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CONTINUITY STRUCTURE OF $f \longmapsto \cup_{x\in I}\omega(x,f)$ AND $f \longmapsto \{\omega(x,f) : x \in I\}$

Abstract

Let the maps Λ and Ω be defined on $C(I,I)$ so that $f \longmapsto \Lambda(f) = \cup_{x\in I}\omega(x,f)$ and $f \longmapsto \Omega(f) = \{\omega(x,f) : x \in I\}$. We characterize those functions at which Λ is continuous, as well as those functions at which Ω is continuous when its domain is restricted to those elements of $C(I,I)$ possessing zero topological entropy.

1 Introduction

At the Twentieth Summer Symposium in Real Analysis, A. M. Bruckner posed several questions regarding the iterative stability of continuous functions as they experience small perturbations, as well as why these questions are of general interest [3]. In particular, how are the set of ω -limit points and the collection of ω-limit sets of a function affected by slight changes in that function? As Bruckner discusses in [3], we may also want to ask these questions when restricting our attention to particular subsets of $C(I,I)$, such as those functions that are in some way nonchaotic, or those functions that satisfy a particular smoothness condition. As one sees from various examples found in [3] and [10], in general, both the set of ω-limit points and the collection of ω-limit sets of a typical function are affected dramatically by arbitrarily small perturbations. We found in [10], however, that by restricting ourselves to certain classes of functions and certain types of ω-limit sets, one gets more positive results.

Theorem 1. *Suppose $\{f_n\} \subset C(I,I), f_n \to f$ uniformly, ω_n is an ω-limit set of f_n for each n, and ω_n converges to ω with respect to the Hausdorff metric.*

1. *As in the statement of Theorem 5, p.3, if ω is a finite set, then ω is an ω-limit set of f.*

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2. *If the topological entropy of f_n is zero for each n, and ω is infinite, then the maximal perfect subset of ω is an ω -limit set of f.*

Theorem 2. *Suppose $f \in C(I, I), f$ has only a finite number of ω-limit sets, and each ω-limit set of f is stable. Then the map Ω taking g to $\Omega(g) = \{\omega(x, g) : x \in I\}$ is continuous at f.*

In this paper we build upon the results of [10] in a couple of ways. First, we describe those continuous functions for which slight perturbations have only a minimal impact upon the set of ω-limit points. In particular, we characterize the points of continuity of the map Λ taking f to $\Lambda(f) = \cup_{x \in I}\omega(x, f)$. We then restrict our attention to those functions possessing zero topological entropy, and characterize the points of continuity of the map Ω with that restriction.

We proceed through several sections. In section 2 we present the notation and definitions we will use throughout the balance of the paper. Section 3 is dedicated to characterizing those functions at which Λ is continuous, and section 4 deals with the continuity of Ω restricted to functions with zero topological entropy. In section 5 we conclude with some open problems and a few observations.

2 Preliminaries

We shall be concerned with the class $C(I, I)$ of continuous functions mapping the unit interval $I = [0, 1]$ into itself, and the iterative properties this class of functions possesses. For f in $C(I, I)$ and any integer $n \geq 1, f^n$ denotes the n^{th} iterate of f. Let $P(f)$ represent those points $x \in I$ that are periodic under f, and if x is a periodic point of period n for which $f^n(x) - x$ is not unisigned in any deleted neighborhood of x, then x is called a stable periodic point; we let $S(f)$ represent the stable periodic points of f, and let $P_n(f) = \{x \in I : f^n(x) = x, f^m(x) \neq x$ whenever $m \mid n\}$ represent the f -periodic points of period n. For each x in I, we call the set of all subsequential limits of the sequence $\{f^n(x)\}_{n=0}^{\infty}$ the ω -limit set of f generated by x, and write $\omega(x, f)$. Let $\Lambda(f) = \cup_{x \in I}\omega(x, f)$ represent the ω-limit points of f, while $\Omega(f) = \{\omega(x, f) : x \in I\}$ denotes the set composed of the ω-limit sets of f. Now, let $\varepsilon > 0$ be given, and take x and y to be any points in $[0, 1]$. An ε-chain from x to y with respect to a function f is a finite set of points $\{x_0, x_1, ..., x_n\}$ in $[0, 1]$ with $x = x_0, y = x_n$ and $\mid f(x_{k-1}) - x_k \mid < \varepsilon$ for $k = 0, 1, ..., n - 1$. We call x a chain recurrent point of f if there is an ε-chain from x to itself for any $\varepsilon > 0$, and write $x \in CR(f)$. We note that for every f in $C(I, I)$, $\Lambda(f) \subseteq CR(f)$.

In addition to the usual, Euclidean metric d on $I = [0,1]$, we will be working in three metric spaces. Within $C(I,I)$ we will use the supremum metric given by $\|f - g\| = \sup\{| f(x) - g(x) |: x \in I\}$. Our second metric space $(\mathcal{K},\mathcal{H})$ is composed of the class of nonempty closed sets $\mathcal{K}$ in I endowed with the Hausdorff metric $\mathcal{H}$ given by $\mathcal{H}(E,F) = \inf\{\delta > 0 : E \subset B_\delta(F), F \subset B_\delta(E)\}$, where $B_\delta(F) = \{x \in I : d(x,y) < \delta, y \in F\}$. This space is compact [4]. Our final metric space $(\mathcal{K}^*,\mathcal{H}^*)$ consists of the nonempty closed subsets of $\mathcal{K}$. Thus, $K \in \mathcal{K}^*$ if K is a nonempty family of nonempty closed sets in I such that K is closed in $\mathcal{K}$ with respect to $\mathcal{H}$. We endow $\mathcal{K}^*$ with the metric $\mathcal{H}^*$ so that K_1 and K_2 are close with respect to $\mathcal{H}^*$ if each member of K_1 is close to some member of K_2 with respect to $\mathcal{H}$, and vice versa. This metric space is also compact [3]. Our interest in, and the utility of, the spaces $(\mathcal{K},\mathcal{H})$ and $(\mathcal{K}^*,\mathcal{H}^*)$ stem from the following two theorems from [1] and [2], respectively.

Theorem 3. *For any f in $C(I,I)$, the set $\Lambda(f)$ is closed in I.*

Theorem 4. *For any f in $C(I,I)$, the set $\Omega(f)$ is closed in $(\mathcal{K},\mathcal{H})$.*

To a large extent, our work investigates the iterative stability of $f \in C(I,I)$ under small perturbations by studying the continuity structure of the maps $\Lambda : (C(I,I), \| \circ \|) \to (\mathcal{K},\mathcal{H})$ given by $f \longmapsto \Lambda(f)$, and $\Omega : (C(I,I), \| \circ \|) \longrightarrow (\mathcal{K}^*,\mathcal{H}^*)$ given by $f \longmapsto \Omega(f)$.

In much of the sequel we will restrict our attention to a closed subset $\mathcal{E}$ of $C(I,I)$ composed of those functions f having zero topological entropy, denoted by $\mathbf{h}(f) = 0$. The reader is referred to Theorem A of [7] for an extensive list of equivalent formulations of topological entropy zero. For our purposes, it suffices to note that every periodic orbit of a continuous function with zero topological entropy has cardinality of a power of two. The following theorem, due to Smital [9], sheds considerable light on the structure of infinite ω-limit sets for functions with zero topological entropy.

Theorem 5. *If ω is an infinite ω-limit set of $f \in C(I,I)$ possessing zero topological entropy, then there exists a sequence of closed intervals $\{J_k\}_{k=1}^\infty$ in $[0,1]$ such that*

1. *for each $k, \{f^i(J_k)\}_{i=1}^{2^k}$ are pairwise disjoint, and $J_k = f^{2^k}(J_k)$.*
2. *for each $k, J_{k+1} \cup f^{2^k}(J_{k+1}) \subset J_k$.*
3. *for each $k, \omega \subset \cup_{i=1}^{2^k} f^i(J_k)$.*
4. *for each k and $i, \omega \cap f^i(J_k) \neq \emptyset$.*

We make the following definitions with Smital's Theorem in mind. Let ω be an infinite compact subset of I, and let f map ω into itself. We call f a simple map on ω if ω has a decomposition $S \cup T$ into compact portions that f exchanges, and f^2 is simple on each of these portions. From Smital's Theorem one sees that every map f with zero topological entropy is simple on each of its infinite ω-limit sets. Let $\{J_k\}_{k=1}^{\infty}$ be a nested sequence of compact periodic intervals with respect to ω and f as described in Smital's Theorem. Every set of the form $\omega \cap f^i(J_k)$ is periodic of period 2^k, and we call each such set a periodic portion of rank k. This system of periodic portions of ω, or of the corresponding periodic intervals, is called the simple system of ω with respect to f. Now, let $K = \cap_{n=1}^{\infty} \cup_{i=1}^{2^n} f^i(J_n)$. We note that every nondegenerate component of K is a wandering interval of f with a trajectory that is contained in K.

3 The Continuity of $\Lambda : C(I,I) \longrightarrow \mathcal{K}$

The main result of this section, Theorem 6, characterizes those functions $f \in C(I,I)$ at which the map $\Lambda : (C(I,I), \| \circ \|) \longrightarrow (\mathcal{K}, \mathcal{H})$ is continuous.

Theorem 6. *Λ is continuous at f if and only if $\overline{S(f)} = CR(f)$.*

This result follows from Lemmas 8 and 9, as Lemma 8 characterizes those continuous functions at which Λ is upper semicontinuous, and Lemma 9 characterizes those continuous functions at which Λ is lower semicontinuous. In the proof of Lemma 8, it is helpful to recall the following result from [1].

Lemma 7. *If $x \in CR(f)$, then any open neighborhood of f in $C(I,I)$ contains a function g for which $x \in P(g)$.*

Lemma 8. *Let $f \in C(I,I)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ so that $\Lambda(g) \subset B_\varepsilon(\Lambda(f))$ whenever $\|f - g\| < \delta$ if and only if $\Lambda(f) = CR(f)$.*

Proof. Suppose $\Lambda(f) = CR(f)$. Since $CR : (C(I,I), \| \circ \|) \longrightarrow (\mathcal{K}, \mathcal{H})$ given by $g \mapsto CR(g)$ is upper semicontinuous, for any $\varepsilon > 0$ there is a $\delta > 0$ so that $CR(g) \subset B_\varepsilon(CR(f))$ whenever $\|f - g\| < \delta$ [1]. By hypothesis, we have that $\Lambda(f) = CR(f)$, so that $\Lambda(g) \subset CR(g) \subset B_\varepsilon(\Lambda(f))$, and our conclusion follows.

Now, let us suppose that $x \in CR(f) - \Lambda(f)$. Then there exists $\{f_n\} \subset C(I,I)$ so that $f_n \longrightarrow f$ and $x \in P(f_n)$ for each n, so that $x \in \lim \Lambda(f_n) - \Lambda(f)$. □

Lemma 9. *Let $f \in C(I,I)$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ so that $\Lambda(f) \subset B_\varepsilon(\Lambda(g))$ whenever $\|f - g\| < \delta$ if and only if $\overline{S(f)} = \Lambda(f)$.*

PROOF. The sufficiency of our lemma follows immediately from the definition of a stable periodic orbit, and the compactness of $\Lambda(f)$. As for the necessity, let us suppose $\overline{S(f)}$ is a proper subset of $\Lambda(f)$, and let J be an open interval in $[0,1]$ for which $\Lambda(f)\cap J \neq \emptyset$, but $\overline{S(f)} \cap J = \emptyset$. If $P(f)\cap J \neq \emptyset$, then there exists K an open interval contained in J for which $P(f)\cap K \subseteq P_n(f)$, for some natural number n. If $P(f)\cap J = \emptyset$, set $K = J$. In either case, then, there exists $\{f_n\} \subset C(I,I)$ and an open interval K for which $K \cap \Lambda(f) \neq \emptyset$, but $P(f_n)\cap K = \emptyset$ for all natural numbers n, and $f_n \longrightarrow f$. Since we may take f_n to be piecewise monotonic on I, $\Lambda(f_n) = \overline{P(f_n)}$ for each n [1], so that by taking a subsequence of $\{f_n\}$ if necessary, we have $\Lambda(f_n) = \overline{P(f_n)} \to F$ in $(\mathcal{K},\mathcal{H})$, with $F\cap K = \emptyset$. Thus, $\Lambda(f)$ is not contained in $\lim \Lambda(f_n)$. □

4 The Continuity of $\Omega \mid \mathcal{E}$

In this section we characterize those functions with zero topological entropy at which the map $\Omega \mid \mathcal{E}$ is continuous. It is interesting to note that these functions can be characterized in exactly the same way as those found in Theorem 6; Lemma 14 sheds some light on why this should be the case. Our main result is the following theorem.

Theorem 10. *$\Omega \mid \mathcal{E}$ is continuous at f if and only if $\overline{S(f)} = CR(f)$.*

We begin our development of Theorem 10 with a couple rather technical lemmas.

Lemma 11. *Suppose each periodic orbit of $f \in C(I,I)$ is of order 2^m for some $m \leq n$. If each of the necessarily periodic ω-limit sets of f is stable, then $P(f)$ is nowhere dense in $[0,1]$.*

PROOF. Let $F(f) = \{x \in I : f(x) - x = 0\}$. Then $F(f)$ is closed, and since $f(x)-x$ is not unisigned in any deleted neighborhood of $x \in F(f), F(f)$ must be nowhere dense. Similarly, $F(f^2)$ is closed, and since $f^2(x)-x$ is not unisigned in any deleted neighborhood of $x \in F(f^2)-F(f), F(f^2)$ is nowhere dense. In general, then, $F(f^{2^k})$ is nowhere dense for $k = 0,1,...,n$. But $F(f^{2^n}) = P(f)$. □

Lemma 12. *Suppose $f \in C(I,I)$ has points of period 2^n for all natural numbers n , but no others. If $S(f) = P(f)$, then $P(f)$ is nowhere dense.*

PROOF. As in the proof of Lemma 11, $F(f^{2^n})$ is nowhere dense for each $n \in \mathbb{N}$. Since $P(f) = \cup_{n=1}^{\infty} F(f^{2^n})$, it follows that $P(f)$ is of the first category, so that $P(f)$ is in fact nowhere dense by Proposition 4.1 of [8]. □

We now need to recall another result from [8], rewritten in terms of our current notation. What this theorem allows us to do is ε -approximate every ω-limit set of a function g with one of its 2^k-cycles of length no more than $2^{N(\varepsilon)}$ for some $N(\varepsilon)$ in $\mathbb{N}$, whenever g is sufficiently close to a particularly well behaved function f.

Theorem 13. *Suppose $f \in C(I,I), \mathbf{h}(f) = 0$ and $P(f)$ is nowhere dense with $intK = \emptyset$ for any simple system of f. Then, for any $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $\delta(\varepsilon) > 0$ so that the following condition holds: If $\|f-g\| < \delta(\varepsilon)$, then for any $\omega \in \Omega(g)$ there exists a 2^k cycle $p \in \Omega(g)$ such that $k \leq n(\varepsilon)$ and $\mathcal{H}(\omega, p) < \varepsilon$.*

Once we have verified our next result, we will actually prove a bit more than the sufficiency of Theorem 10 with Proposition 15.

Lemma 14. *Suppose $f \in C(I,I)$, and $\mathbf{h}(f) = 0$. Then $S(f) = P(f)$ and $intK = \emptyset$ for all simple systems of f if and only if $\overline{S(f)} = CR(f)$.*

Proof. Suppose $S(f) = P(f)$, and $intK = \emptyset$ for all simple systems of f. Then f is nonchaotic in the sense of Li and Yorke, so that $P(f)$ is dense in $\Omega(f)$, and $\overline{S(f)} = \Lambda(f)$ [5]. Moreover, since the chain recurrent set of any function with zero topological entropy is the union of its ω-limit points together with the wandering intervals found in its simple systems, one has $\Lambda(f) = CR(f)$, too.

Now, let us suppose that $\overline{S(f)} = \Lambda(f) = CR(f)$. Since $\Lambda(f) = CR(f)$, we have $intK = \emptyset$ for all the simple systems of f. Since $\overline{S(f)} = \Lambda(f)$, we have $P(f) \subset \overline{S(f)}$, so that $P(f) = S(f)$. □

Proposition 15. *Suppose $f \in C(I,I)$, $\mathbf{h}(f) = 0$, $S(f) = P(f)$ and $intK = \emptyset$ for all simple systems of f . Then $\Omega : (C(I,I), \|\circ\|) \longrightarrow (\mathcal{K}^*, \mathcal{H}^*)$ is continuous at f.*

Proof. Let $\varepsilon > 0$. With Theorem 13 in mind, choose $n \in \mathbb{N}$ and $\delta_1 > 0$ so that the following condition holds: If $\|f - g\| < \delta_1$, then for any $\omega \in \Omega(g)$ there exists $p \in \Omega(g)$ a 2^k cycle so that $k \leq n$, and $\mathcal{H}(\omega, p) < \frac{\varepsilon}{2}$. We now take $\delta_2 > 0$ so that if $\|f - g\| < \delta_2$, then for any 2^k cycle $p \in \Omega(g)$, with $k = 0, 1, ..., n$, there exists $q \in \Omega(f)$ a 2^m cycle, $m \leq k$, such that $\mathcal{H}(q, p) < \frac{\varepsilon}{2}$. Set $\delta = \min\{\delta_1, \delta_2\}$. If $\|f - g\| < \delta$, then for any $\omega \in \Omega(g)$ there exists a 2^m cycle $q \in \Omega(f)$ so that $\mathcal{H}(\omega, q) < \varepsilon$. It follows from Theorem 4 that if $f_n \longrightarrow f, \omega_n \in \Omega(f_n)$ for each n, and $\omega_n \longrightarrow \omega$, then $\omega \in \Omega(f)$. It remains to show that if $\omega \in \Omega(f)$ and $\{f_n\} \subseteq C(I,I)$ for which $f = \lim_{n\to\infty} f_n$, then there exists an appropriate sequence $\{\omega_n\} \subseteq (\mathcal{K}, \mathcal{H})$ where $\omega_n \in \Omega(f_n)$ for each n, so that $\omega = \lim_{n\to\infty} \omega_n$. If $\omega \in \Omega(f)$ is finite, then ω is stable, and our conclusion follows [10]. If ω is infinite, then ω can be approximated by 2^n cycles, each of which is stable, since $intK = \emptyset$ for the simple system containing ω. □

In order to prove the necessity of Theorem 10, we develop the construction found in our next result.

Lemma 16. *$\Lambda \mid \mathcal{E}$ is discontinuous at f whenever $intK \neq \emptyset$ for some simple system of f.*

PROOF. Let $\varepsilon > 0$, and suppose there is some simple system of f for which $intK \neq \emptyset$; let J be a wandering interval found in this simple system, with c the midpoint of J. Since $\Lambda(f) \cap intK = \emptyset$, it follows that c is not an ω -limit point of f. Now, choose n so that $\frac{1}{2^n} < \varepsilon$, and let K_n represent the 2^n f-periodic intervals, each with period 2^n, found in the simple system containing J. From our choice of n, K_n must contain an interval L such that the length of $f(L)$ is less than ε. Moreover, there exists $m \leq 2^n - 1$ so that $f^m(c)$ is contained in L; say $f^m(c) = d$. We now develop a function g in $C(I,I)$ by perturbing f on L to get a function that is monotonic there, and for which $g^{2^n - m}(d) = c$. It follows, then, that $\mathbf{h}(g) = 0, \|f - g\| < \varepsilon$ and $c \in P(g)$. We conclude that $\Lambda \mid \mathcal{E}$ cannot be continuous at f. □

We are now in a position to prove the section's main result.

PROOF OF THEOREM 10. The sufficiency of our theorem follows from Proposition 15; if Ω is continuous at f, then $\Omega \mid \mathcal{E}$ must necessarily be continuous there, too, provided $\mathbf{h}(f) = 0$. Now, suppose that $intK \neq \emptyset$ for a simple system of f. Then $\Lambda \mid \mathcal{E}$ is not continuous at f by Lemma 16, so that $\Omega \mid \mathcal{E}$ is not continuous there, either. If $P(f) \neq S(f)$, but $intK = \emptyset$ for all the simple systems of f, then there exists a periodic $\omega \in \Omega(f)$ such that $\omega \subset P(f) - \overline{S(f)}$, and the discontinuity of $\Lambda \mid \mathcal{E}$ follows [10]. □

5 Conclusions

While we have been able to answer a couple of Bruckner's queries by characterizing the points of continuity of Λ and $\Omega \mid \mathcal{E}$, obvious questions remain. In particular, how can one characterize the points of continuity of the map $\Omega : C(I,I) \longrightarrow \mathcal{K}^*$ without any domain restrictions? A partial answer is provided by the following proposition, which we present without proof.

Proposition 17. *Let $f \in C(I,I)$ with positive topological entropy, and take $X \subset I$ so that $f^n(X) = X$, and $f^n \mid X$ is semiconjugate to the shift operator σ on two symbols, for some natural number n. If X possesses a component with nonempty interior, then Ω is not continuous at f.*

The proof of this result rests on showing that $\Lambda(f) \neq CR(f)$, so that Λ, and therefore Ω, are not continuous there. But what about continuous functions f for which $\mathbf{h}(f) > 0$, $S(f) = P(f)$, and $\overline{S(f)} = \Lambda(f) = CR(f)$? An example of such a function is the double hat map $h : [-1, 1] \longrightarrow [-1, 1]$, where $h(-1) = h(0) = h(1) = 0, h(\frac{1}{2}) = -h(-\frac{1}{2}) = 1$, and h is extended linearly to all of $[-1, 1]$. It is currently unknown whether or not Ω is continuous at h, and since Λ is continuous there, the proof techniques used in section 4 as well as Proposition 17 are not applicable.

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