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DARBOUX-LIKE FUNCTIONS WITHIN THE CLASS OF HAMEL FUNCTIONS

Abstract

In this paper we present a discussion of the relations of the classes of Darboux-like functions within the classes of Hamel functions and Sierpiński-Zygmund Hamel functions. We prove that the inclusion relations among Darboux-like classes remain valid in both cases (under the assumption of CH for Sierpiński-Zygmund Hamel functions). In particular, assuming CH we prove the existence of a Sierpiński-Zygmund Hamel function which is connectivity but not almost continuous. In addition, we investigate the cardinal number $\text{Add}(F_1, F_2)$ in the case when one of the families F_1, F_2 is Darboux-like or Sierpiński-Zygmund and the other one is the class of Hamel functions, where $\text{Add}(F_1, F_2)$ is defined as the smallest cardinality of a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ for which there is no $g \in F_1$ such that $g + F \subseteq F_2$.

1 Definitions and Main Results.

The terminology is standard and follows [2]. The cardinality of a set X we denote by $|X|$. In particular $|\mathbb{R}|$ is denoted by \mathfrak{c} . We consider only real-valued functions. No distinction is made between a function and its graph. We write $f|A$ for the restriction of f to the set $A \subseteq \mathbb{R}$. The interior of the set A is denoted by $\text{int}(A)$. For any function g and any family of functions $F \subseteq \mathbb{R}^{\mathbb{R}}$ we define $g + F = \{g + f : f \in F\}$. Given $P \subseteq \mathbb{R}^2$ and $x \in \mathbb{R}$, P_x denotes the set $\{y \in \mathbb{R} : \langle x, y \rangle \in P\}$.

In this paper we investigate the relations among Darboux-like classes of functions within the class of Hamel functions and within the class of Sierpiński-Zygmund Hamel functions. Before we state the results let us recall the defini-

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tions of the classes of functions considered in the article. A function $f: \mathbb{R} \rightarrow \mathbb{R}$ is:

- *additive* if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$;
- *almost continuous* (in sense of Stallings) if each open subset of \mathbb{R}^2 containing the graph of f contains also the graph of a continuous function from \mathbb{R} to \mathbb{R} ;
- a *connectivity* function if the graph of $f|I$ is connected in $I \times \mathbb{R}$ for any interval I or more generally, $f: X \rightarrow \mathbb{R}$ ($X \subseteq \mathbb{R}^n$) is a connectivity function if the graph of $f|Z$ is connected in $Z \times \mathbb{R}$ for any connected subset $Z \subseteq X$;
- *Darboux* if $f[K]$ is a connected subset of \mathbb{R} (i.e., an interval) for every connected subset K of \mathbb{R} ;
- an *extendability* function provided there exists a connectivity function $F: \mathbb{R} \times [0, 1] \rightarrow \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in \mathbb{R}$;
- *Hamel* function if the graph of f is a Hamel basis for \mathbb{R}^2 ;
- *peripherally continuous* if for every $x \in \mathbb{R}$ and for all pairs of open sets U and V containing x and $f(x)$ respectively, there exists an open subset W of U such that $x \in W$ and $f[\text{bd}(W)] \subset V$. Equivalently for every $x \in \mathbb{R}$, there exist two sequences $s_n \nearrow x$ and $t_n \searrow x$ such that both sequences $f(s_n)$ and $f(t_n)$ converge to $f(x)$;
- *Sierpiński-Zygmund* if for every set $Y \subseteq \mathbb{R}$ of cardinality continuum \mathfrak{c} , $f|Y$ is discontinuous.

We use the following symbols to denote these classes: AD - additive, AC - almost continuous, Conn - connectivity, D - Darboux, Ext - extendable, HF - Hamel, PC - peripherally continuous, SZ - Sierpiński-Zygmund. The classes AC, Conn, D, Ext, PC are called Darboux-like (for more information on these classes see [5]). The following diagram presents relations among Darboux-like classes (see [3] or [5]).

$$C \longrightarrow \text{Ext} \longrightarrow \text{AC} \longrightarrow \text{Conn} \longrightarrow \text{D} \longrightarrow \text{PC}$$

Diagram 1.

The arrows in the above diagram represent strict inclusions. Recall here that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is almost continuous if and only if it intersects every *blocking set*; i.e. a closed set $K \subseteq \mathbb{R}^2$ which meets every continuous function and is disjoint with at least one function from \mathbb{R} to \mathbb{R} . The domain of every blocking set contains a non-degenerate connected set. (See [5].) For a function to be connectivity it suffices that its graph intersects every compact connected subset K of \mathbb{R}^2 such that $\text{dom}(K) = \mathfrak{c}$ (see [3, page 208]).

In [3] and [7] the authors investigate the relations between the Darboux-like classes within the additive functions and within the additive Sierpiński-Zygmund functions. In this paper we present a study of the relations between the Darboux-like classes within the class of Hamel functions and within the class of Sierpiński-Zygmund Hamel functions. The first result shows that the strict inclusions from Diagram 1 remain valid within the class of Hamel functions. The proofs of the following main results (Theorems 1, 2, and 3) are presented in the next section.

Theorem 1. *The following holds for the Hamel functions from \mathbb{R} to \mathbb{R} .*

$$C \subsetneq \text{Ext} \subsetneq \text{AC} \subsetneq \text{Conn} \subsetneq D \subsetneq \text{PC}.$$

The following theorem shows the relations among Darboux-like functions in the class of Sierpiński-Zygmund Hamel functions $\text{SZ} \cap \text{HF}$. A similar result was proved for the class $\text{SZ} \cap \text{Add}$ in [7]. Let us recall here that the existence of a Sierpiński-Zygmund function which is Darboux (connectivity or almost continuous) is independent of ZFC (see [1]). Therefore, to show that the relations among the classes AC, Conn, and D are preserved in the class $\text{SZ} \cap \text{HF}$, we will need an additional set-theoretic assumption. Specifically we will assume the Continuum Hypothesis (CH). However, one can show in ZFC that $(\text{SZ} \cap \text{HF} \cap D) \subsetneq (\text{SZ} \cap \text{HF} \cap \text{PC})$ (see Example 9 at the end of this section). Let us also recall here that the equality $\text{SZ} \cap C = \text{SZ} \cap \text{Ext} = \emptyset$ holds in ZFC. Hence we have that $(\text{SZ} \cap \text{HF} \cap C) = (\text{SZ} \cap \text{HF} \cap \text{Ext}) = \emptyset$ in ZFC.

Theorem 2. *Assume the Continuum Hypothesis. The following is true for the Sierpiński-Zygmund Hamel functions from \mathbb{R} to \mathbb{R} .*

$$\text{Ext} \subsetneq \text{AC} \subsetneq \text{Conn} \subsetneq D.$$

The next result gives the values of the cardinal function $\text{Add}(F_1, F_2)$ in the case when one of the families F_1, F_2 is HF and the other one is a Darboux-like class or SZ. As introduced in [9], the cardinal number $\text{Add}(F_1, F_2)$ (F_1, F_2 are proper non-empty subsets of $\mathbb{R}^{\mathbb{R}}$) is the smallest cardinality of a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ for which there is no $g \in F_1$ such that $g + F \subseteq F_2$. Notice here that the function Add has the property of coordinate monotonicity, that is $\text{Add}(G_1, G) \leq \text{Add}(G_2, G)$ and $\text{Add}(G, G_1) \leq \text{Add}(G, G_2)$ for $G_1 \subseteq G_2$. Recall also here that $\text{Add}(F_1, F_2) = 1$ is equivalent to $F_1 \cap F_2 = \emptyset$ (to see this choose F consisting of a constant zero function). Values of Add have been investigated for various pairs of families such as Darboux-like, Sierpiński-Zygmund, and additive functions (see [9]). The function Add has also been studied in a special case when $F_1 = \mathbb{R}^{\mathbb{R}}$. In this situation it is denoted by A so that $A(F) = \text{Add}(\mathbb{R}^{\mathbb{R}}, F)$ for any $F \subseteq \mathbb{R}^{\mathbb{R}}$.

Theorem 3. (i) Let $\mathcal{F} \in \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{PC}\}$. Then $\text{Add}(\text{HF}, \mathcal{F}) = \text{A}(\mathcal{F})$ and $\text{Add}(\mathcal{F}, \text{HF}) = \text{A}(\text{HF})$. Also $\text{Add}(\text{C}, \text{HF}) = \text{Add}(\text{HF}, \text{C}) = 1$.
(ii) $\text{Add}(\text{SZ}, \text{HF}) = \text{A}(\text{HF})$ and $\text{Add}(\text{HF}, \text{SZ}) > \mathfrak{c}$.

Part (ii) gives only a lower bound for the cardinal $\text{Add}(\text{HF}, \text{SZ})$. It is unknown whether $\text{Add}(\text{HF}, \text{SZ}) = \text{A}(\text{SZ})$.

Problem 4. Is $\text{Add}(\text{HF}, \text{SZ})$ equal to $\text{A}(\text{SZ})$ (in ZFC)?

Let us recall here that $\text{A}(\mathcal{F}) > \mathfrak{c}$ for $\mathcal{F} \in \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{SZ}\}$ (see [5]). The precise value of $\text{A}(\mathcal{F})$ may be different in different models of ZFC. It is also known that $\text{A}(\text{PC}) = 2^{\mathfrak{c}}$ and $\text{A}(\text{HF}) = \omega$ (see [11]).

Remark 5. $\text{Add}(\text{AD}, \text{SZ}) \leq \text{Add}(\text{HF}, \text{SZ})$ and $\text{Add}(\text{AD}, \text{HF}) = 1$.

PROOF. To see the above, fix $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$ such that $|\mathcal{F}| < \text{Add}(\text{AD}, \text{SZ})$. Then there exists a $g \in \text{AD}$ such that $g + \mathcal{F} \subseteq \text{SZ}$. Let $h \in \text{HF}$ be a finitely continuous function. Recall from [12] that a function is *finitely continuous* if it is contained in the union of finitely many continuous real functions defined on a subset of \mathbb{R} . Then $h + g \in \text{HF}$ (by [10, Fact 3.1 (i)]) and $(h + g) + \mathcal{F} = h + (g + \mathcal{F}) \subseteq \text{SZ}$. To see $\text{Add}(\text{AD}, \text{HF}) = \text{Add}(\text{HF}, \text{AD}) = 1$ note that $\text{AD} \cap \text{HF} = \emptyset$. The latter follows from the fact that the graph of an additive function is linearly dependent by the definition. \square

Let us comment now on how the class HF of Hamel functions relates to all the other families in terms of inclusion and intersection. It is easy to observe that if $\text{Add}(F_1, F_2) \geq 2$, then $F_1 \cap F_2 \neq \emptyset$. Thus based on the values of Add given by Theorem 3, we conclude that there exists a Hamel function belonging to each of the classes: Ext, AC, Conn, D, PC, and SZ. From [10, Fact 3.1 (iii)] we conclude that $\text{HF} \cap \text{C} = \emptyset$. This shows that none of Ext, AC, Conn, D, or PC is contained in HF. It is obvious that $\text{SZ} \not\subseteq \text{HF}$. Neither is HF contained in SZ. The latter holds because one can construct a Hamel function which is constant on a set of size \mathfrak{c} . This follows easily from the following fact.

Fact 6. Let $X \subseteq \mathbb{R}$ be a set linearly independent over \mathbb{Q} . Then every function $f: X \rightarrow \mathbb{R}$ can be extended onto \mathbb{R} to a Hamel function.

PROOF. Let $g \in \mathbb{R}^{\mathbb{R}}$ be a Hamel function. Define $g': \mathbb{R} \rightarrow \mathbb{R}$ to be an additive extension of $f - (g|_X)$. Notice that $g + g' \in \text{HF}$ by [10, Fact 3.1 (i)]. Since $(g + g')|_X = f$ we conclude that $g + g'$ is a Hamel function extending f . \square

Thus what remains to be determined is whether HF is a subset of one of Ext, AC, Conn, D, or PC. As one might expect this is not the case. Since the class of all peripherally continuous functions contains all the other classes, it

is sufficient to justify that $\text{HF} \not\subseteq \text{PC}$. An example of a Hamel function which is not peripherally continuous can be easily constructed with the use of the following lemma.

Lemma 7. *Let $V \subseteq \mathbb{R}^n$ be a Hamel basis and let $v' \in V$. For each $v \in V$ fix $q_v \in \mathbb{Q}$ such that $q_{v'} \neq -1$. Then the set $V' = \{v + q_v v' : v \in V\}$ is also a Hamel basis.*

PROOF. It is easy to observe that $\text{Lin}_{\mathbb{Q}}(V') = \mathbb{R}^n$. Indeed $V \subseteq \text{Lin}_{\mathbb{Q}}(V')$ since for each $v \in V$ we have $v = (v + q_v v') - \frac{q_v}{1+q_{v'}}(v' + q_{v'} v')$. To see that V' is linearly independent, choose $v_1, \dots, v_k \in V$ and $q_1, \dots, q_k \in \mathbb{Q}$ and assume that $q_1(v_1 + q_{v_1} v') + \dots + q_k(v_k + q_{v_k} v') = 0 \in \mathbb{R}^n$. This implies that $q_1 v_1 + \dots + q_k v_k + q' v' = 0$ for some $q' \in \mathbb{Q}$. If $v_1, \dots, v_k \neq v'$, then obviously $q_1, \dots, q_k = 0$. If one of v_1, \dots, v_k is equal v' (assume $v_k = v'$), then we conclude that $q_1, \dots, q_{k-1} = 0$. This implies that $q_k(v' + q_{v'} v') = 0$. Hence $q_k = 0$ since $q_{v'} \neq -1$. \square

Example 8. There exists a Hamel function $h: \mathbb{R} \rightarrow \mathbb{R}$ which is not peripherally continuous.

PROOF. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be any Hamel function. For each $x \in \mathbb{R} \setminus \{0\}$ choose $q_x \in \mathbb{Q}$ such that $f(x) + q_x f(0) \notin (f(0) - 1, f(0) + 1)$ (note that $f(0) \neq 0$). Now define $h: \mathbb{R} \rightarrow \mathbb{R}$ by $h(0) = f(0)$ and $h(x) = f(x) + q_x f(0)$ for $x \neq 0$. By Lemma 7 h is a Hamel function. Observe also that $\langle 0, h(0) \rangle$ is an isolated point of the graph of h . Hence $h \notin \text{PC}$. \square

Example 9. There exists a Sierpiński-Zygmund Hamel function which is peripherally continuous but not Darboux.

PROOF. Let $h \in \text{SZ} \cap \text{HF}$ ($\text{SZ} \cap \text{HF} \neq \emptyset$ - see the comments following Remark 5). Choose a dense set $A \subseteq \mathbb{R}$ such that $|A| < \mathfrak{c}$ and $h^{-1}(0) \subseteq A$. Such a set exists since $|h^{-1}(0)| < \mathfrak{c}$. Next we use Lemma 7 ($v' = \langle 0, h(0) \rangle$ and $q_{v'} = 0$) to redefine h on the set A so that $h \subseteq \mathbb{R}^2$ is dense and $h^{-1}(0) = \emptyset$. The first condition implies that $h \in \text{PC}$ and the second condition implies that $h \notin \text{D}$. Based on Lemma 7 we have that h is still a Hamel function. Finally, since $|A| < \mathfrak{c}$ we conclude that $h \in \text{SZ}$. \square

2 Proofs of Main Results.

We will start with the proof of Theorem 3. Before proceeding let us restate an analogous theorem for additive functions which was proven in [9].

Theorem 10. [9, Theorem 10] *Let $\mathcal{F} \in \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{PC}, \text{SZ}\}$.*

- (i) Let $\mathcal{F} \in \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{PC}\}$. Then both $\text{Add}(\text{AD}, \mathcal{F}) = \text{A}(\mathcal{F})$ and $\text{Add}(\mathcal{F}, \text{AD}) = \text{A}(\text{AD})$. We also have $\text{Add}(\text{C}, \text{AD}) = \text{Add}(\text{AD}, \text{C}) = 1$.
- (ii) $\text{Add}(\text{SZ}, \text{AD}) = \text{A}(\text{AD})$ and $\text{Add}(\text{AD}, \text{SZ}) > \mathfrak{c}$.

2.1 Proof of Theorem 3.

PROOF. We prove only part (i). The proof of (ii) is very similar. Let $\mathcal{F} \in \{\text{Ext}, \text{AC}, \text{Conn}, \text{D}, \text{PC}\}$. We will show that $\text{Add}(\text{HF}, \mathcal{F}) = \text{A}(\mathcal{F})$. Choose an $F \subseteq \mathbb{R}^{\mathbb{R}}$ such that $|F| < \text{A}(\mathcal{F})$ and choose a function $f \in \text{HF}$. Based on the equality $\text{Add}(\text{AD}, \mathcal{F}) = \text{A}(\mathcal{F})$, there exists a function $g \in \text{AD}$ such that $g + (f + F) \subseteq \mathcal{F}$. Let $f' = g + f$. Observe that $f' \in \text{HF}$ (see [10, Fact 3.1 (i)]) and $f' + F \subseteq \mathcal{F}$. This shows that $\text{Add}(\text{HF}, \mathcal{F}) \geq \text{A}(\mathcal{F})$. The opposite inequality follows from the monotonicity of Add , that is $\text{Add}(\text{HF}, \mathcal{F}) \leq \text{Add}(\mathbb{R}^{\mathbb{R}}, \mathcal{F}) = \text{A}(\mathcal{F})$.

Now we prove $\text{Add}(\mathcal{F}, \text{HF}) = \text{A}(\text{HF})$. It suffices to show $\text{Add}(\mathcal{F}, \text{HF}) \geq \text{A}(\text{HF})$. Choose an $F \subseteq \mathbb{R}^{\mathbb{R}}$ such that $|F| < \text{A}(\text{HF})$. From the definition of A there exists a function $f \in \mathbb{R}^{\mathbb{R}}$ shifting F into HF , e.g. $f + F \subseteq \text{HF}$. Using again the equality $\text{Add}(\text{AD}, \mathcal{F}) = \text{A}(\mathcal{F})$ (recall that $\text{A}(\mathcal{F}) > \mathfrak{c}$) we can find a $g \in \text{AD}$ with the property $g + \{f\} \subseteq \mathcal{F}$. Note that $(g + f) + F = g + (f + F) \subseteq g + \text{HF} = \text{HF}$ and $g + f \in \mathcal{F}$. This proves $\text{Add}(\mathcal{F}, \text{HF}) \geq \text{A}(\text{HF})$.

The equalities $\text{Add}(\text{C}, \text{HF}) = \text{Add}(\text{HF}, \text{C}) = 1$ follow from $\text{HF} \cap \text{C} = \emptyset$ (see [10, Fact 2.3 (iii)]). \square

2.2 Additional Lemmas.

Before we prove Theorems 1 and 2 we will restate some known results and prove an additional lemma. In [13] the author defines a subset of $[0, 1]^2$ which is used to construct an example of a function from $[0, 1]$ to $[0, 1]$ which is connectivity but not almost continuous. In [4] the authors modify this construction to give an example of a function from \mathbb{R} to \mathbb{R} which is connectivity (with some additional properties) but not almost continuous. Specifically they prove the following lemma. Note that the set \bar{C} from the lemma is defined as $\mathbb{Z} + C$, where $C \subseteq [0, 1]$ is a Cantor set of Lebesgue measure $\frac{1}{2}$ (for details see [4, page 4]).

Lemma 11. [4, Lemma 2.1] *Let X be a countable dense subset of $(-1, 1)$. Then there exists an embedding $F = \langle F_0, F_1 \rangle: \mathbb{R} \rightarrow (-1, 1) \times \mathbb{R}$ such that F_0 is non-decreasing and*

- (a) *an open arc $M = F[\mathbb{R}]$ is closed in \mathbb{R}^2 ,*
- (b) *if $Z = F[\bar{C}] \subset M$ then $g \cap Z \neq \emptyset$ for every continuous $g: [-1, 1] \rightarrow \mathbb{R}$,*
- (c) *$Z_x = M_x$ is a singleton for all $x \in (-1, 1) \setminus X$, and*

(d) for each $x \in X$ the section M_x is a non-trivial closed interval and Z_x consists of the two endpoints of that interval.

Let us mention here that the function F from the proof of the above lemma has the following property: the coordinate function F_1 is decreasing on the intervals on which the coordinate function F_0 is constant. Next we will prove an important property of the set Z .

Fact 12. Let K be a compact connected subset of \mathbb{R}^2 with $\text{dom}(K) = \mathbf{c}$. Then

- (a) $\{x\} \times M_x \subseteq K$ for some $x \in X$ or
- (b) $[(I_1 \times I_2) \cap K] \cap Z = \emptyset$ for some non-degenerate intervals I_1 and I_2 such that $\text{dom}[(I_1 \times I_2) \cap K] = I_1$.

PROOF. Let $I_K = \text{dom}(K)$. First observe that either $I_K \setminus \text{dom}(K \cap Z)$ contains a non-degenerate interval or $I_K = \text{dom}(K \cap Z)$. If the first condition holds then obviously (b) is true. So assume that $I_K = \text{dom}(K \cap Z)$. In this case we can conclude that $Z \cap (\text{int}(I_K) \times \mathbb{R}) \subseteq K$. To see this first notice that $Z \cap ((\text{int}(I_K) \setminus X) \times \mathbb{R}) \subseteq K$. Next observe that $Z \cap ((-1, 1) \setminus X) \times \mathbb{R}$ is dense in Z . This follows from the fact that F is continuous ($Z = F[\bar{C}]$) and $F^{-1}[Z \cap ((-1, 1) \setminus X) \times \mathbb{R}]$ is dense in \bar{C} . Finally since Z and K are closed, we conclude that $Z \cap (\text{int}(I_K) \times \mathbb{R}) \subseteq K$.

Now assume that for all $x \in X \cap \text{int}(I_K)$ we have $\{x\} \times M_x \not\subseteq K$, since otherwise the condition (a) holds. Choose $\langle x_0, y_0 \rangle \in M \setminus K$ ($x_0 \in X \cap \text{int}(I_K)$). Since K is closed, there exists $\delta > 0$ such that $((x_0 - \delta, x_0 + \delta) \times \{y_0\}) \cap K = \emptyset$. Next because K is connected we conclude that $\text{dom}[(x_0 - \delta, x_0) \times (-\infty, y_0)) \cap K] = (x_0 - \delta, x_0)$ or $\text{dom}[(x_0, x_0 + \delta) \times (y_0, \infty)) \cap K] = (x_0, x_0 + \delta)$. Indeed if for some $x \in (x_0 - \delta, x_0)$ we have $(\{x\} \times (-\infty, y_0)) \cap K = \emptyset$, then we must have $(\{x'\} \times (y_0, \infty)) \cap K \neq \emptyset$ for every $x' \in (x_0, x_0 + \delta)$. Assume that $\text{dom}[(x_0 - \delta, x_0) \times (-\infty, y_0)) \cap K] = (x_0 - \delta, x_0)$. (In the other case the following argument is very similar.)

Next we claim that there exists $0 < \delta' < \delta$ such that for every $x \in (x_0 - \delta', x_0)$ we have that $M_x \subseteq (y_0, \infty)$. Observe that the claim implies (b), which finishes the proof of the fact. To see the claim assume that there is a sequence $x_n \in \text{int}(I_K)$ such that $x_n \nearrow x_0$ and $M_{x_n} \not\subseteq (y_0, \infty)$. Let $y' = \max M_{x_0}$ and $t' \in (-1, 1)$ be such that $F(t') = \langle x_0, y' \rangle$. Note here that since F_1 is decreasing on the intervals on which F_0 is constant (see the remark after Lemma 11), $F_0(t) < F_0(t') = x_0$ for all $t \in (-1, t')$. Next let $t_n \in (-1, 1)$ be such that $F_0(t_n) = x_n$ and $F_1(t_n) = \min M_{x_n}$. Observe that t_n is an increasing sequence since F_0 is non-decreasing. Hence t_n converges and $\lim t_n = t'$ because otherwise we would have $x_0 = \lim x_n = \lim F_0(t_n) < F(t') = x_0$. Consequently $\lim F_1(t_n) \leq y_0$. On the other hand (from continuity of F_1) we have that $\lim F_1(t_n) = F_1(t') = y' > y_0$, a contradiction. \square

Recall also the following property of extendable functions.

Theorem 13. [4, Theorem 3.1] *If $f: \mathbb{R} \rightarrow \mathbb{R}$ is an extendable function with a dense graph, then for every $a, b \in \mathbb{R}$, $a < b$ and for each Cantor set K between $f(a)$ and $f(b)$ there is a Cantor set C between a and b such that $f[C] \subset K$ and the restriction $f \upharpoonright C$ is continuous strictly increasing.*

2.3 Proof of Theorem 1.

PROOF. $(\text{HF} \cap \text{C}) \subsetneq (\text{HF} \cap \text{Ext})$: This statement easily follows from Theorem 3 (i) and properties of the function Add ($\text{HF} \cap \text{C} = \emptyset$ and $\text{HF} \cap \text{Ext} \neq \emptyset$).

$(\text{HF} \cap \text{Ext}) \subsetneq (\text{HF} \cap \text{AC})$: Denote by $\{B_\alpha \subseteq \mathbb{R}^2 : \alpha < \mathfrak{c}\}$ and $\{C_\alpha \subseteq \mathbb{R} : \alpha < \mathfrak{c}\}$ the collections of all blocking and perfect sets respectively. Also let $\mathbb{R} = \{y_\alpha : \alpha < \mathfrak{c}\}$. Choose an infinite countable dense set $A \subseteq \mathbb{R}$ and sequences $a_\alpha \in C_\alpha, b_\alpha \in C_\alpha, c_\alpha \in \text{dom}(B_\alpha)$ such that the elements of the set A and terms of these sequences are all linearly independent over \mathbb{Q} . The choice is possible since $|C_\alpha| = |\text{dom}(B_\alpha)| = \mathfrak{c}$. Now define a function h as follows: $h|A$ is a dense subset of \mathbb{R}^2 , $h(a_\alpha) = h(b_\alpha)$, and $\langle c_\alpha, h(c_\alpha) \rangle \in B_\alpha$. Next extend h onto \mathbb{R} to a Hamel function by using Fact 6. The function is almost continuous because it intersects every blocking set and is not extendable by Theorem 13.

$(\text{HF} \cap \text{AC}) \subsetneq (\text{HF} \cap \text{Conn})$: Let $\{K_\alpha : \alpha < \mathfrak{c}\}$ be the collection of all compact connected subsets of \mathbb{R}^2 such that $\text{dom}(K_\alpha) = \mathfrak{c}$ and $\{x\} \times M_x \not\subseteq K_\alpha$ for all $x \in X$, where $X \subseteq (-1, 1)$ is a countable linearly independent (over \mathbb{Q}) set which is dense in $(-1, 1)$ and M is the set from Lemma 11 for this X . Construct a linearly independent set $H = \{h_\alpha : \alpha < \mathfrak{c}\} \subseteq \mathbb{R}$ satisfying the following conditions: $\text{Lin}_{\mathbb{Q}}(X) \cap \text{Lin}_{\mathbb{Q}}(H) = \{0\}$ and $(\{h_\alpha\} \times \mathbb{R}) \cap (K_\alpha \setminus Z) \neq \emptyset$ for every $\alpha < \mathfrak{c}$. The existence of such a set follows from Fact 12. Now define $h: \mathbb{R} \rightarrow \mathbb{R}$ by defining it on $H \cup X$ as $\langle x, h(x) \rangle \in M_x \setminus Z_x$ for $x \in X$ and $\langle h_\alpha, h(h_\alpha) \rangle \in (K_\alpha \setminus Z)_{h_\alpha}$ and then extending it to a Hamel function on \mathbb{R} by using Fact 6. Notice that $[h|(H \cup X)] \cap Z = \emptyset$. Now define $h': \mathbb{R} \rightarrow \mathbb{R}$ by modifying h on the set $\text{dom}(h \cap Z)$ as follows: $h'(x) = h(x) + h(0)$. We conclude that $h' \cap Z = \emptyset$ and $h' \cap K \neq \emptyset$ for each compact connected set $K \subseteq \mathbb{R}^2$ such that $\text{dom}(K) = \mathfrak{c}$. The latter implies that $h' \in \text{Conn}$. Based on the condition $h' \cap Z = \emptyset$, we conclude that $h' \notin \text{AC}$. Indeed consider the open set $\mathbb{R}^2 \setminus Z$. It contains h' but does not contain any continuous function since by Lemma 11 (b), Z intersects every continuous function. By Lemma 7 we have that $h' \in \text{HF}$.

$(\text{HF} \cap \text{Conn}) \subsetneq (\text{HF} \cap \text{D}) \subsetneq (\text{HF} \cap \text{PC})$: Let $H \subseteq \mathbb{R}$ be a Hamel basis which is \mathfrak{c} -dense. Define $h: H \rightarrow \mathbb{R}$ such that $h^{-1}(y)$ is dense in \mathbb{R} for every $y \in \mathbb{R}$. Next extend h onto \mathbb{R} to a Hamel function. Now we will define h_1 as follows. Set $h_1|(\mathbb{R} \setminus h^{-1}(1)) \equiv h|(\mathbb{R} \setminus h^{-1}(1))$ and $h_1(x) = h(x) + h(0)$

for all $x \in h^{-1}(1)$. Observe that h_1 is a Hamel function (by Lemma 7), $h_1 \subseteq \mathbb{R}^2$ is dense, and $h_1^{-1}(1) = \emptyset$. Hence $h_1 \in \text{PC} \setminus \text{D}$. To define h_2 we redefine h on the set $E = \text{dom}(h \cap \{(x, x) : x \in \mathbb{R}\})$ in a similar fashion, that is $h_2|(\mathbb{R} \setminus E) \equiv h|(\mathbb{R} \setminus E)$ and $h_2(x) = h(x) + h(0)$ for all $x \in E$. Similarly, we note that $h_2 \in \text{HF}$ by Lemma 7. In addition, $h_2^{-1}(y)$ is dense in \mathbb{R} for every $y \in \mathbb{R}$. Hence $h_2 \in \text{D}$. Finally since $h_2 \cap \{(x, x) : x \in \mathbb{R}\} = \emptyset$, we conclude that $h_2 \in \text{D} \setminus \text{Conn}$. \square

2.4 Proof of Theorem 2.

PROOF. Let $\mathcal{G} = \{g_\alpha : \alpha < \mathfrak{c}\}$ be the set of all continuous functions defined on G_δ subsets of $\mathbb{R} = \{x_\alpha : \alpha < \mathfrak{c}\}$. Choose a countable dense set $X \subseteq (-1, 1)$ which is linearly independent over \mathbb{Q} . Let Z and M be as in Lemma 11 for the set X . In addition let us denote the collection of all continua in \mathbb{R}^2 with uncountably many uncountable vertical sections by $\mathcal{K} = \{K_\alpha : \alpha < \mathfrak{c}\}$ (i.e. $|\{x \in \mathbb{R} : |(K_\alpha)_x| = \mathfrak{c}\}| = \mathfrak{c}$). We will define by induction a sequence of partial functions h_α ($\alpha < \mathfrak{c}$) such that:

- (0) $X \subseteq \text{dom}(h_\alpha)$ and $h_\alpha|X \subseteq M \setminus Z$;
- (i) $h_\beta \subseteq h_\xi$ for $\beta < \xi$;
- (ii) $x_\alpha \in \text{dom}(h_\alpha)$;
- (iii) $\langle 0, x_\alpha \rangle \in \text{Lin}_{\mathbb{Q}}(h_\alpha)$;
- (iv) $h_\alpha \subseteq \mathbb{R}^2$ is linearly independent over \mathbb{Q} ;
- (v) $(g_\alpha \cap h_\xi) \subseteq h_\alpha$, for $\alpha < \xi$;
- (vi) if $\text{dom}(g_\alpha \setminus \bigcup_{\xi < \alpha} g_\xi)$ is of second category, then h_α is dense in $(g_\alpha \setminus \bigcup_{\xi < \alpha} g_\xi)|U_\alpha$, where $U_\alpha \subseteq \mathbb{R}$ is the maximal open set such that $\text{dom}(g_\alpha \setminus \bigcup_{\xi < \alpha} g_\xi)$ is residual in U_α ;
- (vii) $h_\alpha \cap (K_\alpha \setminus Z) \neq \emptyset$.

First we present the construction of the sequence h_α ($\alpha < \mathfrak{c}$). Define h_0 on X such that $h(x) \in M_x \setminus Z_x$. Next if $x_0 \notin \text{dom}(h_0)$, then choose $h_0(x_0) \in \mathbb{R} \setminus \text{Lin}_{\mathbb{Q}}(h_0(X))$. If $\langle 0, x_0 \rangle \notin \text{Lin}_{\mathbb{Q}}(h_0)$, choose $z \notin \text{Lin}_{\mathbb{Q}}(\text{dom}(h_0))$ and define $h_0(z) = h_0(-z) = \frac{1}{2}x_0$. Let U_0 be the maximal open set such that $\text{dom}(g_0)$ is residual in U_0 . Choose a countable linearly independent dense subset $D_0 \subseteq (\text{dom}(g_0) \cap U_0) \setminus \text{Lin}_{\mathbb{Q}}(\text{dom}(h_0))$ and set $h_0|D_0 \equiv g_0|D_0$. Finally, choose $w \in \{x \in \mathbb{R} : |(K_0)_x| = \mathfrak{c}\} \setminus \text{Lin}_{\mathbb{Q}}(\text{dom}(h_0))$ and define $h_0(w) \in (K_0)_w \setminus Z_w$. It is easy to see that h_0 satisfies all the conditions (0)-(vii).

Now assume that the sequence h_ξ has been defined for $\xi < \alpha$. Set $h_\alpha = \bigcup_{\xi < \alpha} h_\xi$. If $x_\alpha \notin \text{dom}(h_\alpha)$, then choose $h_\alpha(x_\alpha) \in \mathbb{R} \setminus \text{Lin}_{\mathbb{Q}}(h_\alpha(\text{dom}(h_\alpha)) \cup \bigcup_{\xi < \alpha} \{g_\xi(x_\alpha)\})$. If $\langle 0, x_\alpha \rangle \notin \text{Lin}_{\mathbb{Q}}(h_\alpha)$, choose $z \notin \text{Lin}_{\mathbb{Q}}(\text{dom}(h_\alpha))$ and define $h_\alpha(z)$ and $h_\alpha(-z)$ so that $h_\alpha(z) + h_\alpha(-z) = x_\alpha$ and $h_\alpha(\pm z) \notin \bigcup_{\xi < \alpha} \{g_\xi(\pm z)\}$.

Now consider the set U_α , the maximal open set in which $\text{dom}(g_\alpha \setminus \bigcup_{\xi < \alpha} g_\xi)$ is residual. As in the case of h_0 , we will select a countable linearly independent

dense subset $D_\alpha \subseteq (\text{dom}(g_\alpha \setminus \bigcup_{\xi < \alpha} g_\xi) \cap U_0) \setminus \text{Lin}_\mathbb{Q}(\text{dom}(h_\alpha))$ and set $h_\alpha|_{D_\alpha} \equiv g_\alpha|_{D_\alpha}$. Finally choose $w \in \{x \in \mathbb{R} : |(K_\alpha)_x| = \mathfrak{c}\} \setminus \text{Lin}_\mathbb{Q}(\text{dom}(h_\alpha))$ and define $h_\alpha(w) \in (K_\alpha)_w \setminus Z_w$. It is easy to see that h_α satisfies all the conditions (0)-(vii).

Define $h = \bigcup_{\alpha < \mathfrak{c}} h_\alpha$. The function h will serve as a starting point for functions justifying each of the parts of the theorem. Obviously $\text{dom}(h) = \mathbb{R}$. Also notice that $h \in \text{HF}$ based on conditions (i), (iii), and (iv). Condition (v) implies that $h \in \text{SZ}$.

$(\text{SZ} \cap \text{HF} \cap \text{Ext}) \not\subseteq (\text{SZ} \cap \text{HF} \cap \text{AC})$: We will argue that h is almost continuous (no extendability function can be in SZ). Let $B \subseteq \mathbb{R}^2$ be any blocking set. There exists a non-degenerate interval $I \subseteq \text{dom}(B)$ and a continuous function g such that $\text{dom}(g)$ is a G_δ dense subset of I and $g \subseteq B$. Let α_0 be the smallest ordinal number with this property; i.e. there exists a non-degenerate interval $I \subseteq \text{dom}(B)$ and a continuous function contained in B and defined on a residual set in I . Then $\text{dom}(g_{\alpha_0} \setminus \bigcup_{\xi < \alpha_0} g_\xi)$ is of second category (since we assume CH). Therefore the open set U_{α_0} is not empty and consequently $D_{\alpha_0} \neq \emptyset$. Hence $(h|_{D_{\alpha_0}}) \cap B = (g_{\alpha_0}|_{D_{\alpha_0}}) \cap B \neq \emptyset$. This implies that $h \in \text{AC}$.

$(\text{SZ} \cap \text{HF} \cap \text{AC}) \not\subseteq (\text{SZ} \cap \text{HF} \cap \text{Conn})$: Consider the set $E_Z = \{x : \langle x, h(x) \rangle \in Z\}$. The properties of the set Z and the fact that $h \in \text{SZ}$ imply that $|E_Z| < \mathfrak{c}$. We will define h_b by redefining h on the set E_Z (that will keep h_b in the class SZ). Set $h_b|_{(\mathbb{R} \setminus E_Z)} \equiv h|_{(\mathbb{R} \setminus E_Z)}$ and $h_b(x) = h(x) + h(0)$ for $x \in E_Z$. By Lemma 7, $h_b \in \text{HF}$. Since $h_b \cap Z = \emptyset$ we have that $h_b \notin \text{AC}$. What remains to show is that $h_b \in \text{Conn}$. To see that, fix a continua $K \subseteq \mathbb{R}^2$ with $|\text{dom}(K)| = \mathfrak{c}$. Then by Fact 12, (a) $\{x\} \times M_x \subseteq K$ for some $x \in X$ or (b) $[(I_1 \times I_2) \cap K] \cap Z = \emptyset$ for some non-degenerate intervals I_1 and I_2 such that $\text{dom}[(I_1 \times I_2) \cap K] = I_1$ (we may assume that I_1, I_2 are closed). If (a) holds, then by condition (0) $h_b \cap K \neq \emptyset$. Now assume that (b) holds. Let $A = \{x \in \mathbb{R} : |K_x| = \mathfrak{c}\}$. If $|A| = \mathfrak{c}$, then by condition (vii) $h_b \cap K \neq \emptyset$. Suppose that $|A| = \omega$. Note that the set $[(I_1 \setminus A) \times I_2] \cap K$ is a Borel set with each vertical section countable. Hence by Lusin Theorem (see [14, Theorem 5.7.2, page 205]), there is a Borel function $g : (I_1 \setminus A) \rightarrow \mathbb{R}$ contained in $(I_1 \times I_2) \cap K$. This implies the existence of a continuous function $g' \subseteq (I_1 \times I_2) \cap K$ defined on a dense G_δ subset of I_1 . Let α_0 be the smallest ordinal number with the property that $g_{\alpha_0}|_{I_1} \subseteq (I_1 \times I_2) \cap K$ and $\text{dom}(g_{\alpha_0})$ is residual in some non-degenerate interval $I \subseteq I_1$. Then $\text{dom}(g_{\alpha_0} \setminus \bigcup_{\xi < \alpha_0} g_\xi)$ is of second category (since we assume CH). Therefore the open set U_{α_0} is not empty and consequently $D_{\alpha_0} \neq \emptyset$. Hence $(h_b|_{D_{\alpha_0}}) \cap [(I_1 \times I_2) \cap K] = (h_b|_{D_{\alpha_0}}) \cap [(I_1 \times I_2) \cap K] = (g_{\alpha_0}|_{D_{\alpha_0}}) \cap [(I_1 \times I_2) \cap K] \neq \emptyset$.

$(\text{SZ} \cap \text{HF} \cap \text{Conn}) \not\subseteq (\text{SZ} \cap \text{HF} \cap \text{D})$: To construct a function witnessing the above property we redefine h on the set $E = \text{dom}(h \cap \{\langle x, x \rangle : x \in \mathbb{R}\})$.

Set $h_c|(\mathbb{R} \setminus E) \equiv h|(\mathbb{R} \setminus E)$ and $h_c(x) = h(x) + h(0)$ for all $x \in E$. We note that $h_c \in \text{HF} \cap \text{SZ}$. In addition $h_c^{-1}(y)$ is dense in \mathbb{R} for every $y \in \mathbb{R}$. Hence $h_c \in \text{D}$. Finally since $h_c \cap \{\langle x, x \rangle : x \in \mathbb{R}\} = \emptyset$, we conclude that $h_c \in (\text{SZ} \cap \text{HF} \cap \text{D}) \setminus \text{Conn}$. \square

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