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## A FRACTAL PLANCHEREL THEOREM

### Abstract

A measure  $\mu$  on  $\mathbb{R}^n$  is called locally and uniformly  $h$ -dimensional if  $\mu(B_r(x)) \leq h(r)$  for all  $x \in \mathbb{R}^n$  and for all  $0 < r < 1$ , where  $h$  is a real valued function. If  $f \in L^2(\mu)$  and  $\mathcal{F}_\mu f$  denotes its Fourier transform with respect to  $\mu$ , it is not true (in general) that  $\mathcal{F}_\mu f \in L^2$  (e.g. [10]). However in this paper we prove that, under certain hypothesis on  $h$ , for any  $f \in L^2(\mu)$  the  $L^2$ -norm of its Fourier transform restricted to a ball of radius  $r$  has the same order of growth as  $r^n h(r^{-1})$  when  $r \rightarrow \infty$ . Moreover we prove that the ratio between these quantities is bounded by the  $L^2(\mu)$ -norm of  $f$  (Theorem 3.2). By imposing certain restrictions on the measure  $\mu$ , we can also obtain a lower bound for this ratio (Theorem 4.3). These results generalize the ones obtained by Strichartz in [10] where he considered the particular case in which  $h(x) = x^\alpha$ .

### 1 Introduction.

We will say that a measure  $\mu$  is locally and uniformly  $h$ -dimensional (or shortly  $\mu$  is an  $h$ -dimensional measure) if and only if

$$\mu(B_r(x)) \leq h(r) \quad \forall x \in \mathbb{R}^n, \forall 0 < r < 1, \quad (1.1)$$

where  $B_r(x)$  is, as usual, the ball of radius  $r$  centered at  $x$ . We consider functions  $h : [0, +\infty] \rightarrow \mathbb{R}$  that are non-decreasing, continuous, and such that

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$h(0) = 0$ . We further require  $h$  to be *doubling*; i.e. there exists a constant  $c > 0$  such that  $h(2x) < ch(x)$ . Such a function  $h$  will be called *dimension function*. A particular example is  $h(x) = x^\alpha$ , which was analyzed by Strichartz in [10]. In that case we will say indistinctly that  $\mu$  is  $h$ -dimensional or that  $\mu$  is  $\alpha$ -dimensional. Allowing  $h$  to be more general has already proven to be useful (see for example [9],[8], [3]) and it enables us to obtain a lower bound on measures which were not included in previous results (see Section 5).

If  $\mu$  is locally and uniformly 0-dimensional, meaning that the measure of any ball of radius one is bounded, then each  $f \in L^2(\mu)$  defines a tempered distribution, mapping each test function  $\varphi$  in the Schwartz space  $\mathcal{S}$  into  $\int f\varphi d\mu$ . Therefore its Fourier transform is also a tempered distribution defined by  $\varphi \mapsto \int \hat{\varphi}f d\mu$  for  $\varphi \in \mathcal{S}$ , where  $\hat{\varphi}$  is the usual Lebesgue Fourier transform. We will denote by  $\mathcal{F}_\mu f$  this ‘distributional’ Fourier transform of an  $f \in L^2(\mu)$ . If  $f \in L^1(\mu) \cap L^2(\mu)$ , then it is easy to see that  $\mathcal{F}_\mu f(\xi) = \int f(x)e^{i\xi x} d\mu(x)$ . See for example [2].

Strichartz proved in [10] that if  $f \in L^2(\mu)$  and  $\mu$  is zero dimensional then  $\mathcal{F}_\mu f$  belongs to  $L^2(e^{-t|\xi|^2})$  for any  $t > 0$  and therefore to  $L^2_{loc}(\mathbb{R}^d)$ . Note that if  $h$  is one of our dimension functions, we have immediately that  $\mu$  is 0-dimensional.

In this paper, our goal is to prove an analogue to Plancherel’s Theorem (in  $L^2(\mathbb{R}^n)$  with the Lebesgue measure) for any  $h$ -dimensional measure  $\mu$ . In fact we are going to show the existence of upper and lower bounds for the ratio between  $r^n h(r^{-1})$  and the norm of the Fourier transform of a function  $f$  in  $L^2(\mu)$  restricted to the ball of radius  $r$ . The hypotheses under which we obtain the existence of the upper bound are more general than the ones we need for the existence of the lower bound.

The  $h$ -dimensional Hausdorff measure is defined as (see for example [9])

$$\mathcal{H}^h(E) = \lim_{\delta \rightarrow 0} \left( \inf \left\{ \sum_{i=1}^{\infty} h(|U_i|) : E \subset \bigcup_{i \geq 1} U_i \text{ and } |U_i| \leq \delta \right\} \right).$$

$\mathcal{H}^h_{\lfloor E}$  will denote its restriction to a set  $E$ .

The  $h$ -lower density of a set  $E$  in  $x$  is (see for example [4])

$$\underline{D}(\mathcal{H}^h_{\lfloor E}, x) = \liminf_{r \rightarrow 0} \frac{\mathcal{H}^h(E \cap B_r(x))}{h(2r)}. \quad (1.2)$$

The upper density is defined by taking lim sup in the above equation. We will introduce one additional definition.

**Definition 1.1.** A set  $E$  will be said to be an  $h$ -regular set if both upper and lower densities are equal to one in  $\mathcal{H}^h$ -almost every point of  $E$ . In symbols

$\underline{D}(\mathcal{H}^h \llcorner_E, x) = \overline{D}(\mathcal{H}^h \llcorner_E, x) = 1$  for  $\mathcal{H}^h$ -almost every point of  $E$ . If the lower density is greater than a positive constant for  $\mathcal{H}^h$ -almost every point of  $E$  we will say that  $E$  is an  $h$ -quasi regular set.

It is evident that the requirement for a set to be regular is more restrictive than the one to be quasi regular. Actually it has already been proven (see [8]) that there only exist regular sets for functions of the form  $x^k$  with  $k$  integer. On the other hand, there are  $h$ -regular sets for any dimensional function  $h$ .

The lower bound that we obtain (see Theorem 4.2) will be stated for the measure  $\mathcal{H}^h$  restricted to an  $h$ -dimensional and quasi regular set. In section 5 we will show an example of a set  $E$  and a function  $h$  such that  $\mathcal{H}^h \llcorner_E$  is  $h$ -dimensional and  $E$  is quasi regular. Additionally we will prove that there does not exist any  $\alpha$  such that  $\mathcal{H}^{\alpha \llcorner_E}$  is  $x^\alpha$ -dimensional and  $E$  is quasi regular simultaneously. This example satisfies the hypothesis of our Theorem 4.1 but does not satisfy the hypothesis of the analogous Theorem 5.5 in [10].

## 2 Some Technical Results.

Any  $h$ -dimensional measure  $\mu$  is locally finite, which means that for  $\mu$ -almost every  $x$  there exists an  $r > 0$  such that  $0 < \mu(B_r(x)) < \infty$ . Therefore, as Strichartz proved in [10], the strong  $(p, p)$  estimate (for  $p > 1$ ) and the weak  $(1, 1)$  estimate hold for the maximal operator, defined for each  $f \in L^1_{loc}(\mu)$  as

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} |f| d\mu. \quad (2.1)$$

More precisely, we have the following theorem.

**Theorem 2.1.** *Let  $\mu$  be a locally finite measure on  $\mathbb{R}^n$ . For each locally integrable function  $f$  we have:*

1.  $\mu(\{x : M_\mu f(x) > s\}) \leq \frac{c_n}{s} \|f\|_1 \quad \forall f \in L^1(\mu).$
2. For  $1 < p \leq \infty$ ,  $\|M_\mu f\|_p \leq c_p \|f\|_p \quad \forall f \in L^p(\mu).$

This theorem has many consequences which will be useful for our work. In particular, we have the following two corollaries.

**Corollary 2.2.** *Let  $f \in L^1(\mu)$ . For  $\mu$ -almost every  $x$ ,  $\lim_{r \rightarrow 0} \frac{1}{\mu(B_r(x))} \int_{B_r(x)} f d\mu = f(x).$*

**Corollary 2.3.** *Let  $E$  be a  $h$ -regular set and let  $f \in L^1(\mu)$ . For  $\mathcal{H}^h$ -almost every  $y \in E$  and for all  $\varepsilon > 0$  there exists  $\delta > 0$  such that*

$$\left| \int_{B_r(y)} f d\mu - h(r)f(y) \right| \leq \varepsilon h(r) \quad \forall r \leq \delta.$$

The proofs of the theorem and these corollaries are straightforward applications of Besicovitch's Covering Theorem and can be found in [10].

We also need the following quite technical lemma, which will allow us to bound the ratio between  $h$  and its dilation by  $r$  ( $h(rt)/h(t)$ ) by a function in the weighted space  $L^1(e^{-cr^2})$ .

**Lemma 2.4.** *Let  $h : [0, \infty) \rightarrow \mathbb{R}$  be a continuous, non-decreasing, and doubling function ( $h(2x) \leq c_d h(x)$ ). Then there exists a constant  $\kappa > 0$  such that  $h(rt) \leq c_d h(t) \max\{1, r^\kappa\} \forall r, t > 0$ .*

PROOF. First note that  $c_d \geq 1$ , since in fact the doubling condition can be restated as  $c_d \geq h(2x)/h(x)$  and this quantity is not smaller than 1 because  $h$  is a non-decreasing function.

If  $r < 1$ , since  $h$  is non-increasing, we have  $h(rt) \leq h(t)$ . If  $r \geq 1$  we choose the only non-negative integer  $k$  such that  $2^{k-1} < r \leq 2^k$ . So  $h(rt) \leq h(2^k t) \leq c_d^k h(t)$ . Observe that  $k$  was chosen such that  $k \leq \frac{\log r}{\log 2} + 1$  and therefore it then follows that  $c_d^k \leq c_d \cdot r^{\log c_d / \log 2}$ . The proof is complete by taking  $\kappa = \log c_d / \log 2$ .  $\square$

Recall that we are dealing with  $h$ -dimensional measures, which means that the measure of the balls of radius  $r < 1$  is bounded. The next lemma provides a control of the measure of the "large" balls, i.e. those balls of radius greater than one for which the estimate (1.1) does not hold.

**Lemma 2.5.** *Let  $\mu$  be a locally  $h$ -dimensional measure on  $\mathbb{R}^n$ . If  $r > 1$ , then  $\mu(B_r(x)) \leq Cr^n$  for some  $C$  independent of  $x$ .*

PROOF. Denote by  $Q$  the minimal cube centered at  $x$  that contains the ball  $B_r(x)$ ; i.e.  $Q = Q(x, r) = \{y \in \mathbb{R}^n : \|x - y\|_\infty < r\} \supset B_r(x)$ . Let  $k$  be the (unique) integer such that  $k - 1 < r\sqrt{n} \leq k$ .  $Q$  can be divided into  $k^n$  smaller cubes of half-side  $\frac{r}{k}$ . Each of these cubes is contained in a ball of radius  $r_0 = \sqrt{n}\frac{r}{k} \leq 1$ . So we obtain  $\mu(B_r(x)) \leq \mu(Q) \leq k^n \mu(B_{r_0}(x')) \leq k^n h(\frac{\sqrt{n}r}{k})$ . Since  $\sqrt{n}\frac{r}{k} \leq 1$ , it follows that  $h(\sqrt{n}\frac{r}{k}) \leq h(1)$ . On the other hand, by the choice of  $k$ ,  $k^n < (r\sqrt{n} + 1)^n \leq r^n(\sqrt{n} + 1)^n$  and we obtain  $\mu(B_r(x)) \leq (\sqrt{n} + 1)^n h(1) r^n$ .  $\square$

### 3 Upper Bounds.

Our first result is an upper estimate for the  $L^2$ -norm of the Fourier transform of a function  $f \in L^2(\mu)$ .

**Theorem 3.1.** *Let  $\mu$  be a locally and uniform  $h$ -dimensional measure, where  $h$  is a dimension function. Suppose that  $h$  defines a dimension not greater than  $n$  in the sense that  $\lim_{t \rightarrow 0} t^n/h(t) = 0$ . Then*

$$\sup_{0 \leq t \leq 1} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 d\xi \leq c \|f\|_2^2 := c \int |f|^2 d\mu \quad \forall f \in L^2(\mu).$$

PROOF. *First Step.* We will prove that

$$\sqrt{t^n} \int |\mathcal{F}_\mu f(\xi)|^2 e^{-t|\xi|^2} d\xi = \pi^{n/2} \iint e^{-|x-y|^2/4t} f(x) \overline{f(y)} d\mu(x) d\mu(y). \quad (3.1)$$

Recall the inverse Fourier transform for the Gaussian function  $\int e^{-t|\xi|^2} e^{ix\xi} d\xi = \sqrt{t^{-n}} \pi^{n/2} e^{-|x|^2/4t}$ . If  $f$  is integrable, then equation (3.1) follows from Fubini's Theorem since

$$\begin{aligned} \sqrt{t^n} \int |\mathcal{F}_\mu f(\xi)|^2 e^{-t|\xi|^2} d\xi &= \sqrt{t^n} \iiint f(x) \overline{f(y)} e^{i(x-y)\cdot\xi} e^{-t|\xi|^2} d\mu(x) d\mu(y) d\xi \\ &= \pi^{n/2} \iint e^{-|x-y|^2/4t} f(x) \overline{f(y)} d\mu(x) d\mu(y). \end{aligned} \quad (3.2)$$

Now consider any  $f \in L^2(\mu)$  (not necessarily integrable). Let us define  $f_k(x) = f(x) \chi_{\{|x| \leq k\}}(x) \chi_{\{|f(x)| \leq k\}}(x)$ . This sequence converges to  $f$  in  $L^2(\mu)$ . Also since each  $f_k$  is in  $L^1(\mu)$ , it satisfies (3.2). Using Beppo Levi's Theorem, we get

$$\iint e^{-|x-y|^2/4t} f_k(x) \overline{f_k(y)} d\mu(x) d\mu(y) \rightarrow \iint e^{-|x-y|^2/4t} f(x) \overline{f(y)} d\mu(x) d\mu(y).$$

Since  $f \in L^2(\mu)$ , it follows that  $\mathcal{F}_\mu f \in L^2(e^{-t|\xi|^2} d\xi)$ . Hence we can apply the dominated convergence theorem to obtain  $\sqrt{t^n} \int |\mathcal{F}_\mu f_k(\xi)|^2 e^{-t|\xi|^2} d\xi \rightarrow \sqrt{t^n} \int |\mathcal{F}_\mu f(\xi)|^2 e^{-t|\xi|^2} d\xi$ , which yields 3.1.

*Second Step.* We will prove that for any  $y \in \mathbb{R}^n$  and  $f \in L^2(\mu)$ ,

$$\frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\mu(x) \leq C M_\mu f(y).$$

Using Fubini on the left hand side of the inequality, we obtain

$$\int e^{-|x-y|^2/4t} f(x) d\mu(x) = \int_0^\infty \frac{r}{2t} e^{-r^2/4t} \int_{B_r(y)} f(x) d\mu(x).$$

Since  $\int_{B_r(y)} f(x) d\mu(x) \leq \mu(B_r(y))M_\mu f(y)$ , it follows that

$$\int e^{-|x-y|^2/4t} f(x) d\mu(x) \leq M_\mu f(y) \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr. \quad (3.3)$$

We need to prove that the last integral is finite. To establish that, we split the integral into two parts, the first for  $r < 1$  (where (1.1) is valid) and the second for  $r \geq 1$  (where Lemma 2.5 can be applied). For  $r < 1$  we use the hypothesis to obtain

$$\int_0^1 e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr \leq \int_0^1 e^{-r^2/4t} \frac{r}{2t} h(r) dr = \frac{1}{2} \int_0^{1/\sqrt{t}} e^{-r^2/4} r h(r\sqrt{t}) dr$$

or equivalently

$$\frac{1}{h(\sqrt{t})} \int_0^1 e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr \leq \frac{1}{2} \int_0^{1/\sqrt{t}} e^{-r^2/4} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} dr.$$

This integral is finite by Lemma 2.4.

For  $r \geq 1$ ,

$$\int_1^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr \leq \int_1^\infty e^{-r^2/4t} \frac{r}{2t} r^n dr = \frac{1}{2} \sqrt{t^n} \int_{1/\sqrt{t}}^\infty e^{-r^2/4} r^{n+1} dr.$$

Since  $\lim_{t \rightarrow 0} t^n/h(t) = 0$ , we deduce that  $\sqrt{t^n}/h(\sqrt{t}) \leq C$  and therefore  $\frac{1}{h(\sqrt{t})} \int_1^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr \leq C$ , with  $C$  independent of  $t$ . This completes the second step of our proof.

*Third (and Final) Step.* We will now prove the thesis. Using the first and second steps we obtain

$$\begin{aligned} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int |\mathcal{F}_\mu f(\xi)|^2 e^{-t|\xi|^2} d\xi &= \pi^{n/2} \int \left( \int e^{-|x-y|^2/4t} f(x) d\mu(x) \right) f(y) d\mu(y) \\ &\leq C \int M_\mu f(y) |f(y)| d\mu(y). \end{aligned} \quad (3.4)$$

The last term is the inner product in the Hilbert space  $L^2(\mu)$ . Thus we can bound it using Cauchy-Schwartz. The  $L^2(\mu)$  norm of  $M_\mu f$  can be bounded by using the (2, 2) estimate in (2.1). Then  $\int M_\mu f(y) |f(y)| d\mu(y) \leq C \|f\|_2^2$  and this, together with (3.4), gives the desired result.  $\square$

**Theorem 3.2.** *Under the hypothesis of Theorem 3.1, for each  $f \in L^2(\mu)$  we have*

$$\sup_{x \in \mathbb{R}^n} \sup_{r \geq 1} \frac{1}{r^n h(r^{-1})} \int_{B_r(x)} |\mathcal{F}_\mu f(\xi)|^2 d\xi \leq C \|f\|_2^2.$$

PROOF. We need to show that for each  $x \in \mathbb{R}^n$ ,

$$\sup_{r \geq 1} \frac{1}{r^n h(r^{-1})} \int_{B_r(x)} |\mathcal{F}_\mu f(\xi)|^2 d\xi \leq C \|f\|_2^2. \quad (3.5)$$

Making the substitution  $t = r^{-2}$  in Theorem 3.1, we obtain exactly (3.5) for  $B_r(0)$ . Furthermore

$$\int_{B_r(x)} |\mathcal{F}_\mu f(\xi)|^2 d\xi = \int_{B_r(0)} |\mathcal{F}_\mu(e^{ix\xi} f)|^2 d\xi \leq C \|e^{ix\xi} f\|_2^2 = C \|f\|_2^2,$$

which yields the theorem.  $\square$

This theorem provides an upper bound but does not tell us whether the limit for  $r \rightarrow \infty$  exists or not. With our definition, if a measure is  $h$ -dimensional it is also  $g$ -dimensional for any  $h \leq g$ . For example, if  $h(x) \geq x^n$ , then the measure  $\mu = \mathcal{H}^h \llcorner_E + \mathcal{L}$  is an  $h$ -dimensional measure. Here  $\mathcal{L}$  is the  $n$ -dimensional Lebesgue measure and  $E$  is a set of  $\mathcal{H}^h$  finite measure. However in this case it is clear that  $\mu$  has two distinct parts. One ‘truly’ is  $h$ -dimensional ( $\mathcal{H}^h \llcorner_E$ ), but the other ( $\mathcal{L}$ ), while by the previous remark can be considered as  $h$ -dimensional, is in fact  $n$ -dimensional.

The next theorem will allow us to split up our measure in order to separate the part of the measure that is ‘exactly’  $h$ -dimensional from the one that can also be seen as having bigger dimension.

**Definition 3.3.** We say that a measure  $\nu$  is null with respect to (another measure)  $\mu$  if and only if  $\mu(E) < \infty \Rightarrow \nu(E) = 0$ . We will denote this by  $\nu \lll \mu$ .

Now we will prove a theorem that is analogous to the Radon-Nikodym Theorem.

**Theorem 3.4.** *Let  $\mu$  a measure on  $\mathbb{R}^n$  without infinitely many atoms and let  $\nu$  be a  $\sigma$ -finite measure on  $\mathbb{R}^n$  absolutely continuous (null) with respect to  $\mu$ . There exists a unique decomposition of  $\nu$ :  $\nu = \nu_1 + \nu_2$ , where  $\nu_1(E) = \int_E f d\mu$  for some measurable and nonnegative function  $f$ , with  $\nu_2 \lll \mu$ .*

PROOF. *Uniqueness.* Let us suppose we have a decomposition  $\nu = \nu_1 + \nu_2$  with  $\nu_1(E) = \int_E f d\mu$  and  $\nu_2 \lll \mu$ . Consider  $E \subset \mathbb{R}^n$ . Let us analyze separately both cases; i.e. when  $E$  is  $\sigma$ -finite for  $\mu$  and when it is not.

If  $E$  is  $\sigma$ -finite for  $\mu$ , then  $E = \cup_{j \geq 1} E_j$  with  $\mu(E_j) < \infty$ . Since  $\nu_2 \lll \mu$  we have  $\nu_2(E_j) = 0$  for all  $j \geq$  and therefore  $\nu_2(E) = 0$ , which gives  $\nu_1(E) = \nu(E)$ . If we have any other decomposition  $\nu = \nu'_1 + \nu'_2$ , then  $\nu'_2(E) = 0 = \nu_2(E)$  and  $\nu'_1(E) = \nu(E) = \nu_1(E)$ .

If  $E$  is not  $\sigma$ -finite for  $\mu$ , then  $\nu_2$  may be positive. However by hypothesis  $\nu$  is still  $\sigma$ -finite and then  $E = \cup_{j \geq 1} \tilde{E}_j$  with  $\nu(\tilde{E}_j) < \infty$ , where  $\tilde{E}_j$  may be chosen disjoint if necessary. Suppose we have another decomposition  $\nu = \nu'_1 + \nu'_2$  with  $\nu'_1(E) = \int_E g \, d\mu$  and  $\nu'_2 \lll \mu$ . In particular,  $\nu_1 - \nu'_1 = \nu'_2 - \nu_2$ . We have that  $(\nu_1 - \nu'_1)(\{x \in \tilde{E}_j : f(x) > g(x)\}) < \infty$ , which by the definition of  $\nu_1$  and  $\nu'_1$  implies that  $\mu(\{x \in \tilde{E}_j : f(x) > g(x)\}) < \infty$ . Since  $\nu_2$  and  $\nu'_2$  are both null with respect to  $\mu$  we have  $\nu_2(\{x \in \tilde{E}_j : f(x) > g(x)\}) = \nu'_2(\{x \in \tilde{E}_j : f(x) > g(x)\}) = 0$ . We can do the same calculation for the complementary set for which  $f(x) < g(x)$  and conclude that  $\nu_2(\tilde{E}'_j) := \nu_2(\{x \in \tilde{E}_j : f(x) \neq g(x)\}) = \nu'_2(\tilde{E}'_j) = 0$  and therefore  $\nu_1(\tilde{E}'_j) = \nu(\tilde{E}'_j) = \nu'_1(\tilde{E}'_j)$ . In  $\tilde{E}_j \setminus \tilde{E}'_j$ ,  $f$  and  $g$  coincide and so  $\nu_1(\tilde{E}_j \setminus \tilde{E}'_j) = \nu_1(\tilde{E}_j \setminus \tilde{E}'_j)$ . Since  $\tilde{E}_j = \tilde{E}'_j \cup (\tilde{E}_j \setminus \tilde{E}'_j)$ , it follows that  $\nu_1$  and  $\nu'_1$  coincide on each  $\tilde{E}_j$  and therefore on  $E$  if the  $\tilde{E}_j$  were chosen disjoint. Now it follows that  $\nu_2 = \nu'_2$ .

*Existence.* Let us consider first the case when  $\nu$  is finite. We define the set  $\mathcal{A} = \{A \subset \mathbb{R}^n : A \text{ is measurable, } \nu(A) > 0, \mu_{\perp A} \text{ is } \sigma\text{-finite.}\}$ . If  $\mathcal{A} = \emptyset$ , then the theorem follows taking  $\nu_2 = \nu$  and  $\nu_1 = 0$ . If  $\mathcal{A} \neq \emptyset$ , define  $a := \sup_{A \in \mathcal{A}} \nu(A)$ . We have that  $a$  is finite, since  $\nu$  is finite. Consider the set sequence  $(A_j)_{j \in \mathbb{N}} \subset \mathcal{A}$  such that  $\nu(A_j) \rightarrow a$ . Let  $B := \bigcup_{j=1}^{\infty} A_j$ . We are going to see that we can take  $\nu_1 = \nu_{\perp B}$  and  $\nu_2 = \nu_{\perp B^c}$ . In fact, since  $\mu_{\perp B}$  is  $\sigma$ -finite, we have  $f$ , the Radon-Nykodim derivative of  $\nu$  with respect to  $\mu_{\perp B}$ . Now we take a set  $E$  such that  $\mu(E) < \infty$ . If  $\nu_2(E) > 0$ , then  $\nu(E \cup B) > a$  which is a contradiction. Therefore  $\nu_2(E) = 0$  and so  $\nu_2 \lll \mu$ .

Let us analyze now the case when  $\nu$  is not finite (but still  $\sigma$ -finite). Let  $(E_j)$  be a collection of measurable sets with  $\nu(E_j) < \infty$  such that  $\cup E_j = E$ . Without loss of generality, we can assume that  $E_j$  are pairwise disjoint. We define  $\nu^j = \nu_{\perp E_j}$  and  $\mu^j = \mu_{\perp E_j}$ . Then  $\nu^j$  is finite and regarding the previous case, we can decompose  $\nu^j = \nu_1^j + \nu_2^j$ . Now  $\nu_1 = \sum_j \nu_1^j$  and  $\nu_2 = \sum_j \nu_2^j$  satisfy the desired property.  $\square$

**Corollary 3.5.** *If  $\mu$  is an  $h$ -dimensional measure, then there exists  $\varphi \geq 0$  and  $\nu \lll \mathcal{H}^h$  such that  $\mu = \varphi d\mathcal{H}^h + \nu$ .*

PROOF. In view of the previous theorem, we only need to prove that  $\mu$  is absolutely continuous with respect to  $\mathcal{H}^h$ . Let us take a set  $E$  with  $\mathcal{H}^h(E) = 0$ . Then for any  $\varepsilon > 0$ , there is a covering  $(U_i)_{i \geq 1}$  of  $E$  with  $\sum_{i=1}^{\infty} h(|U_i|) < \varepsilon$ , where  $|U_i|$  is the diameter of  $U_i$ . Then

$$\mu(E) \leq \sum_{i=1}^{\infty} \mu(U_i) \leq \sum_{i=1}^{\infty} \mu(B_{|U_i|}(x_i)),$$



picking any  $x_i \in U_i$ . Using that  $\mu$  is  $h$ -dimensional and the previous estimate, we have  $\mu(E) \leq \sum_{i=1}^{\infty} h(|U_i|) < \varepsilon$ . Since  $\varepsilon$  is arbitrary,  $\mu(E) = 0$  and the proof is complete.  $\square$

The next technical lemma will be necessary for our construction.

**Lemma 3.6.** *If  $\nu$  is a locally finite measure on  $\mathbb{R}^n$  and  $\nu \lll \mathcal{H}^h$ , then  $\overline{D}_h(\nu, x) := \limsup_{r \rightarrow 0} \frac{\nu(B_r(x))}{h(2r)} = 0$  for  $\mathcal{H}^h$ -almost every  $x$ .*

PROOF. For each  $k \in \mathbb{N}$  we define the sets  $E_k = \{x \in \mathbb{R}^n : \forall \varepsilon > 0, \exists r \leq \varepsilon$  with  $\frac{\nu(B_r(x))}{h(2r)} \geq \frac{1}{k}\}$ . Since  $\{x \in \mathbb{R}^n : \overline{D}_h(\nu, x) > 0\} = \bigcup_{k \geq 1} E_k$ , it is enough to prove that  $\mathcal{H}^h(E_k) = 0$  for all  $k$ .

We can suppose that  $\nu(E_k)$  is finite since  $E_k = \bigcup_{l \geq 1} (E_k \cap B_l(0))$ .

Let  $k$  be fixed and let  $\varepsilon > 0$ . For each  $x \in E_k$ , we can pick an  $r(x) \leq \varepsilon$  such that  $h(2r(x)) \leq k\nu(B_{r(x)}(x))$ .  $\{B_{r(x)}(x)\}_{x \in E_k}$  is a family of balls with uniformly bounded radii. Therefore by Besicovitch's Covering Theorem ([8]) we can take a countable subcover  $\{B_{r_j}(x_j)\}_{j \geq 1}$  of  $E_k$  such that at most  $c(n)$  of the balls intersect at once (i.e.  $\sum \chi_{B_{r_j}} \leq c(n)$ ).

Since  $r_j \leq \varepsilon$ , it follows that  $B_{r_j} \subset E_{k,\varepsilon} := \{x \in \mathbb{R}^n : \text{dist}(x, E_k) \leq \varepsilon\}$ . So we have  $\sum_{j=1}^{\infty} h(2r_j) \leq k \sum_{j=1}^{\infty} \nu(B_{r_j}(x_j)) \leq kc(n)\nu(E_{k,\varepsilon})$  and therefore  $\mathcal{H}^h(E_k) \leq c(n)k\nu(E_{k,\varepsilon})$ .

However since  $E_k \subset \bigcap_{\varepsilon > 0} E_{k,\varepsilon}$  and  $\nu(E_k)$  is finite, we have that  $\mathcal{H}^h(E_k) \leq c(n)k\nu(E_k)$ . In particular,  $\mathcal{H}^h(E_k)$  is finite, which implies  $\nu(E_k) = 0$  by the hypothesis on  $\nu$ .

Using again that  $\mathcal{H}^h(E_k) \leq c(n)k\nu(E_k)$ , we obtain the desired result.  $\square$

We are now able to establish a finer bound for certain  $h$ -dimensional measures (compare with Theorem 3.1 and Theorem 3.2).

**Theorem 3.7.** *Let  $\mu$  be any  $h$ -dimensional measure and let  $\mu = \varphi d\mathcal{H}^h + \nu$  (with  $\nu \lll \mathcal{H}^h$ ) be the decomposition of Theorem 3.4. If  $f \in L^2(\mu)$ , then*

$$\limsup_{t \rightarrow 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 d\xi \leq c \int |f(x)|^2 \varphi(x) d\mathcal{H}^h(x)$$

and

$$\sup_{y \in \mathbb{R}^n} \limsup_{r \rightarrow \infty} \int_{B_r(y)} |\mathcal{F}_\mu f(\xi)|^2 d\xi \leq c \int |f(x)|^2 \varphi(x) d\mathcal{H}^h(x).$$

PROOF. It suffices to prove that  $\lim_{t \rightarrow 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\nu f(\xi)|^2 d\xi = 0$  and  $\lim_{t \rightarrow 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} \mathcal{F}_\nu f(\xi) \mathcal{F}_{\mathcal{H}^h} f(\xi) d\xi = 0$ .

For this proof we will use the maximal operator  $M_\mu$  as defined in (2.1). Doing the same type of computations as the ones used to obtain (3.3), we have

$$\frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\nu(x) \leq \frac{M_\nu f(y)}{h(\sqrt{t})} \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \nu(B_r(y)) dr. \quad (3.6)$$

On the other hand by Lemma 3.6, for  $\mathcal{H}^h$ -almost every  $y$

$$\overline{D}_h(\nu, y) = \limsup \frac{\nu(B_r(y))}{h(2r)} = 0$$

and therefore for all  $\varepsilon > 0$  we can choose  $0 < \delta < 1$  such that  $\nu(B_r(y)) \leq \varepsilon h(r)$ .

We split the integral on the right of (3.6) into two parts,  $\int_0^\delta + \int_\delta^\infty$ . For the first one, using that  $\nu(B_r(x)) \leq \mu(B_r(x))$  and therefore  $\nu$  is  $h$ -dimensional, we obtain

$$\begin{aligned} \frac{M_\nu f(y)}{h(\sqrt{t})} \int_0^\delta e^{-r^2/4t} \frac{r}{2t} \nu(B_r(x)) dr &\leq M_\nu f(y) \varepsilon \int_0^{\delta/\sqrt{t}} e^{-r^2/4} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} dr \\ &\leq c\varepsilon M_\nu f(y) \end{aligned}$$

by hypothesis.

For the second one, we split again yielding

$$\begin{aligned} \frac{M_\nu f(y)}{h(\sqrt{t})} \int_\delta^\infty e^{-r^2/4t} \frac{r}{2t} \nu(B_r(y)) dr \\ \leq M_\nu f(y) c \left( \int_{\delta/\sqrt{t}}^{1/\sqrt{t}} e^{-r^2/4} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} dr + \int_{1/\sqrt{t}}^\infty e^{-r^2/4} r^{n+1} \frac{\sqrt{t}^n}{h(\sqrt{t})} dr \right) \xrightarrow{t \rightarrow 0} 0. \end{aligned}$$

So if we set

$$H(t, y) := \frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\nu(x),$$

we showed that  $\lim_{t \rightarrow 0} H(t, y) = 0$ . Using dominated convergence in the same way that it was used in the first step of the proof of Theorem 3.1, we obtain

$$\begin{aligned} 0 &= \int \lim_{t \rightarrow 0} H(t, y) \overline{f(y)} d\nu(y) = \lim_{t \rightarrow 0} \frac{1}{h(\sqrt{t})} \int e^{-|x-y|^2/4t} f(x) d\nu(x) \overline{f(y)} d\nu(y) \\ &= \lim_{t \rightarrow 0} \frac{\sqrt{t}^n}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\nu f(\xi)|^2 d\xi. \end{aligned}$$

In the same way, if we integrate with respect to  $\mu$ , we obtain

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} \mathcal{F}_\nu f(\xi) \mathcal{F}_\mu f(\xi) d\xi = 0.$$

Now the thesis is a consequence of Theorems 3.1 and 3.2.  $\square$

#### 4 Lower Estimate.

In this section we estimate the lower bound for the  $\mu$ -Fourier transform. We start by the following theorem.

**Theorem 4.1.** *Let  $\mu = \mathcal{H}^h \llcorner_E$  for an  $h$ -regular set  $E$  (see 1.2). Suppose that the function  $h$  satisfies both  $h(t) \leq t^n$  for  $t \geq 1$  and  $\lim_{t \rightarrow 0} \frac{t^n}{h(t)} = 0$ . Also suppose that the limit  $\lim_{t \rightarrow 0} \frac{h(rt)}{h(t)} := p(r)$  exists. Then for  $f \in L^2(\mu)$ ,*

$$\lim_{t \rightarrow 0} \frac{\sqrt{t^n}}{h(\sqrt{t})} \int e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 d\xi = C_{n,h} \int |f|^2 d\mu, \quad (4.1)$$

where  $C_{n,h} = \int_0^\infty e^{-r^2/2} r p(r) dr$ .

PROOF. In view of (3.1), we will estimate

$$\frac{\sqrt{t^n}}{h(\sqrt{t})} \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) d\mu(x) dr.$$

We write the first integral as sum  $\int_0^\delta + \int_\delta^\infty$ . For any  $\delta$  the second term tends to zero, since

$$\begin{aligned} & \frac{1}{h(\sqrt{t})} \int_\delta^\infty e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) d\mu(x) dr \\ & \leq \frac{1}{h(\sqrt{t})} \int_\delta^\infty e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) M_\mu f(y) dr \\ & \leq \frac{1}{h(\sqrt{t})} M_\mu f(y) \left( \int_\delta^1 e^{-r^2/4t} \frac{r}{2t} h(r) dr + \int_1^\infty e^{-r^2/4t} \frac{r}{2t} r^n dr \right) \\ & = \frac{\sqrt{t^n}}{h(\sqrt{t})} M_\mu f(y) \left( \int_{\delta/\sqrt{t}}^{1/\sqrt{t}} r h(r) e^{-r^2/4} dr + \int_{1/\sqrt{t}}^\infty r^{n+1} e^{-r^2/4} dr \right) \xrightarrow{t \rightarrow 0} 0, \end{aligned} \quad (4.2)$$

using that  $\lim_{t \rightarrow 0} \frac{t^n}{h(t)} = 0$ .

To analyze the other integral, note first that since  $E$  is regular by Corollary 2.3, we have that, for  $\mathcal{H}^h$ -almost every  $y \in E$  (fixed) and for all  $\varepsilon > 0$ , there exists  $\delta > 0$  such that

$$\left| \int_{B_r(y)} f \, d\mu - h(r)f(y) \right| \leq \varepsilon h(r) \quad \forall r \leq \delta. \quad (4.3)$$

On the other hand,

$$\int_0^\delta \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} h(r)f(y) \, dr = f(y) \int_0^{2\delta/\sqrt{t}} e^{-r^2} r \frac{h(r\sqrt{t})}{h(\sqrt{t})} \, dr$$

and so, since  $e^{-r^2} r \frac{h(r\sqrt{t})}{h(\sqrt{t})}$  is dominated by  $e^{-r^2} r^{1+\kappa}$  (see Lemma 2.4), we have

$$\int_0^\delta \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} h(r)f(y) \, dr \xrightarrow{t \rightarrow 0} f(y) \int_0^\infty e^{-r^2} r p(r) \, dr.$$

We conclude that

$$\int_0^\delta \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) \, d\mu(x) \, dr \xrightarrow{t \rightarrow 0} C_{n,h} f(y). \quad (4.4)$$

Combining (4.2) and (4.4) we obtain that

$$H(t, y) := \frac{1}{h(\sqrt{t})} \int_0^\infty e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f \, d\mu \, dr \xrightarrow{t \rightarrow 0} C_{n,h} f(y).$$

Since  $H(t, y)$  is dominated by  $f(y) \int_0^\infty e^{-r^2} r p(r) \, dr$  and  $f \in L^2(\mu)$ , it follows that

$$\lim_{t \rightarrow 0} \int H(t, y) \overline{f(y)} \, d\mu(y) = \int \lim_{t \rightarrow 0} H(t, y) \overline{f(y)} \, d\mu(y) = C_{n,h} \int_E |f|^2 \, d\mu.$$

□

Note that equation (4.3), which was very important in our proof, is a reformulation of Corollary 2.2 substituting  $\mu(B_r(y))$  by  $h(r)$ . We are allowed to make this substitution only because  $E$  is a regular set. However this hypothesis on  $E$  is too restrictive.

Actually it has already been proven (see [8]) that there only exist regular sets for functions of the form  $x^k$  with  $k$  integer. So in order for the last theorem to be meaningful, it will be necessary to obtain a result with a weaker

hypothesis. We will therefore consider  $h$ -quasi regular sets, meaning that there exists a constant  $\theta > 0$  such that for  $\mathcal{H}^h$ -almost every  $x \in E$ ,

$$\liminf_{r \rightarrow 0} \frac{\mathcal{H}^h(B_r(x) \cap E)}{h(r)} \geq \theta. \quad (4.5)$$

For this case, instead of the equality in (4.1), we obtain a lower bound.

**Theorem 4.2.** *Let  $\mu = \mathcal{H}_E^h + \nu$ . If  $\nu \lll \mathcal{H}^h$  and  $E$  is  $h$ -quasi regular, then*

$$\liminf_{t \rightarrow 0} \int e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 d\xi \geq c \int_E |f|^2 d\mathcal{H}^h. \quad (4.6)$$

PROOF. By the proof of Theorem 3.7, we can suppose  $\mu = \mathcal{H}_{\perp E}^h$ .

Since  $E$  is quasi regular, there exists  $\delta_1 > 0$  such that if  $r < \delta_1$  then

$$\mu(B_r(x)) \geq ch(r). \quad (4.7)$$

On the other hand, there exists  $\delta_2 > 0$  such that if  $r < \delta_2$  (and  $f(y) \neq 0$ ), then

$$\left| \frac{1}{\mu(B_r(y))} \int_{B_r(y)} f(x) d\mu(x) - f(y) \right| < \varepsilon |f(y)|. \quad (4.8)$$

Taking  $\delta = \delta_{y,\varepsilon}$  (satisfying both estimates), we may write

$$\frac{1}{h(\sqrt{t})} \iint e^{-|x-y|^2/4t} f(x) \overline{f(y)} d\mu(x) d\mu(y) = \int |f(y)|^2 H(y, t, \varepsilon) d\mu(y) + R(t, \varepsilon),$$

where  $H(y, t, \varepsilon) = \int_0^{\delta_{y,\varepsilon}} \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr$  and

$$\begin{aligned} R(t, \varepsilon) &= \int \overline{f(y)} \int_0^{\delta_{y,\varepsilon}} \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} \left( \int_{B_r(y)} f(x) d\mu(x) - f(y) \mu(B_r(y)) \right) dr d\mu(y) \\ &\quad + \int \overline{f(y)} \int_{\delta_{\varepsilon,y}}^{\infty} \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \int_{B_r(y)} f(x) d\mu(x) dr d\mu(y). \end{aligned}$$

We are now going to bound  $|R(t, \varepsilon)|$ . Using (4.8) and the fact that there exists a bound independent of  $t$  for  $\int_0^{\delta} \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} \mu(B_r(y)) dr$ , we can bound the first term by  $C_1 \varepsilon \|f\|^2$ . The second term is bounded by

$$\int |f(y)| \int_{\delta_{\varepsilon,y}}^{\infty} \frac{1}{h(\sqrt{t})} e^{-r^2/4t} \frac{r}{2t} \mu(B_r(y)) dr M_\mu f(y) d\mu(y).$$

Recalling a previous calculation, the integral  $\int_{\delta}^{\infty} \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} \mu(B_r(y)) dr$  can be bounded by  $\varepsilon$  if we take  $t$  small enough. Therefore by Cauchy Schwartz and the (2,2) strong estimate, the second term is bounded by  $C_2 \|f\|^2$ . So both estimates tell us that  $|R(t, \varepsilon)| \leq C \|f\|^2$  for small enough  $t$ .

On the other hand,  $H(y, t, \varepsilon)$  is bounded bellow by  $\int_0^{\delta} \frac{e^{-r^2/4t}}{h(\sqrt{t})} \frac{r}{2t} h(r) dr$ , using (4.7). Substituting and using that  $\liminf_{t \rightarrow 0} \frac{h(r\sqrt{t})}{h(\sqrt{t})} < \infty$ , we conclude that  $\liminf_{t \rightarrow 0} H(y, t, \varepsilon) \geq C_3$ .

Therefore by Fatou's Lemma

$$\liminf_{t \rightarrow 0} \frac{1}{h(\sqrt{t})} \iint e^{-|x-y|^2/4t} f(x)f(y) d\mu(x)d\mu(y) \geq c \int |f|^2 d\mu.$$

□

**Theorem 4.3.** *Let  $\mu$  an  $h$ -dimensional measure such that  $\mu = \mathcal{H}^h \llcorner_E + \nu$  with  $\nu \lll \mathcal{H}^h$  being  $E$   $h$ -quasi regular. Then the following inequality holds:*

$$\liminf_{r \rightarrow \infty} \frac{1}{r^n h(r^{-1})} \int_{B_r(y)} |\mathcal{F}_\mu f(\xi)|^2 d\xi \geq c \int_E |f|^2 d\mathcal{H}^h,$$

where the constant  $c$  does not depend on  $y$ .

PROOF. For any  $\lambda > 0$  such that  $\lambda \leq t|\xi|^2$ , we have  $e^{-t|\xi|^2} \leq e^{-\lambda/2} e^{-(1/2)t|\xi|^2}$ . Then

$$\begin{aligned} & \frac{\sqrt{t}^n}{h(\sqrt{t})} \int_{\{\xi: t|\xi|^2 \geq \lambda\}} e^{-t|\xi|^2} |\mathcal{F}_\mu f(\xi)|^2 d\xi \\ & \leq 2^{n/2} \frac{h((t/2)^{1/2})}{h(\sqrt{t})} \frac{(t/2)^{n/2}}{h((t/2)^{1/2})} e^{-\lambda/2} \int e^{-(1/2)t|\xi|^2} |\mathcal{F}_\mu(\xi)|^2 d\xi \\ & \leq c e^{-\lambda/2} \int_E |f|^2 d\mathcal{H}^h \end{aligned}$$

by Lemma 2.4 and Theorem 3.7. Using 4.6 and picking  $\lambda$  big enough, we obtain

$$\liminf_{t \rightarrow 0} \frac{\sqrt{t}^n}{h(\sqrt{t})} \int_{\{\xi: t|\xi|^2 \leq \lambda\}} e^{-t|\xi|^2} |\mathcal{F}_\mu(\xi)|^2 d\xi \geq \tilde{c} \int_E |f|^2 d\mu,$$

picking the constant  $c$  smaller if needed. Now taking  $t = \lambda/r^2$ , we obtain

$$\frac{h(\lambda^{1/2})}{\lambda^{n/2}} \frac{\sqrt{t}^n}{h(\sqrt{t})} \int_{\{\xi: t|\xi|^2 \leq \lambda\}} e^{-t|\xi|^2} |\mathcal{F}_\mu(\xi)|^2 d\xi \leq c_\lambda \frac{1}{r^n h(r^{-1})} \int_{B_r(0)} |\mathcal{F}_\mu(\xi)|^2 d\xi,$$

where  $c_\lambda$  is such that  $h(r^{-1})/h(\lambda^{1/2}r^{-1}) \leq c_\lambda$ . This completes the proof. □

### 5 An Example.

We conclude the paper by exhibiting an example of a function  $h$  and a set  $E$  such that  $\mathcal{H}_{\perp E}^h$  is  $h$ -dimensional and  $E$  is quasi regular. For this example Theorem 4.3 holds. However, since  $E$  is  $\alpha$ -dimensional but with zero  $\mathcal{H}^\alpha$  measure the results of Strichartz in [10] do not apply. This shows that by considering more general dimension functions we obtained a useful generalization.

Let  $h : [0, \infty) \rightarrow \mathbb{R}$  be a dimension function such that  $h(2x) < 2h(x)$ . Let  $s_k$  be such that  $h(s_k) = 2^{-k}$ . We will construct a set of Cantor type. Consider the two (closed) subintervals of  $[0, 1]$ ,  $I_{1,1}$  and  $I_{1,2}$ , of length  $s_1$  obtained by suppressing the central open interval of length  $1 - 2s_1$ . In each of these intervals we take the two closed subinterval of length  $s_2$  obtained by removing the central interval of length  $s_1 - 2s_2$  this time. (Note that this number is positive because  $h(2x) < 2h(x)$ .) We obtain four intervals denoted by  $I_{2,1}$ ,  $I_{2,2}$ ,  $I_{2,3}$ ,  $I_{2,4}$ . These intervals will be called intervals of step 2. Following in the same manner at each step, we obtain  $2^k$  closed intervals of length  $s_k$ . Our Cantor set will be

$$E = \bigcap_{k \geq 1} \bigcup_{j=1}^{2^k} I_{k,j}.$$

We assign to each interval  $I_{k,j}$  measure  $2^{-k}$  obtaining a probability measure  $\mu$  supported on  $E$ . We can see ([4]) that this measure is  $\mathcal{H}_{\perp E}^h$ .

We are going to show that this set satisfies the hypothesis of the Theorem 4.2, which means essentially that it is  $h$ -quasi regular. It suffices to see that  $\frac{\mu(B(x,\rho))}{h(2\rho)} \geq c$  (where  $c$  is a positive constant) for all  $x \in E$  and for all  $\rho > 0$ .

Given  $x \in E$  and  $\rho > 0$ , denote by  $k$  the minimum integer such that there exists  $j$  between 1 and  $2^k$  satisfying  $I_{k,j} \subset B(x, \rho)$ . By minimality  $s_{k-1} \geq \rho$ . Then

$$\frac{\mu(B(x, \rho))}{h(2\rho)} \geq \frac{\mu(I_{k,j})}{h(2\rho)} = \frac{2^{-k}}{h(2\rho)} \geq \frac{c_d}{2} \frac{1}{2^{k-1}h(\rho)} \geq \frac{c_d}{2} \frac{1}{2^{k-1}h(s_{k-1})} = \frac{c_d}{2},$$

using that  $I_{k,j} \subset B(x, \rho)$ , the definition of  $\mu$ , Lemma 2.4, the minimality of  $k$ , and the definition of  $s_k$ . Therefore (4.5) follows.

We also need to prove that  $\mu = \mathcal{H}_{\perp E}^h$  is an  $h$ -dimensional measure. In fact,  $E \cap B_\rho(x) \subset I_{k-1,j}$  for some  $j$ . Consequently  $\mu(B_\rho(x)) \leq \mu(I_{k-1,j}) = 2^{-(k-1)} = 2h(s_k) \leq h(\rho)$ .

If we take  $h(x) = x^\alpha \log(1/x)$ , then we obtain a set  $E$  of dimension  $\alpha$  but such that  $\mathcal{H}^\alpha(E) = 0$ . Therefore for any  $\alpha$ ,  $E$  will not be  $\alpha$ -quasi regular and hence we cannot apply Strichartz's Theorem.

However since  $E$  is  $h$ -quasi regular for  $h(x) = x^\alpha \log(1/x)$ , we can apply Theorem 4.3.

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## References

- [1] S. Agmon & L. Hörmander, *Asymptotic properties of solutions of differential equations with simple characteristics*, J. Anal. Math., **30** (1976), 1–38.
- [2] John J. Benedetto & Joseph D. Lakey, *The definition of the Fourier transform for weighted inequalities*, J. Funct. Anal., **120(2)** (1994), 403–439.
- [3] C. Cabrelli, F. Mendivil, U. Molter, & R. Shonkwiler, *On the  $h$ -Hausdorff measure of Cantor sets*, Pacific J. Math., **217(1)** (2004), 29–43.
- [4] K. J. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1985.
- [5] K. J. Falconer, *Techniques in Fractal Geometry*, John Wiley & Sons, New York, 1997.
- [6] K-S. Lau, *Fractal measures and mean  $p$ -variations*, J. Funct. Anal., **108(2)** (1992), 427–457.
- [7] K. S. Lau & Y. Wang, *Characterizations of  $\mathcal{L}^p$ -solutions for the two scale dilation equations*, Preprint, 1997.
- [8] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge University Press, Cambridge, 1995.
- [9] C. A. Rogers, *Hausdorff Measures*, Cambridge University Press, Cambridge, UK, 2<sup>nd</sup> Edition, 1998.
- [10] R. Strichartz, *Fourier asymptotics of fractal measures*, J. Funct. Anal., **89** (1990), 154–187.
- [11] R. Strichartz, *Self-similar measures and their Fourier transforms I*, Indiana Univ. Math. J., **39(3)** (1990), 797–817.



- [12] R. Strichartz, *Self-similar measures and their Fourier transforms II*, Trans. Amer. Math. Soc., (1993).
- [13] R. Strichartz, *Self-similarity in harmonic analysis*, J. Fourier Anal. Appl., **1(1)** (1994), 1–37.

