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Andrei E. Ghenciu, Department of Mathematics and Statistics University of Alaska at Fairbanks, Fairbanks, Alaska, 99775. email: ffeag@uaf.edu

GAUSS-LIKE CONTINUED FRACTION SYSTEMS AND THEIR DIMENSION SPECTRUM

Abstract

To the Gauss-like continued fraction expansions we associate a conformal iterated function system whose limit set is of Lebesgue measure equal to 1. We show that the Texan Conjecture holds; i.e. for every $t \in [0,1]$ there exists a subsystem whose limit set has Hausdorff dimension equal to t.

1 Introduction.

It is well known that every irrational number x in the interval [0,1] has a unique standard continued fraction expansion of the form:

$$[a_1, a_2, ..., a_n] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ...}}} \tag{1.1}$$

where each of the a_i's is a positive integer.

Let $N_1 = \{1, 2, 3, ...\}$ and let J_{N_1} be the set of all numbers that can be represented as in (1.1). J_{N_1} is the set of all irrational numbers in [0,1], so it is a set of Lebesgue measure equal to 1. If we start with a subset of N_1, say A, then we define J_A to be the set of all the numbers in $[0, 1]$ that can be represented as in (1.1), where each of the $a_i's$ is now in A. It turns out that if A is a proper subset of N_1, the Lebesgue measure of J_A is zero (see [7]).

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Given $0 < t < 1$, is there a subset A of N_1 so that $HD(J_A) = t$, where $HD(J_A)$ is the Hausdorff dimension of J_A? This was an open problem for several years and it was called the Texan Conjecture. It was answered affirmatively by Mauldin and Urbańksi for $0 < t < \frac{1}{2}$ (see [8]) and later it was answered affirmatively for $0 < t < 1$ by Kesseböhmer and Zhu (see [5]).

To study this problem, all of the above mentioned authors used a conformal iterated function system associated with the standard continued fraction expansions. We address the same question for more general continued fraction expansions. The systems that we study will be called Gauss-like systems, and they are introduced in Section 4.

On the other hand, we note that there exists conformal iterated function systems S and numbers t between 0 and the Hausdorff dimension of J_S, so that no subsystem of S has Hausdorff dimension equal to t. Such an example consisting of similarities was constructed in [8].

In this paper we prove the analogous Texan Conjecture for Gauss-like continued fraction systems, which will be introduced in Section 4. To do this, we construct and study a conformal iterated function system corresponding to the Gauss-like continued fractions.

2 Preliminaries.

In this section we collect some definitions and results from [7] which are used throughout this paper. We start by introducing the concept of an Iterated Function System in the general setting of a compact metric space.

Throughout this paper, if A is a subset of a Euclidean space, by $HD(A)$ we mean the Hausdorff dimension of A.

Let (X, d) be a non-empty compact metric space and let I be a countable set with at least two elements. Let $S = \{\varphi_i : X \to X : i \in I\}$ be a collection of injective contractions from X to X for which there exists $0 < s < 1$ such that

$$d(\varphi_i(x), \varphi_i(y)) \leq sd(x, y)$$

for every $i \in I$ and for every pair of points x, y in X.
Any such collection S of contractions is called an iterated function system, or i.f.s.

Let $I^* = \bigcup_{n \geq 1} I^n$. For every $\omega = (\omega_1, ..., \omega_n)$ in I^n, $n \geq 1$, set

$$\varphi_\omega = \varphi_{\omega_1} \circ \varphi_{\omega_2} \circ ... \circ \varphi_{\omega_n}.$$

For $\omega \in I^* \bigcup I^\infty$ and $n \geq 1$ that does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2...\omega_n$. Given $\omega \in I^\infty$, the compact sets $\varphi_{\omega|n}(X)$, $n \geq 1$, are decreasing and their diameters converge to zero. We have, in fact:

$$\operatorname{diam}(\varphi_{\omega|n}(X)) \leq s^n \operatorname{diam}(X).$$

This implies that the set

$$\bigcap_{n\geq 0} \varphi_{\omega|n}(X)$$

is a singleton.

Definition 2.1. We define the coding map $\pi : I^\infty \to X$ by:

$$\pi(\omega) = \bigcap_{n\geq 0} \varphi_{\omega|n}(X).$$

Then:

$$J = \pi(I^\infty)$$

is the *limit set* associated to the system $S = \{\varphi_i : X \to X : i \in I\}$.

If I is finite, then the limit set is compact.
An iterated function system $S = \{\varphi_i : X \to X : i \in I\}$ is said to satisfy the Open Set Condition if there exists a non-empty open set $U \subset X$ (in the topology of X) such that $\varphi_i(U) \subset U$ for every $i \in I$ and also $\varphi_i(U) \bigcap \varphi_j(U) = \emptyset$ for every pair $i, j \in I, i \neq j$.

An iterated function system S satisfying the Open Set Condition is said to be conformal (abbreviated CIFS) if the following conditions are satisfied:

1. X is a compact connected subset of a Euclidean space $\mathbb{R}^d$ and $U = Int_{\mathbb{R}^d}(X)$.

2. (Cone condition) There exists $\alpha, l > 0$ such that for every $x \in \partial X \subset \mathbb{R}^d$ there exists an open cone $Cone(x, \alpha, l) \subset Int(X)$ with vertex x, central angle of Lebesgue measure α and altitude l.

3. There exists an open connected set $X \subset V \subset \mathbb{R}^d$ such that all maps $\varphi_i, i \in I$ extend to $C^{1+\epsilon}$ diffeomorphisms on V and are conformal on V.

4. (Bounded Distortion Property)
There exists $K \geq 0$ such that $|\varphi'_\omega(y)| \leq K|\varphi'_\omega(x)|$ for every $\omega \in I^*$ and for every pair of points $x, y \in V$. We will call K *a Bounded Distortion Constant.*

Remark 2.1. We note that when $d = 1$ the Cone Condition is not necessary. When $d \geq 2$, the Cone Condition is used for the construction of a conformal measure on the limit set of a CIFS (see [8]).

Let $S=\{\varphi_i\}_{i\in I}$ be a CIFS and let $A\subset I$.
For every $t\geq 0$ and $n\geq 1$, let:

$$\Phi_{n,A}(t)=\sum_{\omega\in A^n}\left\|\varphi'_\omega\right\|^t. \tag{2.1}$$

For every integers $m,n\geq 1$ and for every $t\geq 0$ we have:

$$\Phi_{m+n,A}(t)\leq \Phi_{m,A}(t)\Phi_{n,A}(t). \tag{2.2}$$

We define:

$$P_A(t)=\lim_{n\to\infty}\frac{\ln \Phi_{n,A}(t)}{n}. \tag{2.3}$$

We call P_A the *topological pressure function.*

Let:

$$\lambda_A(t)=e^{P_A(t)};\quad \theta_A=inf\{t\geq 0|P_A(t)<\infty\}. \tag{2.4}$$

We call θ_A the *finiteness parameter* of the subsystem generated by A.

Next we list the most important results concerning the topological pressure function and the connections to the Hausdorff dimension of the limit set. All these results can be found in [7].

Proposition 2.1. *The topological pressure function is non-increasing on* $[0,\infty)$, *strictly decreasing, convex and continuous on* (θ_A,∞). *Also:*

$$\theta_A=inf\{t\geq 0\ \ |\ \ \Phi_{1,A}(t)<\infty\}. \tag{2.5}$$

Theorem 2.1. *Let* $S=\{\varphi_i\}_{i\in I}$ *be a CIFS. Then:*

$$HD(J_I)=inf\{t\geq 0|P_I(t)<0\}=sup\{HD(J_F)|F\subset I finite\}\geq\theta_I \tag{2.6}$$

If there exists a t *so that* $P_I(t)=0$, *then* $t=HD(J_I)$.

Definition 2.2. A CIFS $S=\{\varphi_i\}_{i\in I}$ is called *regular* if there exists $t>0$ so that $P_I(t)=0$.

3 Results for General Conformal Iterated Function Systems.

The next two theorems appear in a slightly different version in [2]. They will be used to prove the main result of this paper. Here, $\mathbb{N}=\{0,1,2,...\}$.

Theorem 3.1. *Let $S=\{\varphi_i\}_{i\in I}$, be a regular conformal iterated function system, with $I \subset \mathbb{N}$. Let $A \subset I$ and let $b \in I$. Let p_b be a positive real number so that*

$$\|\varphi'_{\omega b\varpi}\| \leq p_b \|\varphi'_{\omega\varpi}\| \tag{3.1}$$

for any words ω and ϖ. Then

$$\lambda_{A\cup\{b\}}(t) \leq \lambda_A(t) + p_b^t \tag{3.2}$$

for every $t \in [0, d]$.

Remark 3.1. We note that Theorem 3.1 remains true if we ask that the inequality (3.1.) holds for any finite words ω and ϖ so that no letters in ω are greater than or equal to l (see [2]). This is an essential observation that we will use later.

We also have:

Theorem 3.2. *Let $S=\{\varphi_i\}_{i\in I}$ be a regular conformal iterated function system, where $I \subset \mathbb{N}$. Let $A \subset I$ and let $b \in I \setminus A$. Let r_b be a positive real number so that*

$$\|\varphi'_{\omega b\varpi}\| \geq r_b \|\varphi'_{\omega\varpi}\| \tag{3.3}$$

for any words ω and ϖ. Then

$$\lambda_{A\cup\{b\}}(t) \geq \lambda_A(t) + r_b^t, \tag{3.4}$$

for every $t \in [0, d]$.

Remark 3.2. We note that Theorem 3.2. remains true is we ask that the inequality (3.1.) holds for any finite words ω and ϖ so that no letters in ϖ are greater than or equal to l (see [2]). This will also be used later in the paper.

Remark 3.3. Under the hypothesis of Theorem 3.1., the existence of a p_b for every b is guaranteed by the Bounded Distortion Property. In particular, p_b can be taken to be $K^2 \|\varphi'_b\|$. Similarly, under the hypothesis of Theorem 3.2., r_b can be taken to be $K^{-2} \|\varphi'_b\|$.

If $S=\{\varphi_i\}_{i\in I}$ is a CIFS, then by the Hausdorff dimension spectrum of S, or $spec_{HD}S$, we mean the set of all t, so that t is the Hausdorff dimension of a limit set generated by a subset of I. We say that a CIFS $S=\{\varphi_i\}_{i\in I}$ has full HD spectrum if for every $0 \leq t \leq HD(J_I)$ there exists $A \subset I$ so that $HD(J_A) = t$.

Before we introduce the Gauss-like continued fractions, we would like to state a result that appears in [5] in a slightly different form.

Theorem 3.3. *Let $S=\{\varphi_i\}_{i\in\mathbb{N}}$ be a regular conformal iterated function system. For every $b \in \mathbb{N}$ let $N_b = \{b, b+1, b+2, ...\}$. If for every $b \in \mathbb{N}$, $t \geq 0$ and for every $A \subset \{1, 2, ..., b-1\}$ we have:*

$$\lambda_{A\cup\{b\}}(t) \leq \lambda_{A\cup N_{b+1}}(t) \tag{3.5}$$

then S has full HD spectrum.

4 Gauss-like Continued Fraction Systems.

Let us recall that every irrational number x in the interval [0,1) has a unique standard continued fraction expansion of the form:

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + ...}}} \tag{4.1}$$

where each of the a_i's is a positive integers. The standard continued fraction expansion is determined by the transformation:

$$T(x) = 1/x - [1/x], x \neq 0; T(0) = 0. \tag{4.2}$$

In particular, for every $n \geq 1$:

$$a_n = a_n(x) = [1/T^{n-1}(x)]. \tag{4.3}$$

In [8] the authors derived many properties of subsystems of standard continued fractions with the use of naturally associated conformal iterated function systems consisting of the maps:

$$\varphi_b : [0,1] \to [0,1]; \varphi_b(x) = 1/(b+x) \tag{4.4}$$

with b positive integers.

We follow similar ideas based on the Gauss-like continued fractions expansion which we describe next.

Let N_0 be the set of all non-negative integers and let $k \geq 1$ be a fixed real number.

Every number x in the interval [0,1], except for a countable set, has a unique continued fraction expansion of the form:

$$\cfrac{k}{k + b_1 + \cfrac{k}{k + b_2 + \cfrac{k}{k + b_3 + ...}}} \tag{4.5}$$

where each of the b_i's is a non-negative integer.
For more details about very general continued fractions expansions see [11].
Also, see [1] for applications of Gauss-like continued fraction expansions.
To analyze the above representation, for every $l \geq 0$, we define:

$$\varphi_l : [0,1] \to [0,1]; \quad \varphi_l(x) = \frac{k}{(k+l+x)}. \tag{4.6}$$

Observe that for $k = 1$ we have the same maps as in the case of standard continued fractions (see [8]).
For every $l \geq 0$, we have:

$$|\varphi_l'(x)| = \frac{k}{(k+l+x)^2}. \tag{4.7}$$

In particular, $||\varphi_l'|| = \frac{k}{(k+l)^2} \leq \frac{1}{k}$.

For every $\omega \in N_0^n$, we have:

$$\varphi_\omega(x) = \frac{p_n + xp_{n-1}}{q_n + xq_{n-1}}, \tag{4.8}$$

where $p_n = p_n(\omega)$ and $q_n = q_n(\omega)$ are defined by recursion; (see [11])). Also, (see [11]):

$$p_{i+2}(\omega) = (k + l_i)p_{i+1}(\omega) + kp_i(\omega) \tag{4.9}$$

and

$$q_{i+2}(\omega) = (k + l_i)q_{i+1}(\omega) + kq_i(\omega) \tag{4.10}$$

where $i + 2 \leq |\omega|$ and l_i is the $i + 2$ letter in ω.
Therefore:

$$|\varphi_\omega'(x)| = \frac{k^n}{(q_n + xq_{n-1})^2}. \tag{4.11}$$

In particular:

$$||\varphi_\omega'|| = \frac{k^n}{q_n^2}. \tag{4.12}$$

Now, for every $\omega \in N_0^n$ and for every $x, y \in [0,1]$ we have:

$$\frac{|\varphi_\omega'(x)|}{|\varphi_\omega'(y)|} = (\frac{q_n + yq_{n-1}}{q_n + xq_{n-1}})^2 \leq (\frac{q_n + q_{n-1}}{q_n})^2 \leq 4. \tag{4.13}$$

We will call the system generated by these maps, the Gauss-like c.f. system with parameter k.
Using all the previous relationships we get the following:

Theorem 4.1. *The Gauss-like c.f. system with parameter k is a regular conformal iterated function system. Its limit set consists of all the numbers in [0,1] except for a countable set. The finiteness parameter is $\frac{1}{2}$ and the bounded distortion constant can be taken to be 4.*

We want to remark here that for big values of k the bounded distortion constant can be taken to be close to 1.

Lemma 4.1. *Let $l \geq 1$ be a positive integer and let ω and ϖ be two finite words so that no letter in ϖ is greater than or equal to l. Then:*

$$||\varphi'_{\omega l \varpi}|| \geq ||\varphi'_{\omega\varpi}|| \quad \frac{k}{(k+l+1)^2}.$$

PROOF. We have the following equality:

$$|\varphi'_{\omega l\varpi}(x)| = |\varphi'_{\omega}(\varphi_{l\varpi}(x))| \quad |\varphi'_{l}(\varphi_{\varpi}(x))|. \quad |\varphi'_{\varpi}(x)| \tag{4.14}$$

For every finite word α, and for every $x \in [0,1]$,

$$|\varphi'_{\alpha}(x)| = \frac{k^n}{(q_n(\alpha) + xq_{n-1}(\alpha))^2}.$$

So for every finite word α, $|\varphi'_{\alpha}|$ is a decreasing function.
For every $x \in [0,1]$,

$$|\varphi'_{l}(x)| = \frac{k}{(k+l+x)^2} \geq \frac{k}{(k+l+1)^2}. \tag{4.15}$$

Since no letter in ϖ is greater than or equal to l, we have, for every $x, y \in [0,1]$:

$$\varphi_{l\varpi}(x) \leq \frac{1}{k+l} \leq \varphi_{\varpi}(y). \tag{4.16}$$

The fact that $|\varphi'_{\omega}|$ is decreasing and (4.16) give us:

$$|\varphi'_{\omega}(\varphi_{l\varpi}(x))| \geq |\varphi'_{\omega}(\varphi_{\varpi}(x))|. \tag{4.17}$$

Combining (4.15) and (4.17) we get:

$$|\varphi'_{\omega l\varpi}(x)| \geq |\varphi'_{\omega}(\varphi_{\varpi}(x))| \quad |\varphi'_{\varpi}(x)| \quad \frac{k}{(k+l+1)^2} = |\varphi'_{\omega\varpi}(x)|\frac{k}{(k+l+1)^2}.$$

Therefore:

$$||\varphi'_{\omega l \varpi}|| \geq ||\varphi'_{\omega\varpi}|| \quad \frac{k}{(k+l+1)^2}.$$

□

Lemma 4.2. *Let $l \geq 1$ be a positive integer and let ω and ϖ two finite words so that no letter in ω is greater than or equal to l. Then:*

$$||\varphi'_{\omega l \varpi}|| \leq ||\varphi'_{\omega\varpi}|| \quad \frac{4k}{(k+l+1)^2}.$$

PROOF. Let $F_\omega : [0,1] \to \mathbb{R}$ by:

$$F_\omega(x) = \frac{|\varphi'_{\omega l}(x)|}{|\varphi'_{\omega}(x)|} = k\frac{(q_n(\omega) + xq_{n-1}(\omega))^2}{(q_{n+1}(\omega l) + xq_n(\omega))^2}.$$

We study the following function:
$G_\omega : [0,1] \to \mathbb{R}$ such that

$$G_\omega(x) = \frac{q_n(\omega) + xq_{n-1}(\omega)}{q_{n+1}(\omega l) + xq_n(\omega)}.$$

We have:

$$G'(x) = \frac{q_{n+1}(\omega l)q_{n-1}(\omega) - q_n^2(\omega)}{(q_{n+1}(\omega l) + xq_n(\omega))^2}.$$

Now, using (4.10):

$$\begin{aligned} q_{n+1}(\omega l)q_{n-1}(\omega) - q_n(\omega)^2 &= ((k+l)q_n(\omega) + kq_{n-1}(\omega))q_{n-1}(\omega) - q_n(\omega)^2 \\ &= (k+l)q_n(\omega)q_{n-1}(\omega) - q_n(\omega)^2 + kq_{n-1}(\omega)^2 \\ &= q_n(\omega)((k+l)q_{n-1}(\omega) - q_n(\omega)) + kq_{n-1}(\omega)^2. \end{aligned}$$

Since no letter in ω is greater than or equal to l, and using (4.10), we have $q_n(\omega) \leq (k+l)q_{n-1}(\omega)$ and so:

$$q_{n+1}(\omega l)q_{n-1}(\omega) - q_n(\omega)^2 \geq 0.$$

So G_ω is increasing and therefore F_ω is increasing. Thus:

$$F_\omega(x) \leq F_\omega(1) \leq k\frac{(q_n(\omega) + q_{n-1}(\omega))^2}{(q_{n+1}(\omega l) + q_n(\omega))^2}.$$

Now, since $q_{n-1}(\omega) \leq q_n(\omega)$, $q_{n+1} = (k+l)q_n + kq_{n-1}$ and $k \geq 1$, we have:

$$(k+l+2)q_{n-1}(\omega) \leq (k+l)q_n(\omega) + 2kq_{n-1}(\omega).$$

So:

$$\begin{aligned} (k+l+2)q_{n-1}(\omega) + (k+l+2)q_n(\omega) \leq (k+l)q_n(\omega) + 2kq_{n-1}(\omega) \\ +(k+l+2)q_n(\omega), \end{aligned} \tag{4.18}$$

so,

$$(k+l+2)(q_{n-1}(\omega)+q_n(\omega)) \le 2(k+l)q_n\omega + 2kq_{n-1}(\omega) + 2q_n(\omega),$$

and (4.18) becomes:

$$(k+l+2)(q_{n-1}(\omega)+q_n(\omega)) \le 2(q_{n+1}(\omega l)+q_n(\omega)).$$

Therefore:

$$\frac{q_n(\omega)+q_{n-1}(\omega)}{q_{n+1}(\omega l)+q_n(\omega)} \le \frac{2}{k+l+2}.$$

So, for every $x \in [0,1]$,

$$F_\omega(x) \le F_\omega(1) \le \frac{4k}{(k+l+2)^2}.$$

This implies that for every $x \in [0,1]$

$$|\varphi'_{\omega l}(x)| \le |\varphi_\omega(x)| \frac{4k}{(k+l+2)^2}.$$

Thus:

$$||\varphi'_{\omega l \varpi}|| \le ||\varphi'_{\omega\varpi}|| \quad \frac{4k}{(k+l+2)^2}.$$

□

Theorem 4.2. *The Gauss-like c.f. system with parameter k has full HD-spectrum* $= [0,1]$.

PROOF. We use Theorem 3.3. to prove that the Gauss-like c.f. system has full HD-spectrum. It is enough to show that: $\lambda_{A\cup\{b\}}(t) \le \lambda_{A\cup N_{b+1}}(t)$ for every $t \in [0,1]$. Using Lemma 4.2. we have that:

$$\lambda_{A\cup\{b\}}(t) \le \lambda_A(t) + \frac{(4k)^t}{(k+b+2)^{2t}}.$$

Using Lemma 4.1. we have that:

$$\lambda_{A\cup N_{b+1}}(t) \ge \lambda_A(t) + \sum_{j\ge b+1} \frac{k^t}{(k+j+1)^{2t}}.$$

Our proof is complete if we can show that for every $t \in [0,1]$

$$\frac{(4k)^t}{(k+b+2)^{2t}} \le \sum_{j\ge b+1} \frac{k^t}{(k+j+1)^{2t}}.$$

It is enough to prove that this inequality holds for $t = 1$. Thus we need to show that

$$\sum_{j \geq b+1} \frac{k}{(k+j+1)^2} \geq \frac{4k}{(k+b+2)^2}.$$

But using the integral test, this inequality holds for every $b \geq 1$. □

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