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INVARIANT MEASURABLE STRUCTURES ON GROUPS AND NONMEASURABLE SUBGROUPS

Abstract

For any group G, the notion of an invariant measurable structure on G is introduced. The following question is investigated: does there exist a subgroup of G nonmeasurable with respect to this structure? It is demonstrated that, for an uncountable solvable group G, such a subgroup of G always exists.

Let E be a set, let S be a σ-algebra of subsets of E and let I be a proper σ-ideal of subsets of E, such that $I \subset S$. We shall say that the pair (S, I) determines a measurable structure on E. Elements from S (respectively, from I) are usually called measurable sets (respectively, small sets) with respect to this structure. Such a situation can frequently be met in various domains of mathematics. The best known examples are: the measurability in the Lebesgue sense and the so-called Baire property (see, for instance, [1], [2] and [3]). In the first case, S is the σ-algebra of all Lebesgue measurable subsets of the real line $\mathbb{R}$ and I is the σ-ideal of all Lebesgue measure zero sets in $\mathbb{R}$. In the second case, S is the σ-algebra of all subsets of $\mathbb{R}$ having the Baire property and I is the σ-ideal of all first category sets in $\mathbb{R}$.

A class of morphisms (homomorphisms) can be introduced for measurable structures e.g. in the following way. We say that a surjective mapping

$$\phi \ : \ (E, (S, I)) \to (E', (S', I'))$$

is a homomorphism if, for any set $X \in S'$, we have $\phi^{-1}(X) \in S$ and, for any set $Y \subset E'$, the relation

$$\phi^{-1}(Y) \in I \Leftrightarrow Y \in I'$$

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is fulfilled. We thus obtain a certain subcategory of the standard category consisting of all sets and mappings between them. Further, a measurable structure (S, I) satisfies the countable chain condition if there is no uncountable disjoint family of sets belonging to $S \setminus I$. It can easily be observed that any homomorphic image of a measurable space satisfying the countable chain condition satisfies this condition, too.

Suppose now that the original set E is an uncountable group denoted by the symbol $(G, \cdot)$. We say that a measurable structure (S, I) on G is left invariant (or, simply, invariant) if both classes of sets S and I are invariant with respect to the group of all left translations of G.

The class of groups equipped with invariant measurable structures forms a subcategory of the above-mentioned category when the class of morphisms is restricted to those mappings which simultaneously are group homomorphisms.

Let G be an uncountable group and let (S, I) be an invariant measurable structure on G satisfying the countable chain condition. The following question seems to be of some interest: does there exist a subgroup of G which is not measurable with respect to (S, I)? It turns out that this question admits a positive solution for a sufficiently wide class of groups. In particular, as shown below, any uncountable solvable group G contains a nonmeasurable subgroup.

In order to establish this fact, we need some auxiliary notions and propositions. The following two statements are purely combinatorial.

Lemma 1. *Let (I, S) be a measurable structure satisfying the countable chain condition and let $\{Z_\alpha : \alpha < \omega_1\}$ be an uncountable family of sets belonging to S. Further, let $m > 0$ be a fixed natural number and suppose that, for any m-element subset D of ω_1, the relation*

$$\cap\{Z_\alpha : \alpha \in D\} \in I$$

is fulfilled. Then there exists an uncountable subset A of ω_1 such that $Z_\alpha \in I$ for each ordinal α from A.

Proof. The proof of this lemma is not hard. It can be carried out by induction on m (for details, see [6]). The case $m = 2$ is, in fact, equivalent to the countable chain condition. □

Let Y be a set of cardinality ω_1. Consider a double family

$$(Y_{n,\xi})_{n<\omega,\xi<\omega_1}$$

of subsets of Y. We shall say that this family is an admissible transfinite matrix for Y if it possesses the next two properties:

(a) for each ordinal number $\xi < \omega_1$, the partial family $(Y_{n,\xi})_{n<\omega}$ is increasing by inclusion and

$$card(Y \setminus \cup\{Y_{n,\xi} \ : \ n < \omega\}) \leq \omega;$$

(b) for each natural number n, there exists a natural number $m = m(n)$ such that, for any set $D \subset \omega_1$ with $card(D) = m$, we have the equality

$$\cap\{Y_{n,\xi} \ : \ \xi \in D\} = \emptyset.$$

Lemma 2. *For any set Y with $card(Y) = \omega_1$, there exists an admissible transfinite matrix.*

Proof. The proof of Lemma 2 easily follows from the existence of an Ulam matrix over Y (in connection with this matrix and some of its applications, see e.g. [1], [2] or [3]). Indeed, let

$$(X_{n,\xi})_{n<\omega,\xi<\omega_1}$$

be an arbitrary Ulam matrix for Y. Then we have:

(c) for each ordinal number $\xi < \omega_1$, the set $Y \setminus \cup\{X_{n,\xi} \ : \ n < \omega\}$ is at most countable;

(d) for each natural number n, the partial family $\{X_{n,\xi} \ : \ \xi < \omega_1\}$ is disjoint.

Let us define

$$Y_{n,\xi} = \cup\{X_{k,\xi} \ : \ k \leq n\}$$

for all $n < \omega$ and $\xi < \omega_1$. Then it is not hard to verify that the family $(Y_{n,\xi})_{n<\omega,\xi<\omega_1}$ is an admissible matrix of subsets of Y (namely, for any natural number n, we may put $m(n) = n + 2$). □

Remark 1. Recall that Ulam matrices are usually utilized in order to establish the classical fact that ω_1 is not a real-valued measurable cardinal. In this connection, it should be noted that the existence of an admissible matrix is also sufficient to prove the non-real-valued measurability of ω_1. Moreover, if $(Y_{n,\xi})_{n<\omega,\xi<\omega_1}$ is an arbitrary admissible matrix of subsets of a set Y with $card(Y) = \omega_1$ and (S, I) is a measurable structure on Y satisfying the countable chain condition and such that all countable subsets of Y belong to I, then there are uncountably many sets $Y_{n,\xi}$ nonmeasurable with respect to this structure (the proof can be deduced from Lemma 1). It would be interesting to investigate other relationships between Ulam matrices and admissible matrices.

Starting with Lemmas 1-2 and applying some well-known results concerning the algebraic structure of infinite commutative groups (see, for instance, the classical monograph by Kurosh [4]), we obtain the following statement.

Lemma 3. *Let $(G,+)$ be an arbitrary uncountable commutative group and let (S,I) be an invariant measurable structure on G satisfying the countable chain condition. Then there exists a subgroup of G nonmeasurable with respect to this structure.*

Proof. Let us sketch the proof of this lemma (which plays the key role in further considerations). First, we may assume, without loss of generality, that all countable subsets of G belong to I. Then we represent G in the form

$$G = \cup\{\Gamma_k : k < \omega\},$$

where $\{\Gamma_k : k < \omega\}$ is an increasing (with respect to inclusion) countable family of subgroups of G, such that every group Γ_k can be represented as a direct sum of cyclic groups. Now, only two cases are possible.

1. For all natural numbers k, the inequality $card(G/\Gamma_k) \geq \omega_1$ holds. In this case, taking into account the fact that our σ-ideal I is proper, we easily infer that there exists a natural number p such that $\Gamma_p \notin I$. On the other hand, the group G contains an uncountable family of pairwise disjoint translates of the group Γ_p. Hence Γ_p cannot be measurable with respect to the given structure.

2. There exists a natural number k such that $card(G/\Gamma_k) \leq \omega$. In this case, we fix a k with this property and consider the group Γ_k. If Γ_k is nonmeasurable with respect to our structure, then there is nothing to prove. Suppose now that $\Gamma_k \in S$. Then we obviously have

$$card(\Gamma_k) \geq \omega_1, \quad \Gamma_k \in S \setminus I.$$

Therefore, it suffices to demonstrate that Γ_k contains a subgroup nonmeasurable with respect to the restriction of our measurable structure to Γ_k. Since Γ_k is uncountable and is a direct sum of cyclic groups, we can write

$$\Gamma_k = G_1 + G_2 \qquad (G_1 \cap G_2 = \{0\}),$$

where G_1 and G_2 are some subgroups of Γ_k, and G_1 satisfies the following relations:

(1) $card(G_1) = \omega_1$;

(2) G_1 is a direct sum of cyclic groups.

We may suppose that $G_2 \in S$ (otherwise, there is nothing to prove). Since there are uncountably many pairwise disjoint translates of G_2, we simultaneously have $G_2 \in I$. Let us put

$$G_1 = \sum_{\xi < \omega_1} G_{1,\xi},$$

where each $G_{1,\xi}$ is the cyclic group generated by some element y_ξ (the sum in the equality above is direct). Also, let us denote

$$Y = \{y_\xi \ : \ \xi < \omega_1\}$$

and consider any admissible matrix $(Y_{n,\xi})_{n<\omega,\xi<\omega_1}$ for the set Y whose cardinality is ω_1. For all $n < \omega$ and $\xi < \omega_1$, define

$$H_{n,\xi} = [Y_{n,\xi}] + G_2,$$

where $[Y_{n,\xi}]$ denotes the group generated by $Y_{n,\xi}$. Treating the set Y as a weak analogue of a Hamel basis, we come to the following relations:

(*) for each ordinal $\xi < \omega_1$, the set $\Gamma_k \setminus \cup\{H_{n,\xi} \ : \ n < \omega\}$ belongs to the σ-ideal I;

(**) for any natural number n and for any set $D \subset \omega_1$ with $card(D) = m(n)$, we have

$$\cap\{H_{n,\xi} \ : \ \xi \in D\} = G_2 \in I.$$

Now, it is clear that there exist a natural number n and an uncountable set $B \subset \omega_1$, for which all the groups $H_{n,\xi}$ ($\xi \in B$) do not belong to I. Applying Lemma 1 to the family $(H_{n,\xi})_{\xi\in B}$, we conclude that there are uncountably many groups from this family, nonmeasurable with respect to our structure. Lemma 3 has thus been proved. □

We denote by M the class of all those uncountable groups G which have the property that, for any invariant measurable structure (S, I) on G satisfying the countable chain condition, there exists at least one subgroup of G nonmeasurable with respect to (S, I).

Lemma 4. *Let G and H be two groups and let $\phi \ : \ G \to H$ be a surjective group homomorphism. If $H \in M$, then $G \in M$, too. In other words, the class M is closed under homomorphic pre-images.*

Proof. Indeed, suppose that $H \in M$ and let (S, I) be an arbitrary invariant measurable structure on G satisfying the countable chain condition. We put

$$S' = \{X \subset H \ : \ \phi^{-1}(X) \in S\}, \quad I' = \{X \subset H \ : \ \phi^{-1}(X) \in I\}.$$

Then (S', I') turns out to be an invariant measurable structure on H satisfying the countable chain condition. Consequently, there exists a group $H_0 \subset H$ nonmeasurable with respect to (S', I'). This implies that the group $G_0 = \phi^{-1}(H_0)$ is nonmeasurable with respect to (S, I).

In particular, if $H \in M$ and F is an arbitrary group, then the product group $G = H \times F$ belongs to M. □

Theorem 1. *Any uncountable solvable group belongs to M.*

Proof. Let G be an arbitrary uncountable solvable group and let

$$G = G_k \supset G_{k-1} \supset ... \supset G_1 \supset G_0 = \{e\}$$

denote the composition series for G. We use induction on k. Only two cases are possible.

1. The factor group G_k/G_{k-1} is uncountable. In this case, we have a canonical surjective homomorphism

$$\phi \ : \ G_k \to G_k/G_{k-1},$$

where G_k/G_{k-1} is a commutative group. Hence we may apply Lemmas 3 and 4 which yield at once that $G = G_k \in M$.

2. The factor group G_k/G_{k-1} is at most countable. In this case, it is not hard to see that the relation $G_{k-1} \in M \Rightarrow G_k \in M$ holds true. But G_{k-1} turns out to be an uncountable solvable group with composition series of smaller length. According to the inductive assumption, we get $G_{k-1} \in M$. Consequently, $G_k \in M$ and the theorem has thus been proved. □

Remark 2. It would be interesting to characterize the groups of the class M in purely algebraic terms. Note that M differs from the class of all uncountable groups. For example, the Jónsson group of cardinality ω_1 constructed by Shelah [5] does not belong to M.

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