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WEAK ALMOST ITERABILITY

Abstract

We investigate continuous selfmappings of a real compact interval, which are, in a sense, close to iterable ones, that is embeddable into continuous iteration semigroup. This is a continuation of a research initiated by W. Jarczyk. We present some necessary and sufficient conditions of weak almost iterability and $M-$weak almost iterability. Classes of functions investigated here are generalizations of a class of almost iterable functions introduced by W. Jarczyk. This refers to the problem posed by E. Jen.

1 Introduction.

In this paper we restrict our attention to the continuous selfmappings of a real, compact interval X. We follow M. C. Zdun [7] in posing a definition of a continuous iteration semigroup and iterability. Namely, function $F\colon (0,\infty)\times X\to X$ will be called a *continuous iteration semigroup* if it is continuous and satisfies the *translation equation*

$$F(t,F(s,x)) = F(s+t,x), \qquad s,t\in(0,\infty), x\in X.$$

A continuous function $f\colon X\to X$ which is embeddable into a continuous iteration semigroup is said to be *iterable.* So, for an iterable function f and a

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suitable continuous iteration semigroup F we have satisfied the initial condition

$$F(1,x)=f(x), \qquad x\in X,$$

and, according to the translation equation,

$$F(n,x)=f^n(x), \qquad x\in X,\ n\in\mathbb{N}.$$

This means that iterable functions are those for which the discrete process $(f^n(x))_{n\in\mathbb{N}}$, generated by f, has a version with continuous time. Necessary and sufficient conditions for embeddability of self-mappings of a real compact interval into continuous iteration semigroup were found by M. C. Zdun.

Theorem 1.1 (M. C. Zdun, [7])**.** *A continuous function $f\colon X\to X$ is iterable if and only if there exist points $x_1,x_2\in X$, $x_1\le x_2$, and a continuous function $e\colon X\to[x_1,x_2]$ fulfilling the conditions:*

(i) $x_1\le f(x_1)\le f(x)\le f(x_2)\le x_2, \qquad x\in X,$

(ii) *the function $f_{|[x_1,x_2]}$ increases and every interval, where it is constant, contains a fixed point of f,*

(iii) $e(x)=x, \qquad x\in[x_1,x_2],$

(iv) $f\circ e=f$.

Let us emphasize the following features of iterable functions.

Corollary 1.2. *If $f\colon X\to X$ is an iterable function and a,b are two consecutive fixed points of f, then $f([a,b])=[a,b]$ and, moreover, $\lim_{n\to\infty}f^n(x)=a$, respectively b, for every $x\in(a,b)$ provided $f(x)<x$, respectively $f(x)>x$, for $x\in(a,b)$. Iterable functions have no periodic points of order* 2.

Let us introduce some notation which will be used in what follows. Given $f\colon X\to X$, we write $\mathrm{Per}(f,k)$ for the set of all periodic points of f of order k:

$$\mathrm{Per}(f,k):=\{x\in X:\ f^k(x)=x \text{ and } f^l(x)\neq x \text{ for } l<k\};$$

we denote by a_f and b_f the smallest and the greatest fixed point of f, respectively. Moreover, x^f stands for the $\lim_{n\to\infty}f^n(x)$, if the limit exists.

Inspired by Problem (3.1.12), posed by E. Jen in [6], W. Jarczyk [2] presented a slightly more general concept called *almost iterability*.

A function $f\colon X \to X$ is called *almost iterable* if there exists an iterable function $g\colon X \to X$ such that

$$\lim_{n\to\infty} (f^n(x) - g^n(x)) = 0 \tag{1}$$

for every $x \in X$ and the convergence is uniform on every component of the set $[a_f, b_f] \setminus \mathrm{Per}(f,1)$.

Notice that it is of no use to assume uniform convergence on the whole interval X, since it would imply that $g \equiv f$ (cf. [2, Lemma 3]). On the other hand, by omitting the assumption of uniform convergence between fixed points of f we enlarge the examined class of function, which is desired, since the class of almost iterable functions is rather narrow. That will lead us to introducing the class of weak almost iterable, and further, M- weak almost iterable functions, which are the subject of this paper.

We will need the following characterization of almost iterable function.

Theorem 1.3. (W. Jarczyk; [2, Theorems 1 and 2]) *A continuous function $f\colon X \to X$ is almost iterable if and only if the function $f_{|[a_f,b_f]}$ increases and every interval, where it is constant, contains a fixed point of f, and* $\mathrm{Per}(f,2) = \emptyset$.

A few other approaches to "near iterability" can be found in [3] and [4].

2 Weak Almost Iterability.

As was previously announced, we define *weak almost iterable* functions, as those continuous selfmapping $f\colon X \to X$ for which there exists an iterable function $g\colon X \to X$ such that (1) holds for every $x \in X$ We start with providing the sufficient and necessary condition of weak almost iterability.

Theorem 2.1. *Let $f\colon X \to X$ be a continuous function. Then f is weak almost iterable if and only if* $\mathrm{Per}(f,2) = \emptyset$ *and there exist points $a_i, b_i \in$* $\mathrm{Per}(f,1)$, *$i \in I$, such that $[a_f, b_f] = \bigcup_{i\in I}[a_i, b_i]$ and for every $i \in I$ one of the following possibilities holds:*

(i) *$a_i = b_i$;*

(ii) *$(a_i, b_i) \cap \mathrm{Per}(f,1) = \emptyset$, $f([a_i, b_i]) = [a_i, b_i]$;*

(iii) *$(a_i, b_i) \cap \mathrm{Per}(f,1) = \{c_i\}$, $x < f(x) < b_i$, for $x \in (a_i, c_i)$, $a_i < f(x) < x$ for $x \in (c_i, b_i)$;*

(iv) *$(a_i, b_i) \cap \mathrm{Per}(f,1) = \emptyset$, $b_i = b_f$, $f(x) > x$ for $x \in (a_i, b_i)$;*

(v) $(a_i, b_i) \cap \mathrm{Per}(f,1) = \emptyset$, $a_i = a_f$, $f(x) < x$ *for* $x \in (a_i, b_i)$.

Before we pass to the proof, let us remind one more result, which is due to Šarkovskii.

Theorem 2.2. (Šarkovskii, [5, Theorems 1 and 2]; see also [1, Ch.VI, Proposition 1]) *Let* $f\colon X \to X$ *be a continuous function with no periodic points of order 2. Then every trajectory is bimonotonic, that is* $f(x) > x$ *implies* $f^n(x) > x$ *for every* $n \in \mathbb{N}$ *and the implication remains valid if we replace* $>$ *by any of* $<, \leq, \geq, =$*; and convergent, with an appropriate relation between* x^f *and* x, $>, <, \leq, \geq, =$, *respectively.*

PROOF OF THEOREM 2.1. First assume that $\mathrm{Per}(f,2) = \emptyset$ and the interval $[a_f, b_f]$ is a union of appropriate intervals $[a_i, b_i]$. Let $h\colon X \to X$ be a continuous function such that h strictly increases on the interval $[a_f, b_f]$ and satisfies the conditions:
$h(x) > x$, if $x \in [a_f, b_f]$ and $f(x) > x$,
$h(x) < x$, if $x \in [a_f, b_f]$ and $f(x) < x$,
$h(x) = x$, if $x \in [a_f, b_f]$ and $f(x) = x$,
and $h(x) = f(x)$ for every $x \in X \setminus [a_f, b_f]$.
According to the Theorem 1.3 such a function h is almost iterable, so we can choose an iterable function $g\colon X \to X$, such that $x^h = x^g$ for every $x \in X$. It is easy to notice, using Theorem 2.2, that $x^f = x^h$ for every $x \in X$.

To see the necessary condition, let us choose an iterable function $g\colon X \to X$ such that (1) holds for every $x \in X$. Of course, $\mathrm{Per}(f,2) = \emptyset$ and $\mathrm{Per}(f,1) = \mathrm{Per}(g,1)$. Let us notice also, that

$$(f((a,b)) \setminus [a,b]) \cap \mathrm{Per}(f,1) = \emptyset, \tag{2}$$

for every two consecutive fixed points a, b of f, since for every $x \in (a,b)$ we have $x^g \in \{a,b\}$. Let $a, b \in \mathrm{Per}(f,1)$, $a < b$ and $(a,b) \cap \mathrm{Per}(f,1) = \emptyset$. If $f([a,b]) = [a,b]$, then we have (ii). Assume then, that $f([a,b]) \setminus [a,b] \neq \emptyset$ and, for instance, $f(x) > x$ for $x \in (a,b)$. By (2) we infer that either $b = b_f$ and (iv) holds, or there is a point $c \in \mathrm{Per}(f,1)$ such that $(b,c) \cap \mathrm{Per}(f,1) = \emptyset$ and $f(x) < c$ for $x \in (a,b)$. Considering the latter possibility we conclude that $f(x) > a$ for $x \in (b,c)$ (again from (2) with b and c instead of a and b, respectively). Theorem 2.2 let us deduce that $f(x) < x$ for $x \in (b,c)$. Putting $a_i = a$, $b_i = c$ and $c_i = b$ we get (iii). If $a \in \mathrm{Per}(f,1)$ is both-hand side accumulation point of the set $\mathrm{Per}(f,1)$ it is enough to put $a = a_i = b_i$ to get (i). This ends the proof. $\square$

3 M-weak Almost Iterable Functions.

In order to extend the class of weak almost iterable functions we will consider pointwise convergence in the condition (1) only for x's from some large set. More precisely, from a dense set, as this approach includes both sets large in a sense of Lebesgue measure and in a sense of category.

Put $\mathcal{M} := \{M \subset X;\ \mathrm{cl}(X \setminus M) = X\} = \{M \subset X;\ \mathrm{int}\, M = \emptyset\}$. Let $M \in \mathcal{M}$. Function $f\colon X \to X$ is called *M-weak almost iterable* if there exists an iterable function $g\colon X \to X$ such that (1) holds for every $x \in X \setminus M$. We will give the necessary condition of M- weak almost iterability under the assumption that f^2 is not turbulent. Let us remind that the continuous map f is said to be *turbulent* if there exist compact intervals J and K with at most one common point such that

$$J \cup K \subset f(J) \cap f(K).$$

Actually, the assumption f^2 is not turbulent is much weaker than $\mathrm{Per}(f, 2) = \emptyset$, the former says that f cannot have periodic points of odd orders, while the latter implies that the only periodic points of f are fixed points. Anyway, we have the following:

Theorem 3.1. (cf. [1, Chapter VI, Proposition 3]) *If f^2 is not turbulent, then every convergent trajectory is bimonotonic, that is $f(x) > x$ implies $f^n(x) > x$ for every $n \in \mathbb{N}$ and the implication remains valid if we replace $>$ by any of $<, \leq, \geq, =$.*

We start with the following

Lemma 3.2. *Let $f\colon X \to X$ and $g\colon X \to X$ be continuous functions such that f^2 and g^2 are not turbulent. If there exists a set $M \in \mathcal{M}$ such that* (1) *holds for every $x \in X \setminus M$, then*

$$f(x) > x \implies g(x) \geq x,$$

$$f(x) < x \implies g(x) \leq x$$

for every $x \in X$.

Proof. Assume that $f(x_0) > x_0 > g(x_0)$ for an $x_0 \in X$. Then the same inequalities holds for x's from a neighbourhood U_{x_0} of x_0 and for $x \in U_{x_0} \setminus M$, due to Theorem 3.1, we have $x^f > x > x^g$, which contradicts the condition (1). $\square$

Now we differentiate some types of fixed points. Namely, let $x \in \mathrm{Per}(f,1)$, then

$$x \in \alpha(f) \Leftrightarrow \bigvee_{\delta>0} \Big(\bigwedge_{y\in(x-\delta,x)} y \le f(y) \le x \ \text{ or } \bigwedge_{y\in(x,x+\delta)} x \le f(y) \le y \Big);$$

$$x \in \beta(f) \Leftrightarrow \bigvee_{\delta>0} \Big(\Big(\bigwedge_{y\in(x-\delta,x)} f(y) \le y \ \text{ and } \bigwedge_{y\in(x,x+\delta)} f(y) \ge y \Big) \text{ or } \Big(\bigwedge_{y\in(x-\delta,x)} f(y) \ge y \ \text{ and } \bigwedge_{y\in(x,x+\delta)} f(y) \le y \Big) \Big);$$

$$x \in \gamma(f) \Leftrightarrow \ x \notin \alpha(f) \text{ and } \bigvee_{\delta>0} \Big(\bigwedge_{y\in(x-\delta,x+\delta)} f(y) \ge y \ \text{ or } \bigwedge_{y\in(x-\delta,x+\delta)} f(y) \le y \Big).$$

Next, lemmas establish some results concerning defined classes of fixed points.

Lemma 3.3. *For a continuous function $f\colon X \to X$ we have*

$$\mathrm{Per}(f,1) = \alpha(f) \cup \mathrm{cl}\,\beta(f) \cup \gamma(f).$$

PROOF. Fix $x \in \mathrm{Per}(f,1)$. Observe that either there exists a $\delta > 0$ such that the sign of $f - \mathrm{id}$ is constant in the left-hand side neighbourhood and right-hand side neighbourhood of x and either

$$(f(y) - y)(f(z) - z) \ge 0, \qquad y \in (x-\delta, x),\ z \in (x, x+\delta),$$

or

$$(f(y) - y)(f(z) - z) \le 0, \qquad y \in (x-\delta, x),\ z \in (x, x+\delta);$$

or such a δ does not exist. We conclude that either $x \in \gamma(f) \cup \alpha(f)$, or $x \in \beta(f)$; or $x \in \mathrm{cl}\,\beta(f)$, respectively. $\square$

Lemma 3.4. *Let $f, g\colon X \to X$ be continuous functions such that f^2 and g^2 are not turbulent. If there exists a set $M \in \mathcal{M}$ such that* (1) *holds for every $x \in X \setminus M$, then*

$$\mathrm{cl}(\alpha(f) \cup \beta(f)) \subset \mathrm{Per}(g,1).$$

Moreover, if g is an iterable function, then

$$\gamma(f) \cap \mathrm{Per}(g,1) \subset \{a_g, b_g\} \tag{3}$$

and

$$\operatorname{Per}(g,1) \setminus \{a_g, b_g\} \subset \operatorname{Per}(f,1). \tag{4}$$

Proof. Suppose that $x \in \alpha(f)$. Without loss of generality we can assume that $y \le f(y) \le x$ for y's from some left-hand side neighbourhood L_x of x. Making use of density of the set $X \setminus M$ we can choose a sequence $(x_n)_{n\in\mathbb{N}}$ of points of $(X \setminus M) \cap L_x$ convergent to x. We have

$$x_n \le x_n^f \le x, \qquad n \in \mathbb{N},$$

which yields that also $\operatorname{Per}(g,1) \ni x_n^g = x_n^f \to x$ as $n \to \infty$. Since $\operatorname{Per}(g,1)$ is closed we get $x \in \operatorname{Per}(g,1)$.

Now, fix an arbitrary $x \in \beta(f)$. Suppose that $g(x) \neq x$. There is no loss of generality in assuming that $g(x) < x$. Under this assumption we have $g(y) < y$ for y's close to x and, by Lemma 3.2, $f(y) \le y$ for such y's. Thus, in fact, $x \in \alpha(f)$, and by what we have already proved, $\alpha(f) \subset \operatorname{Per}(g,1)$. Therefore, assumption $g(x) \neq x$ leads to contradiction, hence $\beta(f) \subset \operatorname{Per}(g,1)$. We have also $\operatorname{cl}(\alpha(f) \cup \beta(f)) \subset \operatorname{Per}(g,1)$, as $\operatorname{Per}(g,1)$ is closed.

Finally, to show the assertion (3), assume that g is iterable and fix an arbitrary $x \in \gamma(f) \cap \operatorname{Per}(g,1)$. Suppose that $f(y) \ge y$ in a neighbourhood of x. Choose $\varepsilon > 0$ such that for every $y \in (x-\varepsilon, x)$ condition $x < f(y)$ implies $f(y) \le f^2(y)$. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence of points of $(X \setminus M) \cap (x-\varepsilon, x)$ convergent to x such that $f(y_n) > x$. We have

$$y_n^g = y_n^f \ge f(y_n) > x.$$

If $x \neq a_g$, then $y_n \in (a_g, x)$, for n large enough, whence $y_n^g \le x$ which is impossible. The third assertion follows from $\operatorname{Per}(g,1) \setminus \{a_g, b_g\} \subset \alpha(g) \cup \operatorname{cl}\beta(g)$ and the already proved part of this Lemma. The proof is completed. □

As we can see from the above Lemma, the fixed points from $\operatorname{cl}(\alpha(f) \cup \beta(f))$ have good properties, as they are also the fixed points of a function g for which (1) holds. Next Remark shows that the differentiable functions have only such fixed points.

Remark 3.1. *Let $f\colon X \to X$ be a continuous function differentiable at least at fixed points. Then*

$$\operatorname{Per}(f,1) \subset \alpha(f) \cup \operatorname{cl}\beta(f).$$

PROOF. In view of Lemma 3.3 it suffices to show that $\gamma(f) = \emptyset$. Assume, on the contrary, that there is an $x \in \gamma(f)$ with, for instance, $f(y) \geq y$ in a neighbourhood of x. Then, obviously, $f'_+(x) \geq 1$ and, since in every left-hand side neighbourhood of x there is y with $f(y) > x$, we have $f'_-(x) \leq 0$, contrary to assumption of differentiability at x. □

4 Necessary and Sufficient Conditions of M-weak Almost Iterability.

The aim of this section is to provide necessary and sufficient conditions under which f is M-weak almost iterable. To shorten the notation, let $\mathcal{R}(f)$ stands for the union $\mathrm{cl}(\alpha(f) \cup \beta(f)) \cup \{a_f, b_f\}$ and $\mathcal{S}(f) := \mathrm{Per}(f,1) \setminus \mathcal{R}(f)$. Moreover, if $x \in \mathrm{Per}(f,1)$, then the set $\mathcal{A}_f(x) := \{y \in X;\ y^f = x\}$ is called the basin of attraction of a fixed point x. For $a, b \in \mathcal{R}(f)$, $a \leq b$, we distinguish the following cases:

(i) $a = b$;

(ii) $(a,b) \cap \mathcal{R}(f) = \emptyset$, $f([a,b]) = [a,b]$;

(iii) $(a,b) \cap \mathcal{R}(f) = \{c\}$, $x \leq f(x) \leq b$, for $x \in (a,c)$, $a \leq f(x) \leq x$ for $x \in (c,b)$;

(iv) $(a,b) \cap \mathcal{R}(f) = \emptyset$, $b = b_f$, $f(x) \geq x$ for $x \in (a,b)$;

(v) $(a,b) \cap \mathcal{R}(f) = \emptyset$, $a = a_f$, $f(x) \leq x$ for $x \in (a,b)$.

We start with necessary conditions.

Theorem 4.1. *Let $f\colon X \to X$ be a continuous function such that f^2 is not turbulent and $a_f, b_f \in \mathrm{cl}(\alpha(f) \cup \beta(f))$. If there exist an iterable function $g\colon X \to X$ and a set $M \in \mathcal{M}$ such that* (1) *holds for every $x \in X \setminus M$, then there exist $a_i, b_i, c_i \in \mathcal{R}(f)$, where I is a set of indexes, such that $[a_f, b_f] = \bigcup_{i \in I} [a_i, b_i]$ and for every $i \in I$ one of the cases* (i)–(v) *holds. Moreover, $f^{-n}(\{a_i, b_i\}) \cap (a_i, b_i) \subset M$ for every $n \in \mathbb{N}$ and $i \in I$ such that case* (iii) *holds, $\mathcal{A}_f(x) \subset M$ for $x \in \mathcal{S}(f)$, and $x \in M$ if the limit $\lim_{n\to\infty} f^n(x)$ does not exist.*

PROOF. Note that here, actually, we have $\mathcal{R}(f) = \mathrm{cl}(\alpha(f) \cup \beta(f))$ and $\mathcal{S}(f) \subset \gamma(f)$. According to Lemma 3.4, $\mathcal{R}(f) \subset \mathrm{Per}(g,1)$. Notice that $(a,b) \cap \mathcal{R}(f) = \emptyset$ for some $a, b \in X$ forces $f(x) \geq x$ for every $x \in (a,b)$ or $f(x) \leq x$ for every

$x \in (a, b)$. Indeed, function $f - \mathrm{id}$ can change the sign only at points from $\mathcal{R}(f)$. The rest of the proof will be divided into a few steps.

(**I**) *Let $a, b \in \mathcal{R}(f)$, $a < b$ and $(a, b) \cap \mathcal{R}(f) = \emptyset$. If $f(x) \geq x$ for every $x \in (a, b)$, then $g(x) > x$ and $x^g = b$ for $x \in (a, b)$, and if $f(x) \leq x$ for every $x \in (a, b)$, then $g(x) < x$ and $x^g = a$ for $x \in (a, b)$.*

We will show the first implication. Assume that $g(\hat{x}) < \hat{x}$ for an $\hat{x} \in (a, b)$. Then $g(x) < x$ for x from a subinterval $(\bar{a}, \bar{b})$ of (a, b). By Lemma 3.2 we infer that $f(x) \leq x$ for $x \in (\bar{a}, \bar{b})$, whence $f(x) = x$ for $x \in (\bar{a}, \bar{b})$. This contradicts $(a, b) \cap \mathcal{R}(f) = \emptyset$. Now assume that $g(\hat{x}) = \hat{x}$ for an $\hat{x} \in (a, b)$. We have $\hat{x} \notin \{a_g, b_g\}$, since $a, b \in \mathrm{Per}(g, 1)$. By (4), $\hat{x}$ is also a fixed point of f. Since $(a, b) \cap \mathcal{R}(f) = \emptyset$, we get $\hat{x} \in \gamma(f)$. Now, using again Lemma 3.4, assertion (3), we conclude that $\hat{x} \in \{a_g, b_g\}$, a contradiction.

(**II**) *Let $a, b, c, d \in \mathcal{R}(f)$ be such that $a < b$, $c < d$, $[(a, b) \cup (c, d)] \cap \mathcal{R}(f) = \emptyset$, $(a, b) \neq (c, d)$ and $f((a, b)) \cap (c, d) \neq \emptyset$. Then*

$$(f(x) - x)(f(y) - y) \leq 0, \qquad x \in (a, b), y \in (c, d).$$

To see this, assume, for instance, that $f(x) \geq x$ for $x \in (a, b)$. If $f(x) \geq x$ for $x \in (c, d)$, then $f^2(x) \geq f(x)$ for an $x \in (a, b) \cap (X \setminus M)$, which implies, in view of Theorem 3.1, $x^g = x^f \geq f(x) > c$ for this x. But $x^g = b$ from (**I**), a contradiction.

(**III**) *Let $a, b \in \mathcal{R}(f)$, $a < b$ and $(a, b) \cap \mathcal{R}(f) = \emptyset$. Then*

$$\operatorname{int} \mathrm{Per}(f, 1) \cap (f((a, b)) \setminus [a, b]) = \emptyset.$$

Assume, on the contrary, that the assertion does not hold. Then there is an subinterval $I_0 \subset [a, b]$, such that $f(x) \in \mathrm{Per}(f, 1) \setminus [a, b]$ for $x \in I_0$, whence, by density of the set $X \setminus M$, we have $I_0 \cap (X \setminus M) \neq \emptyset$ and therefore there exists $x \in (a, b) \cap (X \setminus M)$ such that $f(x) \in \mathrm{Per}(f, 1) \setminus [a, b]$, which implies $x^f = f(x) \notin [a, b]$ whereas $x^g \in [a, b]$.

(**IV**) *Let $a, b, c, d \in \mathcal{R}(f)$ be such that $a < b$, $c < d$ and $[(a, b) \cup (c, d)] \cap \mathcal{R}(f) = \emptyset$. If $f((a, b)) \cap (c, d) \neq \emptyset$, then $b = c$ or $a = d$.*

We can assume, without loss of generality, that

$$f(x) \geq x, \qquad x \in (a, b). \tag{5}$$

Then we have $a < b \leq c < d$ and it is enough to prove that $b = c$. Suppose, on the contrary, that $b < c$. Using (**II**), in view of (5), it is easy to see that

$$f(x) \leq x, \qquad x \in (b, c). \tag{6}$$

Choose an open, nonempty interval $J \subset (c,d) \cap f((a,b))$. Put $K := f^{-1}(J) \cap (a,b)$ and

$$A := \{x \in J;\ x^f < c\}, \quad B := \{x \in J;\ x^f \geq c\}, \qquad C := J \setminus (A \cup B)$$
$$A_0 := f^{-1}(A) \cap K, \quad B_0 := f^{-1}(B) \cap K, \qquad C_0 := f^{-1}(C) \cap K.$$

Obviously, $A \cup B \cup C = J$ and $A_0 \cup B_0 \cup C_0 = K$. Notice also that since for $x \in C$ the sequence $(f^n(x))_n \in \mathbb{N}$ does not converge, we have

$$C \subset M. \tag{7}$$

Furthermore,

$$A \subset M, \tag{8}$$

as $x^g \geq c$ for every $x \in A$, since $A \subset J \subset (c,d)$. Moreover, since $B_0 \subset M$, as for every $x \in B_0$ we have $x^f \geq c$ and $x^g \leq b$, and $C_0 \subset M$, we deduce that $A_0 \not\subset M$. Fix an $x_0 \in A_0 \setminus M$. We have $x_0^f = x_0^g$. By (**I**) and (5), we conclude that $x_0^g = b$. Fix an arbitrary positive $\varepsilon < \frac{c-b}{2}$ and choose a positive $\delta < \min\{\varepsilon, b-a\}$ in such a way that

$$f((b-\delta, b+\delta)) \subset (b-\varepsilon, b+\varepsilon). \tag{9}$$

Since $x_0^f = b$ we can find an $n_0 \in \mathbb{N}$ with $f^{n_0+1}(x_0) \in (b-\delta, b+\delta)$. Let J_0 be a neighbourhood of $f(x_0)$ such that $f^{n_0}(J_0) \subset (b-\delta, b+\delta)$.

Fix an $x \in J_0$ such that the limit $\lim_{n\to\infty} f^n(x)$ exists. Making use of (5) and (9) we infer that either there exists an $N \geq n_0$ with $f^N(x) \in (b, b+\varepsilon)$ or

$$f^n(x) \in (b-\delta, b], \qquad n \geq n_0. \tag{10}$$

In both cases we get $x^f < c$. Indeed, in the first case, according to (6), we have $f^{N+1}(x) \leq f^N(x)$, whence, due to Theorem 3.1, we obtain $x^f \leq f^N(x) < b+\varepsilon < c$. In the second case (10) implies $x^f \leq b$. Hence, we have showed that $J_0 \cap B = \emptyset$. Observe that (cf. (7) and (8))

$$\emptyset \neq J_0 \cap J = J_0 \cap (A \cup B \cup C) = (J_0 \cap A) \cup (J_0 \cap B) \cup (J_0 \cap C) \subset M,$$

which yields that M has nonempty interior, which is a contradiction.

(**V**) *Let $a, b \in \mathcal{R}(f)$ and $a < b$. If $b < b_f$ and $((a, b_f)) \cap \mathcal{R}(f) = \{b\}$, then $f(x) \leq b_f$ for $x \in (a,b)$, whereas $a_f < a$ and $((a_f, b)) \cap \mathcal{R}(f) = \{a\}$ implies $f(x) \geq a_f$ for $x \in (a,b)$.*

Assume, for instance, that the first possibility holds.
If $f((a,b)) \cap (b_f, \sup X] \neq \emptyset$, then $f(x) \geq x$ for $x \in (a,b)$ and, by (**II**), $f(x) \leq x$ for $x \in (b, b_f)$. Since $f(x) < x$ for $x \in (b_f, \sup X]$, $((a, b_f)) \cap \mathcal{R}(f) = \{b\}$

and $b_f \in \mathcal{R}(f)$ we infer that $b_f \in \alpha(f)$, more precisely $b_f \le f(x) < x$ for $x \in (b_f, b_f + \delta)$, for a positive δ. Evidently, $x^f = b_f$ for $x \in (b_f, b_f + \delta)$. The density of the set $X \setminus M$ allows us to choose a point $x \in (a, b) \setminus M$ such that $f(x) \in (b_f, b_f + \delta)$. By (**I**) we obtain $b = x^g = x^f = b_f$, a contradiction.

(**VI**) Let $a, b \in \mathcal{R}(f)$, $a < b$ and $(a, b) \cap \mathcal{R}(f) = \emptyset$. Then

$$\operatorname{int}(f([a, b]) \setminus [a, b]) \cap \mathcal{R}(f) = \emptyset.$$

By (**III**) and (**IV**) we infer that the set $(f([a, b]) \setminus [a, b]) \cap \mathcal{R}(f)$ has at most one element. If there exists a $c \in \operatorname{int}(f([a, b]) \setminus [a, b]) \cap \mathcal{R}(f)$, then, due to (**IV**), $c = b_f$, which contradicts (**V**).

Finally, we will present the partition of $[a_f, b_f]$ into a union of intervals $[a_i, b_i]$ satisfying one of the conditions (i)-(v). In order to do it choose an arbitrary $x \in [a_f, b_f]$. We have the following possibilities.

• $x \in \mathcal{R}(f)$.

We put $a_i = b_i = x$; interval $[a_i, b_i]$ fulfills condition (i).

• $x \notin \mathcal{R}(f)$.

Then there exist $a, b \in \mathcal{R}(f)$ with $(a, b) \cap \mathcal{R}(f) = \emptyset$ and $x \in (a, b)$. Assume, for instance, that $f(x) \ge x$ for $x \in (a, b)$. Here we differentiate further.

• $f([a, b]) = [a, b]$.

We put $a_i = a$ and $b_i = b$ to get interval $[a_i, b_i]$ satisfying (ii).

• $f([a, b]) \setminus [a, b] \neq \emptyset$.

According to (**VI**), $\operatorname{int}(f([a, b]) \setminus [a, b]) \cap \mathcal{R}(f) = \emptyset$. We have the following possibilities.

• There exists a fixed point from $\mathcal{R}(f)$ greater than b.

Let us denote $d := \min(\mathcal{R}(f) \cap (b, \infty)$. According to (**II**) we have $f(x) \le x$ for $x \in (b, d)$. Moreover, making use of (**VI**), we infer that $f(x) \ge a$ for $x \in (b, d)$. It is enough to put $a_i = a$, $b_i = d$ and $c_i = b$ to get interval $[a_i, b_i]$ satisfying (iii). The last part of the assertion (iii) follows from the fact that $x^g = c_i$ for $x \in (a_i, b_i)$, which is due to (**I**).

• There are no fixed points from $\mathcal{R}(f)$ greater than b.

Then $b = b_f$. Putting $a_i = a$ and $b_i = b$ we obtain the interval $[a_i, b_i]$ satisfying (iv).

Lemma 3.4 implies $\mathcal{S}(f) \cap \operatorname{Per}(g, 1) = \emptyset$, whence $\mathcal{A}_f(x) \subset M$ for every $x \in \mathcal{S}(f)$. The last assertion is obvious. $\square$

Now we are going to provide the sufficient conditions of M-weak almost iterability.

Theorem 4.2. *Let $f \colon X \to X$ be a continuous function such that* $\operatorname{Per}(f, 2) \cap (X \setminus [a_f, b_f]) = \emptyset$. *Assume that there exist $a_i, b_i, c_i \in \mathcal{R}(f)$, $i \in I$, such that $[a_f, b_f] = \bigcup_{i \in I} [a_i, b_i]$ and for every $i \in I$ one of the cases* (i)–(v) (*from page*

366) *holds. Put* $Z_i := \bigcup_{n\in\mathbb{N}} f^{-n}(\{a_i, b_i\}) \cap (a_i, b_i)$, $Z_i' := \bigcup_{n\in\mathbb{N}} f^{-n}(Z_i)$ *for* i *such that* (iii) *holds and* $Z_i = Z_i' := \emptyset$ *otherwise. Then there exists an iterable function* g*, such that (1) holds for every* $x \in X \setminus M$*, where*

$$M := \bigcup_{i\in I}(Z_i \cup Z_i') \cup \bigcup_{x\in\mathcal{S}(f)} \mathcal{A}_f(x) \cup \{x \in X;\ \lim_{n\to\infty} f^n(x) \textit{ does not exist}\}.$$

Particularly, if $M \in \mathcal{M}$*, then* f *is* M*-weak almost iterable.*

PROOF. We start with a construction of an almost iterable function h. Namely, let $h\colon X \to X$ be a continuous function such that $h(x) = f(x)$ for every $x \in (X \setminus [a_f, b_f]) \cup \mathcal{R}(f)$, $h_{|_{[a_f,b_f]}}$ is strictly increasing and

$$\begin{aligned}
&h(x) > x, \text{if } f(x) \geq x \text{ for } x \in (a_i, b_i) \text{ and (ii) holds},\\
&h(x) < x, \text{if } f(x) \leq x \text{ for } x \in (a_i, b_i) \text{ and (ii) holds},\\
&h(x) > x, \text{for } x \in (a_i, c_i) \text{ where (iii) holds},\\
&h(x) < x, \text{for } x \in (c_i, b_i) \text{ where (iii) holds},\\
&h(x) > x, \text{for } x \in (a_i, b_i) \text{ where (iv) holds},\\
&h(x) < x, \text{for } x \in (a_i, b_i) \text{ where (v) holds}.
\end{aligned}$$

Since $\mathrm{Per}(h, 2) = \emptyset$ and $h_{|_{[a_f,b_f]}} = h_{|_{[a_h,b_h]}}$ strictly increases, it follows from Theorem 1.3 that h is almost iterable. Let $g\colon X \to X$ be an iterable function such that $x^g = x^h$ for every $x \in X$. Put $M := \bigcup_{i\in I}(Z_i \cup Z_i') \cup \bigcup_{x\in\mathcal{S}(f)} \mathcal{A}_f(x) \cup \{x \in X;\ \lim_{n\to\infty} f^n(x)$ does not exist$\}$. To finish the proof, it suffices to show that $x^f = x^h$ for $x \in X \setminus M$.

Fix an $x \in [a_f, b_f] \setminus M$. It is worth pointing out that $x^f \in \mathcal{R}(f)$, so $x^f = a_i, b_i$ or c_i for some $i \in I$. If $x \in \mathcal{R}(f)$, then $x^f = x = x^h$. Otherwise $x \in (a_i, b_i)$, where one of (ii)-(v) is satisfied. In the case (ii), since the interval $[a_i, b_i]$ is invariant under f, we get $x^f = a_i$ or $x^f = b_i$, depending on the sign of $f - \mathrm{id}$ on this interval, according to Theorem 2.2. As $(f-\mathrm{id})(h-\mathrm{id}) > 0$, we get the assertion. Considering case (iii) notice that again $[a_i, b_i]$ is invariant under f so, $x^f \in \{a_i, b_i, c_i\}$. Assume, for instance, that $x \in (a_i, c_i)$. If there exists a natural number n_0 such that $f^{n_0}(x) > c_i$, then $x < f(x)$ and $f^{n_0}(x) > f^{n_0+1}(x)$. Therefore, using once more Theorem 2.2, we conclude that $x^f \in (x, f^{n_0}(x))$, and consequently, $x^f = c_i$. Note, that we have also used the assumption $x \notin Z_i$. Otherwise, $x < f^n(x) < c_i$, whence $x^f = c_i$. Obviously, $x^h = c_i$. Passing to case (iv) , observe that $x < f(x)$, whence

$x < x^f$. But $(x, \sup X] \cap \mathcal{R}(f) = \{b_f\}$. Thus $x^f = b_f$, we also have $x^h = b_f$. We proceed analogously in the case (v).

If $x \in X \setminus ([a_f, b_f] \cup M)$, then either $f^n(x) \in X \setminus [a_f, b_f]$ for every $n \in \mathbb{N}$ whence $f^n(x) = h^n(x)$ and also $x^f = x^h$, or let $n_0 \in \mathbb{N}$ be the smallest natural number such that $f^{n_0}(x) \in [a_f, b_f]$. We have $f^k(x) = h^k(x) \in X \setminus [a_f, b_f]$ for $k = 1, 2, \ldots, n_0 - 1$, whence $f^{n_0}(x) = h^{n_0}(x) \in [a_f, b_f]$. In fact $f^{n_0}(x) = h^{n_0}(x) \in [a_f, b_f] \setminus M$ and the assertion follows from what we have already shown. $\square$

5 Convergence in Measure.

As we already mentioned, assumption $M \in \mathcal{M}$ includes the case when M is of Lebesgue measure zero. One can ask, if and how the class of M- weak almost iterable functions will increase if we change the convergence in (1) from convergence almost everywhere into convergence in measure. Namely, we consider a measure μ such that

$$\begin{aligned} &\mu \text{ is a finite Borel measure on the interval } X, \text{ vanishing at points} \\ &\text{and positive on every measurable subset of } X \text{ with nonempty interior.} \end{aligned} \tag{H}$$

We are going to prove a result which says that under some quite natural conditions, satisfying $f^n - g^n \to 0$, μ-almost everywhere is equivalent to convergence in this measure.

We follow [1] in posing the following definitions.

Let $f\colon X \to X$ be a continuous function. We say that $x \in X$ is *approximately periodic* if for every $\varepsilon > 0$ there exist a periodic point y and a natural number N such that

$$|f^n(x) - f^n(y)| < \varepsilon, \qquad n \geq N.$$

A function f is *uniformly nonchaotic* if every point of X is approximately periodic. [1, Chapter VI, Corollary 26, Proposition 27 and Lemma 28] justifies, in a sense, assuming in what follows that f is uniformly nonchaotic.

Theorem 5.1. *Suppose* (H). *Let $f\colon X \to X$ be a uniformly nonchaotic function and $g\colon X \to X$ an iterable function such that*

$$\lim_{n\to\infty} (f^n - g^n) = 0 \textit{ in measure } \mu.$$

Then $\lim_{n\to\infty}(f^n(x) - g^n(x)) = 0$ for μ-almost every $x \in X$.

Proof. Notice that

$$X = \{x \in X; \quad x^f = x^g\} \cup X' \cup X'',$$

where

$$X' := \{x \in X; \quad x^f \neq x^g\}$$

and

$$X'' := \{x \in X; \quad \lim_{n\to\infty} f^n(x) \text{ does not exist}\}.$$

Let us consider X'. We have

$$X' = \bigcup_{m=1}^{\infty} X_m,$$

where

$$X_m := \Big\{x \in X; \quad \big|x^f - x^g\big| > \frac{1}{m}\Big\}.$$

For every $x \in X_m$ there exists $n \in \mathbb{N}$ such that

$$\big|f^k(x) - g^k(x)\big| \geq \frac{1}{3m}, \qquad k \geq n.$$

Hence

$$X_m \subset \bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} \Big\{x \in X; \quad \big|f^k(x) - g^k(x)\big| \geq \frac{1}{3m}\Big\}.$$

We can estimate

$$\begin{aligned}\mu(X_m) &\leq \mu(\bigcup_{n\in\mathbb{N}} \bigcap_{k=n}^{\infty} \Big\{x \in X; \quad \big|f^k(x) - g^k(x)\big| \geq \frac{1}{3m}\Big\}) = \\ &= \lim_{n\to\infty} \mu(\bigcap_{k=n}^{\infty} \Big\{x \in X; \quad \big|f^k(x) - g^k(x)\big| \geq \frac{1}{3m}\Big\}) \leq \\ &\leq \lim_{n\to\infty} \mu(\Big\{x \in X; \quad |f^n(x) - g^n(x)| \geq \frac{1}{3m}\Big\}) = 0.\end{aligned}$$

Now we pass to X''. Fix a point $x \in X''$. Let m be a natural number such that

$$\frac{1}{m} < \frac{\limsup_{n\to\infty} f^n(x) - \liminf_{n\to\infty} f^n(x)}{5}.$$

Since f is uniformly nonchaotic we can find a point $x^* \in \operatorname{Per} f$ and a positive integer K such that

$$\big|f^k(x) - f^k(x^*)\big| < \frac{1}{3m}, \qquad k \geq K. \tag{11}$$

If $x^* \in \mathrm{Per}(f,1)$, then

$$\left|f^k(x) - x^*\right| < \frac{1}{3m}, \qquad k \geq K,$$

whence $\limsup_{n\to\infty} f^n(x) - \liminf_{n\to\infty} f^n(x) \leq \frac{2}{3m}$, contrary to the choice of a number m. Therefore $x^* \in \mathrm{Per}(f,l)$ for a $l \geq 2$. Notice also that there exist positive integers $k_1, k_2 \geq K$ such that

$$\left|f^{k_1}(x) - \liminf_{n\to\infty} f^n(x)\right| < \frac{1}{m}, \tag{12}$$

$$\left|f^{k_2}(x) - \limsup_{n\to\infty} f^n(x)\right| < \frac{1}{m}. \tag{13}$$

From the choice of m and conditions (11), (12) and (13), we get

$$\begin{aligned} \left|f^{k_1}(x^*) - f^{k_2}(x^*)\right| &\geq \limsup_{n\to\infty} f^n(x) - \liminf_{n\to\infty} f^n(x) + \\ &- \left|f^{k_1}(x^*) - f^{k_1}(x)\right| - \left|f^{k_2}(x^*) - f^{k_2}(x)\right| + \\ &- \left|f^{k_1}(x) - \liminf_{n\to\infty} f^n(x)\right| - \left|f^{k_2}(x) - \limsup_{n\to\infty} f^n(x)\right| \geq \\ &\geq \frac{5}{m} - \frac{1}{3m} - \frac{1}{3m} - \frac{1}{m} - \frac{1}{m} > \frac{2}{m}. \end{aligned}$$

Hence, there is an $i \in \{0, \ldots, l-1\}$ such that

$$\left|x^g - f^i(x^*)\right| > \frac{1}{m}. \tag{14}$$

We have already shown that for every $x \in X''$ there exist a periodic point $x^* \in \mathrm{Per}(f,l)$ of period $l \geq 2$ and positive integers $m, K \in \mathbb{N}$ and $i \in \{0, \ldots, l-1\}$ such that (11) and (14) hold. Since $x^* \in \mathrm{Per}(f,l)$, by (14) and condition (11), we get

$$\begin{aligned} \left|f^{i+kl}(x) - g^{i+kl}(x)\right| \geq \left|f^{i+kl}(x^*) - x^g\right| - \left|f^{i+kl}(x) - f^{i+kl}(x^*)\right| + \\ - \left|x^g - g^{i+kl}(x)\right| \geq \frac{1}{m} - \frac{1}{3m} - \frac{1}{3m} = \frac{1}{3m}, \end{aligned}$$

for k sufficiently large. Whence

$$X'' = \bigcup_{l\geq 2} \bigcup_{m\in\mathbb{N}} \bigcup_{i\in\{1,\ldots,l-1\}} \bigcup_{K'\in\mathbb{N}} X_{l,m,i,K'},$$

where

$$X_{l,m,i,K'} := \Big\{x \in X''; \quad \big|f^{i+kl}(x) - g^{i+kl}(x)\big| \geq \frac{1}{3m} \text{ for } k \geq K'\Big\}.$$

Moreover

$$X_{l,m,i,K'} \subset \bigcap_{k \geq K'} \Big\{x \in X; \quad \big|f^{i+kl}(x) - g^{i+kl}(x)\big| \geq \frac{1}{3m}\Big\}.$$

Thereafter

$$\mu(X_{l,m,i,K'}) \leq \lim_{k\to\infty} \mu(\Big\{x \in X; \quad \big|f^{i+kl}(x) - g^{i+kl}(x)\big| \geq \frac{1}{3m}\Big\}) = 0,$$

which implies $\mu(X'') = 0$. □

It is worth pointing out that the topological equivalent of the above Theorem also holds true. Namely, we have

Remark 5.1. *Let $f\colon X \to X$ be a uniformly nonchaotic function and $g\colon X \to X$ an iterable function such that from every subsequence $(f^{n_k} - g^{n_k})_{k\in\mathbb{N}}$ of the sequence $(f^n - g^n)_{n\in\mathbb{N}}$ we can choose a subsequence $(f^{n_{k_m}} - g^{n_{k_m}})_{m\in\mathbb{N}}$ such that $\lim_{m\to\infty}(f^{n_{k_m}}(x) - g^{n_{k_m}}(x)) = 0$ for every $x \in X \setminus M$, where M is a set of first category. Then $\lim_{n\to\infty}(f^n(x) - g^n(x)) = 0$ for every $x \in X \setminus L$, where L is a set of first category.*

Proof. With notation from the previous proof we have X_m is a subset of a meager set, whence X' is also a set of first category. Considering the subsequence $(f^{i+kl} - g^{i+kl})_{k\in\mathbb{N}}$ leads to conclusion that the set $X_{l,m,i,K'}$ is a subset of a meager set. Thereafter X'' is a set of first category. □

References

[1] L. S. Block and W. A. Coppel, *Dynamics in One Dimension*, Lecture Notes in Math., **1513**, Springer-Verlag, Berlin, 1992.

[2] W. Jarczyk, *Almost iterable functions*, Aequationes Math., **42(2-3)** (1991), 202–219.

[3] B. Przebieracz, *Approximately iterable functions*, Grazer Math. Ber., **351** (2007), 139–157.

[4] B. Przebieracz, *The closure of the set of iterable functions*, Aequationes Math., **75** (2008), 239–250.

[5] A. N. Šarkovskiĭ, *On cycles and the structure of a continuous mapping*, (Russian), Ukraïn. Mat. Zh., **17** (1965), 104–111.

[6] Gy. Targoński, *New Directions and Open Problems in Iteration Theory*, Forschungszentrum Graz, Ber. Math.-Stat. Sekt., **229**, Graz, 1984.

[7] M. C. Zdun, *Continuous and Differentiable Iteration Semigroups*, Pr. Nauk. Uniw. Śl. Katow., **308**, Wydawn. Uniw. Śląskiego, Katowice, 1979.