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ON EXTENDABLE DERIVATIONS

Sometimes it's as easy to prove a stronger result . . .
Kenneth R. Kellum

Abstract

There are derivations $f : \mathbb{R} \to \mathbb{R}$ which are almost continuous in the sense of Stallings but not extendable. Every derivation $f : \mathbb{R} \to \mathbb{R}$ can be expressed as the sum of two extendable derivations, as the discrete limit of a sequence of extendable derivations and as the limit of a transfinite sequence of extendable derivations. Analogous results hold for additive functions.

Let us establish some terminology to be used. By $\mathbb{R}$ and $\mathbb{Q}$ we denote the fields of all reals and rationals, respectively. Let F be a subfield of $\mathbb{R}$. An element $a \in \mathbb{R}$ is called algebraic over F, if $p(a) = 0$ for some polynomial $p \in F[x]$, $p \neq 0$. For $A \subset \mathbb{R}$, $\mathbb{Q}(A)$ denotes the extension of $\mathbb{Q}$ by the set A, i.e., the smallest subfield of $\mathbb{R}$ containing $\mathbb{Q} \cup A$. The algebraic closure of $A \subset \mathbb{R}$ is the set $\operatorname{algcl}(A)$ of all algebraic elements over $\mathbb{Q}(A)$. Notice that $|\operatorname{algcl}(A)| < \mathfrak{c}$ whenever $|A| < \mathfrak{c}$. A set $A \subset \mathbb{R}$ is algebraically independent over $\mathbb{Q}$ if for all $n < \omega$, $p \in \mathbb{Q}[x_1, \ldots x_n]$, $p \neq 0$, and $a_1, \ldots a_n \in A$, we have $p(a_1, \ldots, a_n) \neq 0$. A set $A \subset \mathbb{R}$ is an algebraic base of $\mathbb{R}$ over $\mathbb{Q}$ if A is algebraically independent and $\operatorname{algcl}(A) = \mathbb{R}$. (An algebraic base is often called *transcendental*.) Recall that every algebraically independent over $\mathbb{Q}$ set $A \subset \mathbb{R}$ can be extended to an algebraic base of $\mathbb{R}$ [9, Theorem 4.10.1, p. 102].

A function $f : \mathbb{R} \to \mathbb{R}$ is:

- *additive* ($f \in \mathrm{Add}$) if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$;
- *a derivation* ($f \in \mathrm{Der}$) if f is additive and $f(xy) = xf(y) + yf(x)$ for all $x, y \in \mathbb{R}$ ([9], p. 346);

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- *almost continuous in the sense of Stallings* ($f \in$ ACS), if every open neighbourhood of f in $\mathbb{R}^2$ contains also a continuous function $g : \mathbb{R} \to \mathbb{R}$;
- *extendable* ($f \in$ Ext) if there is a connectivity function $F \colon \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $F(x,0) = f(x)$ when $x \in \mathbb{R}$ ([14], see also [4]).

Recall also that a function $f : X \to Y$, where X and Y are topological spaces, is *connectivity*, if the restriction $f{\restriction}C \colon C \to Y$ is a connected subset of $X \times Y$ whenever C is a connected subset of X (see [4]). Remark that if $g \in$ Der then $g{\restriction}\mathbb{Q} = 0$ [9, Lemma. 14.1.3, p. 347], thus Der is a proper subclass of Add. Similarly, it is well-known that Ext is a proper subclass of ACS (see [4]). The Jones' example of an additive function with the big graph (see [9, Theorem 12.4.5, p. 290]) is ACS but not Ext [12]. Thus Add $\cap$ ACS $\setminus$ Add $\cap$ Ext $\neq \emptyset$. In the first part of this note we remark that an easy modification of Jones' construction gives an example of derivation which is ACS and non Ext. In the second part we will consider algebraic properties of the classes Der $\cap$ Ext and Add $\cap$ Ext. Recall that algebraic properties of the class Add $\cap$ ACS have been investigated by Z. Grande [6]. He proved that: • every additive function $f : \mathbb{R} \to \mathbb{R}$ is the sum of two additive almost continuous functions; • every $f \in$ Add is the pointwise limit of a sequence $(f_n)_n \subset$ Add $\cap$ ACS; • every $f \in$ Add is the limit of a transfinite sequence of Add $\cap$ ACS functions. (Recall that $f : \mathbb{R} \to \mathbb{R}$ is a limit of transfinite sequence $f_\alpha : \mathbb{R} \to \mathbb{R}$, $\alpha < \omega_1$, if for each $x \in \mathbb{R}$ there is $\alpha < \omega_1$ such that $f_\beta(x) = f(x)$ for all $\beta \geq \alpha$ [13].) E. Strońska proved recently analogous results for almost continuous derivations [15]. We will show that almost continuity in [6] and in [15] can be replaced by extendability.

1 Almost Continuous Derivation which is not Extendable.

Notice that $f{\restriction}\mathrm{algcl}\,(\mathbb{Q}) = 0$ for every $f \in$ Der [9, Lemma 14.1.4, p. 347]. Following [15], our constructions of derivations will be based on the following facts (see [9, Theorem 14.2.1, p. 352]).

Fact 1. *Let A be an algebraic base of $\mathbb{R}$ over $\mathbb{Q}$. Then for any $g : A \to \mathbb{R}$ there exists a unique derivation $h : \mathbb{R} \to \mathbb{R}$ such that $h{\restriction}A = g$.*

Fact 2. *If $f, g \in$ Der, $A \subset \mathbb{R}$ is algebraically independent and $f{\restriction}A = g{\restriction}A$ then f agrees with g on* algcl (A).

Lemma 3. *There exists an algebraically independent set $A \subset \mathbb{R}$ that meets each perfect set $P \subset \mathbb{R}$ on a set of size $\mathfrak{c}$.*

Proof. List all perfect sets in a sequence $\{P_\alpha : \alpha < \mathfrak{c}\}$. Let $a_0 \in P \setminus \mathbb{Q}$. Fix $\alpha < \mathfrak{c}$ and suppose we have chosen a_β for $\beta < \alpha$ such that $a_\beta \in P_\beta$ and the set $A_\alpha = \{a_\beta : \beta < \alpha\}$ is algebraically independent over $\mathbb{Q}$. Since $|\mathrm{algcl}\,(A_\alpha)| < \mathfrak{c}$, we have $P_\alpha \setminus \mathrm{algcl}\,(A_\alpha) \neq \emptyset$. Choose $a_\alpha \in P_\alpha \setminus \mathrm{algcl}\,(A_\alpha)$. Set $A = \{a_\alpha : \alpha < \mathfrak{c}\}$, then A is algebraically independent and it meets each perfect set. Now, since each perfect set P can be decomposed onto $\mathfrak{c}$ many perfect sets, $|A \cap P| = \mathfrak{c}$. □

Theorem 4. *There exists an almost continuous derivation $f : \mathbb{R} \to \mathbb{R}$ which is not extendable.*

Proof. Let $\{K_\alpha : \alpha < \mathfrak{c}\}$ be a sequence of all closed subsets $K \subset \mathbb{R}^2$ with $\mathrm{dom}(K) = \mathfrak{c}$, where $\mathrm{dom}(K)$ denotes the x-projection of K. Then for each $\alpha < \mathfrak{c}$, $\mathrm{dom}(K_\alpha)$ includes a perfect set. Let A be an algebraically independent set with $|P \cap A| = \mathfrak{c}$ for each perfect set P. Fix $f_0 : A \to \mathbb{R}$ such that $f_0 \cap K_\alpha \neq \emptyset$, i.e., there is $a_\alpha \in A$ with $\langle a_\alpha, f_0(a_\alpha)\rangle \in K_\alpha$, for each $\alpha < \mathfrak{c}$. By Fact 1, there is a derivation $f : \mathbb{R} \to \mathbb{R}$ with $f{\restriction}A = f_0$. Since f_0 meets each blocking set in $\mathbb{R}^2$, $f \in \mathrm{ACS}$ ([8], see also [11]). Now, observe that f is unbounded on each perfect set, thus $f{\restriction}P$ is continuous for no perfect set P. Consequently, f is not extendable ([5], see also [4]). □

2 Properties of the Classes $\mathrm{Der} \cap \mathrm{Ext}$ and $\mathrm{Add} \cap \mathrm{Ext}$.

Notice that the classes Add and Der are closed under sums, pointwise limits and transfinite limits. We will prove that each $f \in \mathrm{Der}$ can be represented as:

- the sum of two $\mathrm{Ext} \cap \mathrm{Der}$ functions;
- the pointwise limit of a sequence of $\mathrm{Ext} \cap \mathrm{Der}$ functions;
- the limit of a transfinite sequence of functions from the class $\mathrm{Ext} \cap \mathrm{Der}$.

Similar results hold for the class $\mathrm{Ext} \cap \mathrm{Add}$. In proofs we will use the method of negligible sets. ([2], see also [4, Section 7.2]). Recall that if $\mathcal{K}$ is the class of functions from $\mathbb{R}$ to $\mathbb{R}$ and $k \in \mathcal{K}$, then the set $M \subset \mathbb{R}$ is k-negligible with respect to $\mathcal{K}$, provided $f \in \mathcal{K}$ for every function $f : \mathbb{R} \to \mathbb{R}$ which agrees with k on $\mathbb{R} \setminus M$. (This is the same as saying that every function $f : \mathbb{R} \to \mathbb{R}$ obtained by arbitrarily redefining k on M is still a member of $\mathcal{K}$.)

Lemma 5. ([3, Proposition 4.3]) *For every c-dense meager F_σ set $M \subset \mathbb{R}$ there exists $g \in \mathrm{Ext}$ such that $\mathbb{R} \setminus M$ is g-negligible with respect to* Ext*.*

Lemma 6. ([7]) *For every c-dense set $M \subset \mathbb{R}$ there exists $g \in \mathrm{ACS}$ such that $\mathbb{R} \setminus M$ is g-negligible with respect to* ACS*.*

Lemma 7. ([10, Theorem 1]) *Let* $(I_n)_n$ *be a sequence of all open intervals with rational endpoints. There exists a sequence of pairwise disjoint perfect sets* $(P_n)_n$ *such that* $P_n \subset I_n$ *and* $\bigcup_{n<\omega} P_n$ *is algebraically independent.*

In the proofs below, let $(P_n)_n$ be sequence of perfect sets as in Lemma 7 and A be an algebraic base of $\mathbb{R}$ over $\mathbb{Q}$ which includes all P_n's.

Theorem 8. *For every family* $\mathcal{F}$ *of derivations with* $|\mathcal{F}| \le \mathfrak{c}$ *there exists* $g \in \mathrm{Der} \cap \mathrm{Ext}$ *such that* $f + g \in \mathrm{Ext}$ *for each* $f \in \mathcal{F}$.

PROOF. We can assume that $|\mathcal{F}| = \mathfrak{c}$ and $0 \in \mathcal{F}$. Let $\mathcal{F} = \{f_\alpha : \alpha < \mathfrak{c}\}$. Decompose each P_n onto c-many perfect sets $P_{n,\alpha}$, $\alpha < \mathfrak{c}$. For each $\alpha < \mathfrak{c}$ set $F_\alpha = \bigcup_{n<\omega} P_{n,\alpha}$. Then F_α is a c-dense meager F_σ set, so by Lemma 5 there is $g_\alpha \in \mathrm{Ext}$ such that $\mathbb{R} \setminus F_\alpha$ is g_α-negligible (with respect to Ext). Define $h : A \to \mathbb{R}$ by

$$h(x) = \begin{cases} g_\alpha(x) - f_\alpha(x) & \text{for } x \in F_\alpha\ \alpha < \mathfrak{c}, \\ 0 & \text{for } x \in A \setminus \bigcup_{\alpha<\mathfrak{c}} F_\alpha. \end{cases}$$

Let $g : \mathbb{R} \to \mathbb{R}$ be the derivation such that $g \restriction A = h$. Then, for any $\alpha < \mathfrak{c}$, $(g+f_\alpha)\restriction F_\alpha = g_\alpha \restriction F_\alpha$, thus $g + f_\alpha \in \mathrm{Ext}$. Since $f_\alpha = 0$ for some α, $g \in \mathrm{Ext}$. □

Notice that an analogous result concerning additive almost continuous functions has been proved by D. Banaszewski [1].

Corollary 9. *Every derivation is the sum of two extendable derivations.*

PROOF. Fix $f \in \mathrm{Der}$. Applying Theorem 8 to the family $\{0, f\}$ we obtain $g \in \mathrm{Ext} \cap \mathrm{Der}$ such that $h = f + g \in \mathrm{Ext} \cap \mathrm{Der}$. Then $f = h - g$. □

Corollary 10. *There are discontinuous extendable derivations.*

PROOF. Remark that if $g \in \mathrm{Der}$ is continuous then $g = 0$ [9, Theorem. 14.1.1, p. 348]. Let $f \in \mathrm{Der}$ be discontinuous and let $f_0, f_1 \in \mathrm{Der} \cap \mathrm{Ext}$ be such that $f = f_0 + f_1$. Then at least one of f_0, f_1 is not continuous (cf [15, Remark 1]). □

Theorem 11. *Every derivation* $f : \mathbb{R} \to \mathbb{R}$ *is the limit of a sequence of extendable derivations.*

PROOF. Fix $f \in \mathrm{Der}$. For each n, let $P_{n,i}$, $i < \omega$, be a decomposition of P_n onto ω many perfects sets, and let $F_i = \bigcup_{n<\omega} P_{n,i}$. The sets F_i are c-dense, meager and F_σ, so there exist $g_i \in \mathrm{Ext}$ such that for each i the set $\mathbb{R} \setminus F_i$ is g_i-negligible. Set $h_i : A \to \mathbb{R}$,

$$h_i(x) = \begin{cases} g_i(x) & \text{for } x \in F_i, \\ f(x) & \text{for } x \in A \setminus F_i. \end{cases}$$

Let $f_i : \mathbb{R} \to \mathbb{R}$ be the derivation such that $f_i \restriction A = h_i$. Since $f_i \restriction F_i = g_i \restriction F_i$, f_i are extendable. Observe that $\lim_{i\to\infty} f_i = f$. In fact, fix $x \in \mathbb{R}$. There exists a finite subset A_x of A such that $x \in \operatorname{algcl}(A_x)$. Fix n_0 such that $A_x \subset (A \setminus \bigcup_n P_n) \cup \bigcup_{i<n_0} P_i$. Then $f_i \restriction A_x = f \restriction A_x$ for $i \geq n_0$. Thus Fact 2 yields $f_i(x) = f(x)$ for $i \geq n_0$, so $\lim_i f_i(x) = f(x)$. □

Notice that the sequence $(f_n)_n$ constructed in the proof of Theorem 11 has the following property: for each $x \in \mathbb{R}$ there is $n < \omega$ with $f_i(x) = f(x)$ for all $i \geq n$. Thus $(f_n)_n$ converges discretely to f.

Theorem 12. *Every derivation $f : \mathbb{R} \to \mathbb{R}$ is the limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ of extendable derivations.*

Proof. Fix $f \in \mathrm{Der}$. Decompose each P_n onto c-many perfect sets $P_{n,\alpha}$, $\alpha < \mathfrak{c}$. For each $\alpha < \omega_1$ set $F_\alpha = \bigcup_{n<\omega} P_{n,\alpha}$. Let g_α be an extendable function such that $\mathbb{R} \setminus F_\alpha$ is g_α-negligible. Define $h_\alpha : A \to \mathbb{R}$ by

$$h_\alpha(x) = \begin{cases} g_\alpha(x) & \text{for } x \in F_\alpha, \\ f(x) & \text{for } x \in A \setminus F_\alpha. \end{cases}$$

Let $f_\alpha : \mathbb{R} \to \mathbb{R}$ be the derivation such that $f_\alpha \restriction A = h_\alpha$. Since $f_\alpha \restriction F_\alpha = g_\alpha \restriction F_\alpha$, $f_\alpha \in \mathrm{Ext}$. As in the proof of Theorem 11 we verify that $\lim_\alpha f_\alpha = f$. □

Theorem 13. *Let $f \in \mathrm{Add}$. Then*

1. *f is the sum of two $\mathrm{Add} \cap \mathrm{Ext}$ functions;*
2. *f is the discrete limit of a sequence of $\mathrm{Add} \cap \mathrm{Ext}$ functions;*
3. *f is a limit of a transfinite limit of $\mathrm{Add} \cap \mathrm{Ext}$ functions.*

Proof. Proofs of all those statements are the same as proofs of Corollary 9 and Theorems 11, 12. We have to use Lemma 6 instead of Lemma 7. □

Observe that for any $a \neq 0$ the function $f : x \mapsto ax$ belongs to the class $\mathrm{Ext} \cap (\mathrm{Add} \setminus \mathrm{Der})$.

Corollary 14. *There exists discontinuous function $f \in \mathrm{Ext} \cap (\mathrm{Add} \setminus \mathrm{Der})$.*

Proof. Fix a discontinuous $f \in \mathrm{Add} \setminus \mathrm{Der}$. Then f is the limit of a sequence $(f_n)_n \in \mathrm{Add} \cap \mathrm{Ext}$. We may assume that all f_n are discontinuous. (In fact, otherwise f is the limit of continuous additive, i.e., linear, functions, thus f is continuous.) Since $f \notin \mathrm{Der}$, there is $n < \omega$ such that $f_n \notin \mathrm{Der}$. □

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