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# TWICE PERIODIC MEASURABLE FUNCTIONS

**Abstract**

In this note we prove that, for $a, b \in (0,1)$ and $f$ a measurable function mapping $[0,1]$ to $\mathbb{R}$, the following statements are equivalent:

(i) $f(x) = f(x-a)$ a.e. in $[a,1]$ and $f(x) = f(x-b)$ a.e. in $[b,1]$ implies that $f$ is a.e. constant in $[0,1]$.

(ii) $a + b \leq 1$ and $a/b$ is irrational.

Dealing with periods of measurable functions it is well known that, for a periodic real-valued function defined on $\mathbb{R}$, *either* there exists the smallest positive period $t_0$ and all periods are of the form $nt_0$ where $n$ is any integer, *or* the set of the periods is dense. Moreover, if a measurable function has a dense set of periods then it is a.e. constant. On the other hand, if a twice periodic measurable real-valued function is defined on the interval $[0,1]$, no results like the above, imposing conditions on the periods of the function, seem to exist in the literature.

Denote by $\mathcal{F}_a$ the set of measurable functions $f\colon [0,1] \to \mathbb{R}$ such that $f(x) = f(x-a)$ a.e. in $[a,1]$ where $a \in (0,1)$, and let $\mathcal{C}$ be the set of functions mapping $[0,1]$ to $\mathbb{R}$ which are constant a.e. in $[0,1]$. The result we shall prove is the following:

**Theorem 1.** *If $a, b \in (0,1)$ then $a+b \leq 1$ and $a/b$ is irrational* $\Leftrightarrow \mathcal{F}_a \cap \mathcal{F}_b = \mathcal{C}$.

Let us first prove an auxiliary result. For any function $f\colon [0,1] \to \mathbb{R}$ let $H_f$ be the set of points $a \in (0,1)$ such that $f(x) = f(x-a)$ a.e. in $[a,1]$.

**Lemma 2.** *Let $f\colon [0,1] \to \mathbb{R}$ and suppose $a, b \in H_f$, $a + b \leq 1$, and $a/b$ is irrational. Then $H_f$ is dense in $[0,1]$.*

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PROOF. Assume $a < b$. Let $a_0 := a$, $b_0 := b$ and define recursively $a_{n+1} := \min\{a_n, b_n - a_n\}$ and $b_{n+1} := \max\{a_n, b_n - a_n\}$ $(n \in \mathbb{N} \cup \{0\})$. Let us show that $a_n \to 0$, $n \to \infty$. The fact that the limits of $a_n$ and $b_n$ exist and are nonnegative follows since the sequences are decreasing and nonnegative. If $x = \lim a_n$ and $y = \lim b_n$, using that $a_n + b_n = b_{n-1}$ we get

$$x + y = \lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} b_{n-1} = y$$

implying that $x = 0$. Now, one can easily show that $\{a_n\} \subset H_f$ which in turn implies (we omit the trivial proof) that $H_f$ is dense. $\square$

PROOF OF THEOREM.

"$\Rightarrow$": Suppose $f\colon [0,1] \to \mathbb{R}$ measurable is not a.e. constant but $a, b \in H_f$, $a + b \leq 1$ and $a/b$ is irrational. Then the inverse image of some interval is a set $A$ with measure $0 < m(A) < 1$. By the Lebesgue density theorem there is an interval $I$ (with length less than $\epsilon > 0$) where the density of $A$ is less than $\epsilon$. If $h$ is in $H_f$ and $I + h$ (or $I - h$) is in $(0,1)$ then the intersection of $A$ and $I$ is congruent to the intersection of $A$ and $I + h$ (or $I - h$), so the density is the same in $I + h$ (or $I - h$). By lemma, $H_f$ is dense, and so we can cover almost the whole $(0,1)$ interval (with an exception of finitely many intervals with total length less than $\epsilon$) with disjoint translates using translations from $H_f$. Since in each translate the density is less than $\epsilon$ and only less than $\epsilon$ is uncovered we get that $m(A) < 2\epsilon$ for any $\epsilon > 0$, which is a contradiction.

"$\Leftarrow$": Suppose $0 < a < b < 1$ are such that $a + b > 1$ (the $a/b$ rational case is quite obvious). Let $A_0 := [1 - b, a)$. If $A_k$ is in $[0, 1 - a)$ then let $A_{k+1} := A_k + a$; if $A_k$ is in $[b, 1)$ then let $A_{k+1} := A_k - b$; if neither then let $m := k$ and stop. It is easily seen that if $x \in A_i \cap A_j$ $(i < j)$ then either $x - a$ or $x + b$ is in $A_{i-1}$ and $A_{j-1}$. Repeating this, we get that $A_{j-i}$ intersects $A_0 = [1-b, a)$ which cannot happen by definition. Thus $A_0, A_1, \ldots$ are disjoint intervals with length $a+b-1 > 0$ and so $m$ must be finite. Let $B_m$ be a proper subinterval of the intersection of $A_m$ and $[1-a, b)$ and, going backwards, define $B_k$ as a subinterval of $A_k$ such that $B_{k+1} = B_k + a$ or $B_k - b$ in the same way as in the definition of $A_k$. Then the characteristic function of the union of $B_0, B_1, \ldots, B_m$ is in $\mathcal{F}_a$ and $\mathcal{F}_b$ but not a.e. constant. $\square$

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