Real Analysis Exchange
Vol. (), , pp. 879–886

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MONOTONE AND DISCRETE LIMITS OF CONTINUOUS FUNCTIONS

Abstract

In this note we prove that for a quite large class of topological spaces every upper semi-continuous function, which is a discrete limit of continuous functions, it is also a pointwise decreasing discrete limit of continuous functions. This question was motivated by a paper of Zbigniew Grande. He asked that whether it be true for the topology of right hand continuity on the real line. He gave a partial answer showing that under some additional condition imposed on the function the answer is affirmative.

Throughout the paper we will use the following notation, for any function $f : X \to \mathbb{R}$, we denote the level set $A(f < c) = \{x \in A : f(x) < c\}$, and similarly for other relations. Recall that a sequence of function f_n converges discretely to f provided that $\cup_n(\cap_{k>n} f_k) = f$, see [2]. We will use the trick due to Urysohn, which makes it possible to define a continuous function through its level sets.

Theorem 1. *Let (X, τ) be topological space, with the property that every open topological subspace is normal. Also, let $f : X \to [0, 1]$ upper semi-continuous, $A \subset X$ closed subset. If $f|_A$ is continuous then there is a continuous function $g : X \to [0, 1]$ such that $g \geq f$ and $f|_A \subset g$.*

Proof. First we prove the statement for function $f : X \to [0, 1)$.

To do this let $\{r_k : k \in \mathbb{N}\}$ be an enumeration of rational numbers in $[0, 1]$, such that $r_0 = 0$ and $r_1 = 1$. We define a sequence (A_k) such that

(1) A_k is open,

(2) $A_k \subset X(f < r_k)$,

(3) $A_k \cap A = A(f < r_k)$,

Key Words: upper-semi continuous function, discrete limit
Mathematical Reviews subject classification: 26A03, 26A15, 26A21, 26A99
Received by the editors March 1, 1999

(4) if $l < k$ and $r_l < r_k$ then $\overline{A_l} \subset A_k$,

(5) if $l < k$ and $r_l > r_k$ then $\overline{A_k} \subset A_l$,

(6) $\overline{A_k \cap A} \subset \overline{A_k} \cap A \subset A(f \leq r_k)$,

(7) and $\overline{A_k} \setminus A \subset X(f < r_k)$.

Assume that we have already defined (A_k) for all k, and let

$$g(p) = \inf \{r_k \ : \ p \in A_k\}.$$

Property (2) ensures that $f \leq g$. From property (1), (4) and (5) g is continuous and finally from property (3) $f|_A = g|_A$.

Now let $A_0 = \emptyset$ and $A_1 = X$. It is obvious that for $k = 0, 1$ all the above properties are fulfilled.

Assume that A_l is defined for $l < k$ satisfying (1)–(7). Then $\tilde{X} = X \setminus A(f = r_k)$ is an open subset, so $(\tilde{X}, \tau|_{\tilde{X}})$ is normal. The closed subsets of $\tilde{X}$

$$B_k = \tilde{X} \cap \left(A(f \leq r_k) \cup \bigcup_{l<k: r_l < r_k} \overline{A_l} \right)$$

and

$$C_k = \tilde{X} \cap \left(X(f \geq r_k) \cup \bigcup_{l<k: r_l > r_k} (X \setminus A_l) \right)$$

are disjoint by (3), (4), (5), (6) and (7). $\tilde{X}$ is normal so they can be separated by disjoint open sets in $\tilde{X}$, in other words: there is an open set $G_k \subset \tilde{X}$ such that it is larger then B_k and its closure (in $\tilde{X}$) is disjoint from C_k. Put $A_k = X(f < r_k) \cap G_k$.

Then property (1) and (2) are clear for A_k. Since

$$G_k \supset \tilde{X} \cap A(f \leq r_k) = A(f < r_k)$$

property (3) also holds.

$A_k \cap \tilde{X} \subset G_k$ therefore $\overline{A_k} \cap \tilde{X}$ is disjoint from C_k. This fact implies property (7). To see (6) observe that $\overline{A_k} \cap A \supset \overline{A_k \cap A}$ and

$$\overline{A_k} \cap A \subset \overline{G_k} \cap A \subset (A \setminus C_k) \cup (A \setminus \tilde{X}) \subset A(f < r_k) \cup A(f = r_k).$$

We have to show that (4) and (5) also hold. Since $\overline{A_l} \setminus A \subset X(f < r_l)$ for $l < k$ we have that if $r_l < r_k$ then

$$\overline{A_l} = (\overline{A_l} \setminus A) \cup (\overline{A_l} \cap A) \subset X(f < r_l) \cup A(f \leq r_l) \subset X(f \leq r_l) \subset X(f < r_k).$$

On the other hand,

$$\overline{A_l} \subset \overline{A_l} \setminus A \cup A(f \leq r_l) \subset \tilde{X},$$

so by the choice of G_k, $\overline{A_l} \subset G_k$. It follows that $\overline{A_l} \subset G_k \cap X(f < r_k) = A_k$ and (4) is shown.

Finally, to see (5) let $l < k$ such that $r_l > r_k$ and observe that

$$\overline{A_k} \setminus A \subset \overline{G_k} \cap \tilde{X}$$

which is disjoint from C_k so $\overline{A_k} \setminus A$ does not meet $\tilde{X} \setminus A_l$. So

$$\overline{A_k} = (\overline{A_k} \setminus A) \cup (\overline{A_k} \cap A) \subset A_l \cup A(f \leq r_k) \subset A_l \cup A(f < r_l) \subset A_l.$$

So the sequence (A_n) can be defined and the theorem is proved for the special case when f does not take the value 1.

For the general case let $f' = (1/2)f$ and g' be a continuous function such that $f'|_A \subset g'$ and $f' \leq g'$. Then $g = \min(2g', 1)$ is continuous, g extends $f|_A$ and $f \leq g$. This completes the proof. □

Of course not all topological space satisfies the condition of the previous theorem, e.g let $X = 2^{\aleph_1}$ be the topological power of the compact discrete space $\{0, 1\} = 2$, where $\aleph_1$ is the first uncountable cardinal. X is compact and Hausdorff, so normal. However, it is also a folklore among topologists that for any point $e \in X$ the subspace $X \setminus \{e\}$ is not normal. This shows that the condition we have found in Theorem 1 is not necessary, since in the paper of Grande [1], Theorem 1 —proposed by the referee— states the same implication for σ-compact spaces.

When someone deals with limits of continuous functions, perfectly normal spaces behave nicely. The next lemma states that for perfectly normal spaces the previous theorem applies.

Lemma 2. *Let (X, τ), be a perfectly normal topological space, if $G \in \tau$ then $(G, \tau|_G)$ is normal.*

PROOF. Let F_1 and F_2 closed subsets of X such that $F_1 \cap F_2 \cap G = \emptyset$. We have to prove that there are open sets $S_1, S_2 \subset G$ such that $S_1 \cap S_2 = \emptyset$, $F_i \cap G \subset S_i$ for $i = 1, 2$.

S_1 and S_2 will be level sets of a continuous function $f : G \to [0, 2]$. So first we define this function. Let $A = F_1 \cup F_2$. Since the space is perfectly normal there are continuous functions $f_i : F_i \to [0,1]$ such that $F_i(f_i = 1) = F_i \setminus G$. Put $f = f_1 \cup (2 - f_2)$. f is a function on A since both f_1 and f_2 takes 1 on $A \setminus G$. It is continuous because it is bounded and its graph is closed in $X \times \mathbb{R}$. f can be extended to X to a continuous function, we denote this extension also f.

Now $S_1 = G(f < 1)$ and $S_2 = G(f > 1)$ are open subsets of G and separate $F_1 \cap G$ and $F_2 \cap G$. □

We can state somewhat more.

Lemma 3. *Let (X, τ) be a topological space such that for any upper semi-continuous function $f : X \to [0,1]$ and closed set A there is a continuous function $g : X \to [0,1]$ such that $f|_A \subset g$ and $f \leq g$, provided that $f|_A$ is continuous. Then X is normal.*

PROOF. Let F_1 and F_2 be disjoint closed sets. Put $f = \chi_{F_1}$ the characteristic function of F_1. f is upper semi-continuous, $f|_{F_1 \cup F_2}$ is continuous, so by the hypothesis there is a continuous function g such that $f|_{F_1 \cup F_2} \subset g$. Clearly, g separates F_1 and F_2. □

The two lemmas and the theorem have the following

Corollary 4. *Let (X, τ) be a perfect topological space (i.e. every open set is F_σ in X). Then the following three statements are equivalent*

(i) for any upper semi-continuous function $f : X \to [0,1]$ and closed set A there is a continuous function $g : X \to [0,1]$ such that $f|_A \subset g$ and $f \leq g$, provided that $f|_A$ is continuous,

(ii) (X, τ) is normal,

(iii) for any open set $G \subset X$, $(G, \tau|_G)$ is normal.

The following statement is almost obvious from the definitions and it was also used in [1].

Lemma 5. *Let X be a normal topological space and $f : X \to \mathbb{R}$ be a function. f is a discrete limit of continuous functions if and only if there is a sequence (A_k) of closed subset of X such that $f|_{A_k}$ is continuous and $\cup_k A_k = X$*

Theorem 6. *Let (X, τ) be topological space, with the property that every open topological subspace is normal. Also, let $f : X \to \mathbb{R}$ be an upper semi–continuous function which is discrete limit of a sequence of continuous functions. Then there is a sequence (f_n) of continuous functions such that $f_{n+1} \leq f_n$ and f_n tends to f discretely.*

Proof. By usual trick we trace back the general case to the one when actually $f : X \to (0,1)$. Composing every functions with a continuous increasing bijection of $(0,1)$ to $\mathbb{R}$ it is clear that it is enough to show that f_n can be chosen in such a way that beside the listed properties in the statement we have $f_n : X \to (0,1)$.

By lemma 5 there are closed sets A_n such that $f|_{A_n}$ continuous and $\cup_n A_n = X$. Forming finite unions we can also assume that $A_n \subset A_{n+1}$. Then by theorem 1 there are continuous functions $g'_n : X \to (0,1]$ such that $f|_{A_n} \subset g'_n$ and $g'_n \geq f$. Put

$$g_n = \sum_{k=n}^{\infty} \frac{1}{2^{k+1-n}} g'_k.$$

Then $g_n : X \to (0,1)$, $g_n \geq f$ and it extends $f|_{A_n}$, So $f_n = \min_{k \leq n} g_k$ is pointwise decreasing and tends to f discretely. □

The topology examined by Zbigniew Grande in [1] is the topology of the continuity from the right, i.e. for which the sets $[a,b)$, $a < b$ constitutes a base. In what follows this topology will be denoted by τ.

Although, the topology τ is not second countable, it is hereditary Lindelöf therefore every topological subspace is normal. Sometimes such space is called completely or hereditary normal. It is also well known that τ is perfect.

In [1] Grande defined three properties, (1), (1)' and (2) of upper from the right semicontinuous functions such that if f is a discrete limit of continuous functions from the right then it satisfies (1). He proved that (1) implies (1)' which is equivalent to (2), and that (1)' does not imply (1). Eventually he proved that any upper from the right semi continuous function fulfilling property (1)' is a pointwise decreasing discrete limit of a sequence of from the right continuous functions. He asked whether condition (1)' can be replaced with condition (1). The answer follows from the above theorem. To see this, we repeat first property (1) from [1]. All the topological notions without the prefix τ refer to the natural topology of the real line.

A function $f : [0,1) \to \mathbb{R}$ satisfies property (1) if *every non-empty perfect set A has a non-empty portion $I \cap A$ such that $f|_{B \cap I}$ is τ-continuous, where B is the set of not τ-isolated points of A.*

Lemma 7. *Let $f : [0,1) \to \mathbb{R}$ be a function satisfying property (1). Then there is a sequence A_n of τ-closed sets such that $f|_{A_n}$ is τ-continuous and $\cup_n A_n = [0,1)$.*

Proof. We will use the following simple facts.

1. for a closed set A the set of the accumulation points of A, denoted by A' is a perfect subset of A and $A \setminus A'$ is at most countable.

2. If A is any set then the set of τ-isolated points of A is at most countable.

By transfinite recursion we define a sequence $\{I_\alpha \,:\, \alpha<\xi\}$ of open intervals (in $[0,1)$) with rational endpoints, beside this we will define a sequence C_α of the same type such that each C_α is at most countable.

Let $H_0=[0,1)$. H_0 is perfect in the natural topology of $[0,1)$. So by property (1) there is a non-empty open interval I_0 with rational endpoint such that $f|_{I_0}$ is τ-continuous.

Assume that we have already defined I_β for $\beta<\alpha$. Let $S_\alpha=[0,1)\setminus\bigcup\{I_\beta \,:\, \beta<\alpha\}$ and $H_\alpha=S'_\alpha$ be the set accumulation points of S_α.

$H_\alpha\subset S_\alpha$, since this later is closed, and H_α is perfect. If H_α is empty then we do not continue the sequence. Otherwise, property (1) provides an interval I_α that meets H_α. Put

$$C_\alpha=(S_\alpha\setminus H_\alpha)\cup\{x\in H_\alpha \,:\, x \text{ is } \tau\text{-isolated}\}\,.$$

Then C_α is countable and $f|_{I_\alpha\cap(H_\alpha\setminus C_\alpha)}$ is τ-continuous.

So the sequence $\{I_\alpha, C_\alpha \,:\, \alpha<\xi\}$ is defined. Observe that the map $\alpha\mapsto I_\alpha$ is injective, since I_α meets a set (H_α) which is disjoint from all I_β, $\beta<\alpha$. This means that ξ is a countable ordinal.

Note, also, that $H_\alpha\setminus C_\alpha$ is τ-closed. Therefore $I_\alpha\cap(H_\alpha\setminus C_\alpha)$ is τ-F_σ, as well as $\bigcup\{C_\alpha \,:\, \alpha<\xi\}$. To finish the proof let us choose τ-closed sets $A_{\alpha,n}$ such that

$$I_\alpha\cap(H_\alpha\setminus C_\alpha)=\bigcup_{n=1}^{\infty}A_{\alpha,n}.$$

And finally let us enumerate the countable family

$$\{A_{\alpha,n} \,:\, \alpha<\xi, n\in\mathbb{N}\}\cup\{\{p\} \,:\, p\in\cup_\alpha C_\alpha\}$$

into a sequence (A_n).

It is clear from the construction that $\cup_n A_n=[0,1)$ and $f|_{A_n}$ is τ-continuous for each n. $\square$

Corollary 8. *Let $f:[0,1)\to\mathbb{R}$ be a function. f is a discrete limit of τ-continuous functions if and only if f satisfies property (1).*

PROOF. It was proved in [1] that if f is a discrete limit of τ-continuous functions then it satisfies property (1).

To prove the opposite implication, let A_n be a sequence of τ-closed sets such that $f|_{A_n}$ τ-continuous and $\cup_n A_n=[0,1)$. The topology τ is normal so there are τ-continuous functions f_n such that $f|_{\cup_{k=1}^n A_k}\subset f_n$. It is clear that f_n tends to f discretely. $\square$

Now we can answer in the affirmative the problem posed by Grande.

Corollary 9. *Assume $f : [0, 1) \to \mathbb{R}$ satisfies property (1) and τ-upper semi-continuous. Then there is a sequence of τ-continuous functions $f_n \geq f_{n+1}$ tending to f discretely.*

Proof. By lemma 7 there are τ-closed sets A_n such that $f|_{A_n}$ τ-continuous and $\cup_n A_n = [0, 1)$. By lemma 5 we can apply theorem 6 and the proof is complete. □

References

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