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ON DISCRETE LIMITS OF SEQUENCES OF FUNCTIONS SATISFYING SOME SPECIAL APPROXIMATE QUASICONTINUITY CONDITIONS

Abstract

In this article we investigate some properties of discrete limits of sequences of functions satisfying some special approximate quasicontinuity conditions.

Let $\mathbb{R}$ be the set of all reals. In article [2] the authors introduced the notion of the discrete convergence of sequences of functions and investigated the discrete limits in different families; for example, in the family $\mathcal{C}$ of all continuous functions.

We will say that a sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, discretely converges to the limit f ($f = d - \lim_{n\to\infty} f_n$) if

$$\forall_x \exists_{n(x)} \forall_{n>n(x)} f_n(x) = f(x).$$

For any family $\mathcal{P}$ denote by $B_d(\mathcal{P})$ the family of all discrete limits of sequences of functions taken from the family $\mathcal{P}$.

In [2] the class $B_d(\mathcal{C})$ is described and the authors observe that every strictly increasing function F whose set of discontinuity points is dense does not belong to the discrete Baire system generated from $\mathcal{C}$ with discrete convergence.

In this article we will investigate the discrete limits of sequences of functions satisfying some special conditions introduced in [3].

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Recall that x is a density point of a set $A \subset \mathbb{R}$ if there is a measurable (in the Lebesgue sense) set $B \subset A$ such that

$$\lim_{h \to 0^+} \frac{\mu(B \cap (x-h, x+h))}{2h} = 1,$$

where μ denotes Lebesgue measure in $\mathbb{R}$.

The family

$$T_d = \{A \subset \mathbb{R}; \forall_{x \in A}\ x \text{ is a density point of } A\}$$

is a topology called the density topology ([1] and [8]).

If T_e denotes the Euclidean topology in $\mathbb{R}$, then the continuity of functions from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$ is called approximate continuity.

The following conditions were introduced in [3].

A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the condition:

(s_1) for every positive real η for each point x and for each set $U \in T_d$ including x there is point $u \in U$ of continuity of f such that $|f(u) - f(x)| < \eta$;

(s_2) for every positive real η for each point x and for each set $U \in T_d$ including x there is an open interval I such that

$$U \cap I \neq \emptyset, \quad f(I \cap U) \subset (f(x) - \eta, f(x) + \eta) \quad \text{and} \quad I \cap U \subset C(f),$$

where $C(f)$ denotes the set of all continuity points of f;

(s_3) for every positive real η for each point x and for each set $U \in T_d$ including x there is a point $u \in U$ of approximate continuity of f such that $|f(u) - f(x)| < \eta$;

(s_4) for every positive real η for each point x and for each set $U \in T_d$ including x there is an open interval I such that

$$U \cap I \neq \emptyset, \quad f(U \cap I) \subset (f(x) - \eta, f(x) + \eta) \quad \text{and} \quad I \cap U \subset A(f),$$

where $A(f)$ denotes the set of all points at which f is approximately continuous.

In [3] it was observed that a function f satisfies condition (s_1) if and only if it is strongly quasicontinuous at each point x; **i.e.,** for every positive real η and for every set $U \in T_d$ including x there is an open interval I such that

$$I \cap U \neq \emptyset \quad \text{and} \quad f(I \cap U) \subset (f(x) - \eta, f(x) + \eta),$$

satisfies condition (s_3) if and only if it is T_d-quasicontinuous at each point x; **i.e.,** for every positive real η and for every set $U \in T_d$ including x there is a nonempty set $V \subset U$ belonging to T_d such that $f(V) \subset (f(x) - \eta, f(x) + \eta)$.

The definition of strong quasicontinuity was introduced in [4], where it is also proved that every strongly quasicontinuous function f is almost everywhere continuous. T_d-quasicontinuous functions were investigated in [5].

It is obvious that condition (s_2) implies condition (s_4) and that condition (s_4) implies (s_3) and (s_1).

Remark 1. *For a given nonempty set $U \in T_d$ and functions $f, g : \mathbb{R} \to \mathbb{R}$ we suppose that f and g satisfy condition (s_1) and $U \subset A(f)$. If there is a set $V \subset cl(U)$ dense in the closure $cl(U)$ of the set U such that $f(x) = g(x)$ for each point $x \in V$, then $f(x) = g(x)$ for each point $x \in U$.*

PROOF. Since the restricted functions $f \restriction V$ and $g \restriction V$ are equal, $f(x) = g(x)$ for each point $x \in C(f) \cap C(g) \cap U$. But f and g are almost everywhere continuous; so $f(x) = g(x)$ for almost all points $x \in U$. Assume, to the contrary, that there is a point $u \in U$ such that $f(u) \neq g(u)$. Let $3\eta = |f(u) - g(u)|$. There is a closed set

$$A \subset U \cap \{x; |f(u) - f(x)| < \eta\} \cap \{x; f(x) = g(x)\}$$

such that u is a density point of A. Since g satisfies condition (s_1), the point u is not any density point of the interior $\text{int}(\{x; |g(x) - g(u)| \geq \eta\})$ of the set $\{x; |g(x)-g(u)| \geq \eta\}$. So u is a density point of the set $A \cap \{x; |f(x)-f(u)| < \eta\}$ and u is not a density point of the set $\{x; |g(x) - g(u)| \geq \eta\}$. Thus

$$A \cap \{x; |f(x) - f(u)| < \eta\} \cap \{x; |g(x) - g(u)| < \eta\} \neq \emptyset$$

and there is a point $w \in A$ such that $|f(w) - f(u)| < \eta$ and $|g(w) - g(u)| < \eta$. Since $f(w) = g(w)$, this implies

$$3\eta = |f(u) - g(u)| \leq |f(u) - f(w)| + |g(w) - g(u)| < 2\eta,$$

which is a contradiction and thus completes the proof. □

Theorem 1. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of functions f_n, $n = 1, 2, \ldots$, satisfying condition (s_1), then it satisfies the following condition*

(i_1) *for each nonempty set $A \in T_d$ there is an open interval I such that $I \cap A \neq \emptyset$, the restricted function $f \restriction (I \cap A)$ is almost everywhere continuous and for each positive real η and every point $x \in I \cap A$ there is an open interval $J \subset I \cap (x - \eta, x + \eta)$ for which*

$$J \cap A \neq \emptyset \text{ and } f(J \cap A) \subset (f(x) - \eta, f(x) + \eta).$$

PROOF. Let $A \in T_d$ be a nonempty set. For $n = 1, 2, \dots$ let

$$A_n = \{x \in cl(A); f_k(x) = f(x) \text{ for } k \geq n\}.$$

Observe that $\text{cl}(A) = \bigcup_{n=1}^{\infty} A_n$ and $A_n \subset A_{n+1}$ for $n \geq 1$. There is a positive integer m such that the set A_m is of the second category in $\text{cl}(A)$. So there is an open interval I with $I \cap A_m \neq \emptyset$ and $I \cap \text{cl}(A) \subset \text{cl}(I \cap A_m)$. Put $B = \bigcap_{k \geq m} C(f_k)$ and observe that for $k \geq m$ and $x \in I \cap B \cap \text{cl}(A)$ we have $f_k(x) = f(x)$. Since the functions f_n, $n \geq 1$, are almost everywhere continuous, the set $(I \cap A) \setminus B$ is of measure zero. But

$$f \restriction (A \cap I \cap B) = f_k \restriction (A \cap I \cap B) \text{ for } k \geq m,$$

so the restricted function $f \restriction (I \cap A)$ is almost everywhere continuous.

Assume, to the contrary, that there is a positive real η and a point $x \in I \cap A$ such that for every open interval $J \subset I \cap (x - \eta, x + \eta)$ such that $J \cap A \neq \emptyset$ there is a point $u \in J \cap A$ at which $|f(u) - f(x)| \geq \eta$. Let $j \geq m$ be an integer such that $f_k(x) = f(x)$ for $k \geq j$. Since $x \in (x - \eta, x + \eta) \cap I \cap A \in T_d$ and the function f_j satisfies condition (s_1), there is an open interval $J \subset I \cap (x - \eta, x + \eta)$ such that

$$J \cap A \neq \emptyset \text{ and } f_j(J \cap A) \subset (f_j(x) - \frac{\eta}{2}, f_j(x) + \frac{\eta}{2}).$$

Let $u \in J \cap A$ be a point for which $|f(u) - f(x)| \geq \eta$ and let $i \geq j$ be an integer such that $f_k(u) = f(u)$ for $k \geq i$. Since the function f_i satisfies condition (s_1) and since $u \in J \cap A \in T_d$, there is an open interval $K \subset J$ such that

$$K \cap A \neq \emptyset \text{ and } f_i(K \cap A) \subset (f(u) - \frac{\eta}{2}, f(u) + \frac{\eta}{2}).$$

But the restricted functions $f_i \restriction (J \cap A)$ and $f(J \cap A)$ are almost everywhere equal; so there is a point $w \in J \cap A \cap B$ at which the equality $f(w) = f_i(w) = f_j(w)$ holds. Consequently,

$$\eta \leq |f(u) - f(x)| \leq |f_i(u) - f_i(w)| + |f_j(w) - f_j(x)| < \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

and the contradiction completes the proof. □

Since the functions satisfying condition (s_1) are almost everywhere continuous, for the function f from the last theorem there is an F_σ-set E of measure zero such that the restricted function $f \restriction (\mathbb{R} \setminus E)$ is the discrete limit of a sequence of continuous functions on $\mathbb{R} \setminus E$ ([6]).

The next result follows from the proof of the last theorem.

Corollary 1. *If the function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of functions f_n, $n \geq 1$, satisfying condition (s_1), then the set*

$$D_{sq}(f) = \{x; f \text{ is not strongly quasicontinuous at } x\}$$

is nowhere dense.

PROOF. Let I be an open interval. As in the proof of the last theorem, we find an open interval $J \subset I$ a set $E \subset J$ of measure zero and a positive integer m such that $f_k(x) = f(x)$ for $x \in J \setminus E$ and $k \geq m$. The reasoning used in the proof of the last theorem shows that f is strongly quasicontinuous at each point $x \in J$.

Theorem 2. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of functions f_n, $n \geq 1$, satisfying condition (s_4), then it satisfies the following condition.*

(i_2) *For every nonempty set $A \in T_d$ there are an open interval I and a positive integer m such that*

$$A(f) \supset I \cap A \neq \emptyset \text{ and } f_k(x) = f(x) \text{ for } ; x \in A \cap I \text{ and } k \geq m.$$

PROOF. As in the proof of Theorem 1, we find a positive integer m and an open interval I such that $\emptyset \neq I \cap A \subset A(f_m)$ and $\bigcap_{k \geq m}\{x \in I \cap A; f_k(x) = f(x)\}$ is dense in $I \cap A$. Now we use Remark 1 and observe that $f_k(x) = f(x)$ for $x \in I \cap A$ and $k \geq m$. □

As an immediate consequence we obtain the following corollary.

Corollary 2. *For the function f from Theorem 2 the set $D_{sq}(f) \cap (\mathbb{R} \setminus A(f))$ is nowhere dense.*

Theorem 3. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of functions f_n, $n = 1, 2, \ldots$, satisfying condition (s_2), then it satisfies the following condition.*

(i_3) *For each nonempty set $A \in T_d$ there is an open interval I such that $I \cap A \neq \emptyset$ and the restricted function $f \restriction (I \cap A)$ is continuous.*

PROOF. The proof is completely analogous to that of Theorem 2. □

As an immediate consequence we obtain the following.

Corollary 3. *If a function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of functions f_n, $n \geq 1$, satisfying condition (s_2), then the set $\mathbb{R} \setminus C(f)$ is nowhere dense.*

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function whose set of discontinuities is dense. If for all discontinuities x of f we suppose that

$$f(x) = \frac{\lim_{t\to x^+} f(t) + \lim_{t\to x^-} f(t)}{2},$$

then f is not in the Baire system generated by the family of all functions satisfying condition (s_1) and discrete convergence.

Of course, for every quasicontinuous function g the set $\{x; f(x) \neq g(x)\}$ is residual, so the graph of the function f can not covered by a countable family of the graphs of quasicontinuous functions, and consequently f does not belong to the above Baire system.

However if for all discontinuity points x of f we have $f(x) = \lim_{t\to x^+} f(t)$, then f satisfies condition (s_1), but is not the discrete limit of any sequence of functions satisfying condition (s_4), because the set $\mathbb{R} \setminus A(f)$ is dense.

Every almost everywhere continuous function $h : \mathbb{R} \to \mathbb{R}$ which is everywhere approximately continuous and discontinuous on a dense set, satisfies condition (s_4), but is not the discrete limit of any sequence of functions satisfying condition (s_2), because its set of discontinuities is dense.

Theorem 4. *A function $f : \mathbb{R} \to \mathbb{R}$ is the discrete limit of a sequence of functions satisfying condition (s_3) if and only if it is measurable.*

PROOF. The necessity is evident, since all functions satisfying condition (s_3) are measurable. The proof of the sufficiency is the repetition of the proof for the pointwise limits from [7] or [5].

If the function f is measurable, then the set $B = \mathbb{R} \setminus A(f)$ is of measure zero. Let $E \supset B$ be a G_δ-set of measure zero. There are ([7]) measurable sets $B_{n,k}$, $k, n = 1, 2, \ldots$, such that

$B_{n,k} \cap B_{m,l} = \emptyset$ if $(n,k) \neq (m,l)$ and $\mathbb{R} \setminus E = \bigcup_{k,n=1}^{\infty} B_{n,k}$;

if $x \in B_{n,k} \cup E$, then x is not a density point of the set $\mathbb{R} \setminus B_{n,k} \setminus E$, $k, n = 1, 2, \ldots$.

Let (w_n) be an enumeration of all rationals such that $w_n \neq w_m$ for $n \neq m$. For $n \geq 1$ define f_n by

$$f_n(x) = \begin{cases} w_k & \text{if } x \in B_{n,k} \text{ for } k = 1, 2, \ldots \\ f(x) & \text{otherwise on } \mathbb{R}. \end{cases}$$

Then each function f_n, $n \geq 1$, satisfies condition (s_3) and the sequence (f_n) discretely converges to f. $\square$

If C is a Cantor set of positive measure such that for each open interval I with $I \cap C \neq \emptyset$ the set $I \cap C$ is of positive measure and if $B \subset C$ is a countable set such that every point $x \in B$ is a density point of C and $cl(B) = C$, then the function f equal 1 on B and zero otherwise on $\mathbb{R}$ does not satisfy condition (i_1). So, it is not the discrete limit of any sequence of functions satisfying condition (s_1).

Evidently f is the discrete limit of a sequence of almost everywhere continuous functions.

Theorem 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Suppose that there are pairwise disjoint closed sets A_n of measure zero and functions $g_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, satisfying condition (s_j), where $j \in \{1, 2, 4\}$, on the sets $\mathbb{R} \setminus A_n$, such that the restricted function $f \restriction (\mathbb{R} \setminus \bigcup_{n=1}^{\infty} A_n)$ is the discrete limit of the sequence of restricted functions $g_n \restriction (\mathbb{R} \setminus \bigcup_{k=1}^{\infty} A_k)$ and is not a point $x \in \bigcup_{k=1}^{\infty} A_k$ being a density point of the closure $\mathrm{cl}(\bigcup_{k=1}^{\infty} A_k)$. Then f is the discrete limit of a sequence of functions satisfying condition (s_j).*

PROOF. Evidently the set $A = \bigcup_{n=1}^{\infty} A_n$, is nowhere dense. We find pairwise disjoint closed intervals $I_{n,k,m} = [a_{n,k,m}, b_{n,k,m}]$, $k, n, m = 1, 2, \ldots$, such that:

> $I_{n,k,m} \cap cl(A) = \emptyset$ for $m, n, k \geq 1$;
>
> all endpoints $a_{n,k,m}, b_{n,k,m}$ are continuity points of g_n, $k, m, n = 1, 2, \ldots$;
>
> if x is an accumulation point of the set $\{a_{n,k,m}, b_{n,k,m}; k, m = 1, 2 \ldots\}$, then $x \in A_n$, $n \geq 1$;
>
> if $x \in A_n$ an $(m_i)_i$ is a strictly increasing sequence of positive integers, then x is not a density point of the set $\mathbb{R} \setminus \bigcup_{i=1}^{\infty} I_{n,k,m_i}$ for $k, n \geq 1$.

In the interior of each interval $I_{k,n,m}$, $k, m, n \geq 1$, we find a closed interval $J_{n,k,m}$ such that for every point $x \in A_n$ and for every strictly increasing sequence of positive integers m_i, $i = 1, 2, \ldots$, x is not a density point of a set $\mathbb{R} \setminus \bigcup_{i=1}^{\infty} J_{n,k,m_i}$ for $k, n \geq 1$. For a nonempty set $X \subset \mathbb{R}$ and for $x \in \mathbb{R}$ let

$$\mathrm{dist}(x, X) = \inf\{|t - x|; t \in X\}.$$

Let $(w_k)_k$ be an enumeration of all rationals. We will define functions f_n, $n = 1, 2, \ldots$, as follows. Fix a positive integer n. If $x \in J_{i,k,m}$, $i \leq n$, $k, m = 1, 2, \ldots$, and if

$$\max(\mathrm{dist}(a_{i,k,m}, \bigcup_{i \leq n} A_i), \mathrm{dist}(b_{i,k,m}, \bigcup_{i \leq n} A_i) < \frac{1}{n} \qquad (*),$$

then $f_n(x) = w_k$, for $k, m, n \geq 1$. If $\operatorname{dist}(x, \bigcup_{i\leq n} A_i) \geq \frac{1}{n}$ or if $x \in \mathbb{R} \setminus \bigcup_{i\leq n} \bigcup_{k,m\geq 1} \operatorname{int}(I_{i,k,m})$ or if $x \in I_{n,k,m}$ and condition $(*)$ does not hold, then $f_n(x) = g_n(x)$. If $i \leq n$ and the triple (i, k, m) satisfies condition $(*)$, then f_n is linear on the components of the set $I_{i,k,m} \setminus \operatorname{int}(J_{i,k,m})$. Then the functions f_n, $n \geq 1$, satisfy condition (s_j) and the sequence (f_n) discretely converges to f. □

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