(N,x,y). \end{aligned} \tag{75}$$

Set for $j \in \mathbb{Z}_0$ and $s \in I$

$$F_j(s) := \int_0^1 g(s,t)\chi_j(t)\,dt. \tag{76}$$

Now we will prove boundedness in measure on I^2 of the sequence $\{J^{(2,1)}(N,x,y)\}$. We have for a.e. fixed $y \in I$:

$$\begin{aligned}\sum_{j=0}^{\infty} b^{(\chi)}_{\sigma(p_1),j}(g) S^{\Phi}_{q(N)}(\gamma_j,y) &= \sum_{j=0}^{\infty} b^{(\chi)}_{\sigma(p_1)}(F_j) S^{\Phi}_{q(N)}(\gamma_j,y)\\ &= \sum_{j=0}^{\infty} b^{(\chi)}_{\sigma(p_1)}(F_j S^{\Phi}_{q(N)}(\gamma_j,y)).\end{aligned}$$

Introduce the function defined on I^2, $G_N(s,y) := \sum_{j=0}^{\infty} F_j(s) S^{\Phi}_{q(N)}(\gamma_j,y)$. The series $\sum_{j=0}^{\infty} |F_j(s) S^{\Phi}_{q(N)}(\gamma_j,y)|$ converges on I^2 to an integrable function according to Levy's theorem because (see (76), (3), (5))

$$\begin{aligned}\sum_{j=0}^{\infty}\int_0^1\int_0^1 |F_j(s) S^{\Phi}_{q(N)}(\gamma_j,y)|\,ds\,dy &\le \sum_{j=0}^{\infty} \|g\|_{L(I^2)} \frac{\sqrt{j+1}}{32^{0,5j+0,5}} \\ &:= C_5 < \infty.\end{aligned} \tag{77}$$

Here we used also Bessel's inequality. Consequently, the function $G_N(s,y)$ is integrable on I^2. We note also that for a.e. fixed $y \in I$ the series $\sum_{j=0}^{\infty} F_j(s)$ $S^{\Phi}_{q(N)}(\gamma_j,y)$ converges in metric $L(I)$ as a function series in the variable s because the series $\sum_{j=0}^{\infty} \int_0^1 |F_j(s)|\,ds|S^{\Phi}_{q(N)}(\gamma_j,y)|$ converges for a.e. y according to Levy's theorem (see (77)).

Now it is obvious for a.e. fixed $y \in I$ that $J^{(2,1)}(N,x,y) = \sum_{p_1=0}^{N} b^{(\chi)}_{\sigma(p_1)}$ $\big(\sum_{j=0}^{\infty} F_j S^{\Phi}_{q(N)}(\gamma_j,y)\big)\tilde{\chi}_{\sigma(p_1)}(x)$. According to the weak (1,1) type property of rearrangements of the one-dimensional Haar system (see [3], p.771) and (77) we have

$$\begin{aligned}&\mu_2\{(x,y) \in I^2 : |J^{(2,1)}(N,x,y)| \ge R\}\\ &\qquad \le \frac{C_4}{R}\Big\|\sum_{j=0}^{\infty} F_j S^{\Phi}_{q(N)}(\gamma_j,y))\Big\|_{L(I^2)} \le \frac{1}{R}C_6,\end{aligned}$$

where the constants C_4 and C_6 do not depend on N. Therefore, the sequence $\{J^{(2,1)}(N,x,y), N \in \mathbb{N}\}$ is bounded in measure on I^2.

Now we will prove boundedness in measure on I^2 of the sequence $\{J^{(4)}(N,x,y)\}$. We note that (see (72), (3), (5))

$$\|J^{(4)}(N,x,y)\|_{L(I^2)} \le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\int_0^1\int_0^1 |b_{i,j}^{(\chi)}(g)S_{q(N)}^{\Phi}(\gamma_i,x)S_{q(N)}^{\Phi}(\gamma_j,y)|\,dz\,dy$$
$$\le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{32^{0,5i+0,5}}\sqrt{i+1}\frac{1}{32^{0,5j+0,5}}\sqrt{j+1}\|g\|_{L(I^2)}$$
$$:= C_7 < \infty,$$

where C_7 does not depend on N.

The proof of boundedness in measure on I^2 of the sequence $\{J^{(2,2)}(N,x,y)\}$ (see (75)) is similar to that of the sequence $\{J^{(4)}(N,x,y)\}$.

Consequently, (see (75)) the sequence $\{J^{(2)}(N,x,y),\ N\in\mathbb{Z}_0\}$ is bounded in measure on I^2, as is the sequence $\{J^{(3)}(N,x,y),\ N\in\mathbb{Z}_0\}$ for similar reasons. According to Theorem 2 and the properties of Haar-type systems it follows that the sequence $\{I^{(1)}(N,x,y)\}$ (see (73)) is not bounded in measure on E_0. Taking account of (73), (72), and (75), we see that Theorem 3 has been proven.

References

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