Real Analysis Exchange
Vol. 26(1), 2000/2001, pp. 471–484

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APPROXIMATION THEOREMS FOR GENERALIZED RIEMANN INTEGRALS

Abstract

This note concerns the generalized Riemann integrals using free tagged subdivisions, and leading to theories of absolute integration. We first establish an approximation theorem by step functions. This result will allow us to define a natural notion of equi-integrability in the Lebesgue space L^1. Some applications, such as a new characterization of compact parts of L^1, are presented.

We first establish some approximation results of integrable functions by step functions. The framework will be a generalized Riemann integral developed in [Fea 1], equivalent to the McShane integral [9] when functions are valued in a finite dimension Banach space.

Those results lead naturally to concepts of gauges and equi-integrability for the space of class functions L^1. Using these concepts, we then characterize the compact parts of L^1. Some theorems can be expressed in the Lebesgue theory while others require the gauge formalism.

Lastly, we use the approximation theorems to prove the equivalence of a generalized Riemann integral and those of the Lebesgue or Bochner theories.

1 Definitions and Notation

In the following, $(X, \| \ \|)$ refers to a Banach space.

1. *Gauges and Subdivisions*

   - Let $[a, b]$ be an interval of $\mathbb{R}$ $(a < b)$. A gauge on $[a, b]$ is a function δ from $[a, b]$ to $\mathbb{R}_+^*$.

Key Words: Gauge, McShane integral, step functions, equi-integrability
Mathematical Reviews subject classification: 26A39, 28B05
Received by the editors July 7, 2000

- A subdivision of $[a,b]$ is a sequence $x = (x_i)_{0 \leqslant i \leqslant n}$ of $[a,b]$ such that $a = x_0 < \ldots < x_n = b$, and we set $\tau(x) = \underset{1 \leqslant i \leqslant n}{\mathrm{Max}}\,(x_i - x_{i-1})$. A tagged subdivision is a couple (x,c) such that $x = (x_i)_{0 \leqslant i \leqslant n}$ is a subdivision and $c = (c_i)_{1 \leqslant i \leqslant n}$ a sequence of $[a,b]$.
- A free tagged subdivision subordinate to a gauge δ is a tagged subdivision (x,c) which satisfies

$$\forall i \in \{1 \ldots, n\}, \quad c_i - \delta(c_i) \leqslant x_{i-1} < x_i \leqslant c_i + \delta(c_i).$$

  It is a bounded tagged subdivision subordinate to δ if

$$\forall i \in \{1 \ldots, n\}, \quad c_i - \delta(c_i) \leqslant x_{i-1} \leqslant c_i \leqslant x_i \leqslant c_i + \delta(c_i).$$

  Without indication, a tagged subdivision (x,c) subordinate to δ will suppose to be free.
- For a tagged subdivision (x,c) of $[a,b]$, the Riemann sum of $f : [a,b] \to X$ for this subdivision is

$$S_f(x,c) = \sum_i (x_i - x_{i-1}) f(c_i).$$

Let us recall that, for every gauge δ, we can find a tagged subdivision (free or bounded) subordinate to δ [8].

2. *A generalized Riemann integral*

The framework of this article is the integral theory developed in [4].

Theorem and definition. *A function $f : [a,b] \to E$ is called integrable if, for each $\varepsilon > 0$, we can find a gauge δ_ε such that*

$$\sum_i \|(x_i - x_{i-1})(f(c_i) - f(c'_i))\| \leqslant \varepsilon$$

whenever (x,c) and (x,c') are tagged subdivisions of $[a,b]$ subordinate to δ_ε.

For an integrable function f, a gauge satisfying this property for ε is said to be ε-adapted (to f).

In this case, for every family of tagged subdivisions $((x_\varepsilon, c_\varepsilon))_{\varepsilon > 0}$ respectively subordinate to $(\delta_\varepsilon)_{\varepsilon > 0}$, the function $\varepsilon \to S_f(x_\varepsilon, c_\varepsilon)$ has a limit when ε goes to 0 and this limit does not depend on the sequence chosen. By definition, it is the integral of f on $[a,b]$ and it is denoted by $\displaystyle\int_a^b f$.

We know that an almost everywhere null function is integrable with a null integral. Accordingly, the integrals of two integrable functions equal almost everywhere are the same.

The resulting theory is easily seen to be equivalent to the McShane integral when X is a finite dimension space (and also equivalent to the Lebesgue integral). It is equivalent to the Bochner theory in the general case. These properties, and the results used in the following are proved in [4].

3. *Functional spaces*

- A function $f : [a, b] \to X$ is a step function if there exists a subdivision $x = (x_i)_{0 \leqslant i \leqslant n}$ of $[a, b]$ such that every restriction of f to an interval $]x_i, x_{i+1}[$ is constant. In this case, the subdivision x is adapted to f (and f is adapted to x). The set of step functions from $[a, b]$ to X is denoted by $\mathcal{E}([a, b], X)$.
- We denote $E([a, b], X)$ the quotient of $\mathcal{E}([a, b], X)$ by the negligibility relation. The elements of $E([a, b], X)$ are class of functions. If $F \in E([a, b], X)$ and $f \in F$ is a step function adapted to a subdivision x, we shall say that x is adapted to F (and F is adapted to x).
- We denote $\mathcal{L}^1([a, b], X)$ the set of integrable functions and we set $\|f\|_1 = \int_a^b \|f\|$ for every $f \in \mathcal{L}^1([a, b], X)$.
- The set $L^1([a, b], X)$ is the quotient of $\mathcal{L}^1([a, b], X)$ by the negligibility relation. We also denote $\| \ \|_1$ the usual norm for $L^1([a, b], X)$.
- Let $f \in \mathcal{L}^1([a, b], X)$ and $x = (x_i)_{0 \leqslant i \leqslant n}$ be a subdivision of $[a, b]$. We denote by $T(f, x)$ the step function equal to f on the elements of x, and such that

$$\forall i \in \{1, \dots, n\},\ \forall t \in]x_{i-1}, x_i[,\quad T(f, x)(t) = \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} f.$$

- If $F \in L^1([a, b], X)$ and $f \in F$, the class of $T(f, x)$ is also denoted by $T(F, x)$, for every subdivision x of $[a, b]$.

2 The Approximation Theorems

Theorem 1. *Let $f : [a, b] \to X$ be an integrable function and $\varepsilon > 0$. For every gauge δ which is ε-adapted to f, we have*

$$\sum_i \int_{x_{i-1}}^{x_i} \|f(t) - f(c_i)\| dt \leqslant \varepsilon$$

whenever (x, c) *is a free tagged subdivision subordinate to* δ.

PROOF. Let $\varepsilon > 0$ and δ be a gauge which is ε-adapted to f. We fix a free tagged subdivision $(x, c) = ((x_i)_{0 \leqslant i \leqslant p}, (c_i)_{1 \leqslant i \leqslant p})$ subordinate to δ and choose a sequence of gauges $(\delta^n)_{n \in \mathbb{N}}$ such that δ^n is 2^{-n}-adapted to f and which satisfies $\delta^n \leqslant \delta$ for every $n \in \mathbb{N}$.

Let $i \in \{1, \ldots, p\}$. For all $n \in \mathbb{N}$, let (x^{ni}, c^{ni}) be a free tagged subdivision of $[x_{i-1}, x_i]$ subordinate to the restriction of δ^n at this segment. Obviously, we can merge the p tagged subdivisions to provide a tagged subdivision (x^n, c^n) of $[a, b]$ subordinate to δ^n and δ.

We can also consider the free tagged subdivision (x^n, d^n) built by merging the tagged subdivisions (x^{ni}, d^{ni}), $1 \leqslant i \leqslant p$, where for every i the tags of d^{ni} are repetition of c_i. This tagged subdivision (x^n, d^n) is also subordinate to δ.

Accordingly,

$$\sum_i \sum_j (x_j^{ni} - x_{j-1}^{ni}) \left\| f(c_j^{ni}) - f(c_i) \right\| = \sum_i \sum_j (x_j^{ni} - x_{j-1}^{ni}) \left\| f(c_j^{ni}) - f(d_j^{ni}) \right\| \leqslant \varepsilon$$

from which the conclusion follows when n increase to $+\infty$. □

The Saks-Henstock Theorem is an easy consequence of Theorem 1.

Theorem 2 (Saks-Henstock). *Let* $f : [a, b] \to X$ *be an integrable function and* $\varepsilon > 0$. *For every gauge* δ ε*-adapted to* f, *we have*

$$\sum_i \left\| (x_i - x_{i-1}) f(c_i) - \int_{x_{i-1}}^{x_i} f \right\| \leqslant \varepsilon.$$

whenever (x, c) *is a free tagged subdivision subordinate to* δ.

The ideas developed in the proof of Theorem 1 can be used to prove the following approximation theorem by step functions.

Theorem 3. *Let* $f \in \mathcal{L}^1([a, b], X)$. *For all* $\varepsilon > 0$ *there exists a step function* f_ε *such that*

$$\|f - f_\varepsilon\|_1 \leqslant \varepsilon.$$

Moreover, if δ *is a gauge which is* $\frac{\varepsilon}{2}$*-adapted to* f, *we can take* $f_\varepsilon = T(f, x)$ *whenever* (x, c) *is a bounded tagged subdivision subordinate to* δ.

PROOF. The former result is an obvious consequence of the later. Nevertheless, it can be proved from Theorem 1 by choosing a step function with value $f(c_i)$ on every interval $]x_{i-1}, x_i[$. We establish the second result.

Let $\varepsilon > 0$ and δ be a gauge which is $\frac{\varepsilon}{2}$-adapted to f, and fix a bounded tagged subdivision $(x, c) = ((x_i)_{0 \leqslant i \leqslant p}, (c_i)_{1 \leqslant i \leqslant p})$ subordinate to δ. We can choose a sequence of gauges $(\delta^n)_{n\in\mathbb{N}}$ such that δ^n be 2^{-n}-adapted to f and $T(f,x)$ with $\delta^n \leqslant \delta$ for all $n \in \mathbb{N}$.

Let $i \in \{1,\dots,p\}$. For all $n \in \mathbb{N}$, let (x^{ni}, c^{ni}) be a tagged subdivision of $[x_{i-1}, x_i]$ adapted to the restriction of δ^n at this segment. We can merge those p tagged subdivisions to provide a tagged subdivision (x^n, c^n) of $[a,b]$ subordinate to δ^n.

If c_j^{ni} belong to x (as it is x_{i-1} or x_i), then $f(c_j^{ni}) - T(f,x)(c_j^{ni}) = 0$. We let Δ^{ni} denote the set of j such that c_j^{ni} doesn't belong to x, and compute

$$\begin{aligned}
&\sum_{i=1}^{p}\sum_{j}\left\|(x_j^{ni}-x_{j-1}^{ni})(f(c_j^{ni})-T(f,x)(c_j^{ni}))\right\| =\\
&\quad=\sum_{i=1}^{p}\sum_{j\in\Delta^{ni}}\left\|(x_j^{ni}-x_{j-1}^{ni})(f(c_j^{ni})-T(f,x)(c_j^{ni}))\right\|\\
&\quad\leqslant\sum_{i=1}^{p}\sum_{j\in\Delta^{ni}}\left\|(x_j^{ni}-x_{j-1}^{ni})(f(c_j^{ni})-f(c_i))\right\|\\
&\qquad+\sum_{i=1}^{p}\sum_{j\in\Delta^{ni}}\left\|(x_j^{ni}-x_{j-1}^{ni})(f(c_i)-T(f,x)(c_j^{ni}))\right\|\\
&\quad\leqslant\sum_{i=1}^{p}\sum_{j}\left\|(x_j^{ni}-x_{j-1}^{ni})(f(c_j^{ni})-f(c_i))\right\|\\
&\qquad+\sum_{i=1}^{p}\sum_{j}\left\|(x_j^{ni}-x_{j-1}^{ni})\left(f(c_i)-\frac{1}{x_i-x_{i-1}}\int_{x_{i-1}}^{x_i} f\right)\right\|.
\end{aligned}$$

The free tagged subdivision (x^n, c^n) is subordinate to δ as is (x^n, d^n) obtained by setting $d_j^{ni} = c_i$ for every j. Accordingly, the former of the two terms is lesser than $\frac{\varepsilon}{2}$, because δ is $\frac{\varepsilon}{2}$-adapted to f, and the same majoration is obtained for the later term by application of the Saks-Henstock theorem.

From this it follows that $\|f - T(f,x)\|_1 \leqslant \varepsilon$ when n increases to infinity. □

An easy and important consequence is the following approximation theorem.

Theorem 4. *Let $f \in \mathcal{L}^1([a,b], X)$. For every $\varepsilon > 0$ there exists a function $f_\varepsilon \in C^0([a,b], X)$ such that*

$$\int_a^b \|f - f_\varepsilon\| \leqslant \varepsilon.$$

Moreover, using a usual regularization process, we can choose the function f_ε in $C^\infty([a,b],X)$. This theorem is very classical in the Lebesgue theory, but the usual proofs use the Lusin Theorem, itself proved after some delicate topological results (Urysohn's Lemma [10] or Tietze's Extension Theorem [7]).

Theorem 5. *Let $f \in \mathcal{L}^1([a,b],X)$. For every $\varepsilon > 0$, there exists $\eta > 0$ such that*

$$\|f - T(f,x)\|_1 \leqslant \varepsilon$$

whenever $x = (x_i)_{0\leqslant i\leqslant q}$ is a subdivision of $[a,b]$ satisfying $\tau(x) \leqslant \eta$.

PROOF. Let $\varepsilon > 0$. There exits $g \in C^0([a,b],X)$ such that $\|f-g\|_1 \leqslant \dfrac{\varepsilon}{3}$, and hence we can find $\eta > 0$ such that

$$\forall (u,v) \in [a,b]^2, \quad |u-v| \leqslant \eta \Rightarrow |g(u)-g(v)| \leqslant \frac{\varepsilon}{3(b-a)}.$$

Let $x = (x_i)_{0\leqslant i\leqslant q}$ be a subdivision of $[a,b]$ such that $\tau(x) \leqslant \eta$. Then

$$\begin{aligned}\int_{[a,b]} |g - T(g,x)| &= \sum_{i=1}^{q} \int_{x_{i-1}}^{x_i} \left| g(u) - \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} g(v)dv \right| du \\ &\leqslant \sum_{i=1}^{q} \int_{x_{i-1}}^{x_i} \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} |g(u)-g(v)| dv du \\ &\leqslant \frac{\varepsilon}{3}.\end{aligned}$$

Moreover,

$$\int_{[a,b]} |T(g,x) - T(f,x)| \leqslant \sum_{i=1}^{q} (x_i - x_{i-1}) \left[\frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} |g(u) - f(u)| du \right] \leqslant \frac{\varepsilon}{3}.$$

Finally, we have

$$\|f - T(f,x)\|_1 \leqslant \|f-g\|_1 + \|g - T(g,x)\|_1 + \|T(g,x) - T(f,x)\|_1 \leqslant \varepsilon.$$

□

Let us notice that this theorem can be generalized for a finite family $(f_k)_{0\leqslant k\leqslant p}$ of $\mathcal{L}^1([a,b],X)$. For all $\varepsilon > 0$, there exists $\eta > 0$ such that

$$\forall k \in \{0,\ldots,p\}, \quad \|f_k - T(f_k,x)\|_1 \leqslant \varepsilon$$

for every subdivision $x = (x_i)_{0\leqslant i\leqslant n}$ of $[a,b]$ satisfying $\tau(x) \leqslant \eta$.

3 Gauges Adapted to an Element of L^1

In some sense, approximation theorems by step functions are a generalization of an integral using the Riemann definition, and this point of view leads us to give a definition of a gauge which is ε-adapted to an element of $L^1([a,b],X)$.

Definition 6. *A subdivision $x=(x_i)_{0\leqslant i\leqslant p}$ of $[a,b]$ is subordinate to a gauge δ if there exists a set of tags $c=(c_i)_{1\leqslant i\leqslant p}$ such that (x,c) be a bounded tagged subdivision subordinate to δ.*

Definition 7. *Let $F\in L^1([a,b],X)$ and $\varepsilon>0$. A gauge δ is ε-adapted to the class of functions F if*

$$\|F-T(F,x)\|\leqslant\varepsilon$$

whenever x is a subdivision subordinate to δ.

Theorem 8. *Let $F\in L^1([a,b],X)$. For all $\varepsilon>0$, there exists a gauge δ wich is ε-adapted to F.*

In fact, if $f\in F$ and δ is a gauge which is $\varepsilon/2$-adapted to f, (in the function meaning), then δ is a gauge which is ε-adapted to F (in the class of functions meaning). Moreover, for all $\varepsilon>0$, we can find constant gauges which are ε-adapted to F.

PROOF. Those results are straightforward consequences of Theorems 3 and 5. □

4 Application 1: A Characterization of the Compact Parts of L^1

Throughout this section, X will denote a finite dimensional Banach space.

Definition 9. *Let Γ be a part of $L^1([a,b],X)$*

- *The set Γ is equi-integrable on $[a,b]$ if for every $\epsilon>0$, there is a subdivision x of $[a,b]$ such that*

  $$\forall F\in\Gamma,\ \exists F_x\in E([a,b],X)\text{ adapted to }x\text{ such that }\ \|F-F_x\|_1\leqslant\varepsilon.$$

- *The set Γ is uniformly equi-integrable on $[a,b]$ if for every $\epsilon>0$ there is a real $\eta>0$ such that*

  $$\forall F\in\Gamma,\ \exists F_x\in E([a,b],X)\text{ adapted to }x\text{ such that }\ \|F-F_x\|_1\leqslant\varepsilon$$

  whenever x is a subdivision with $\tau(x)\leqslant\eta$.

- *The set* Γ *is strongly equi-integrable on* $[a,b]$ *if for every* $\varepsilon > 0$ *there is a subdivision* x *of* $[a,b]$ *such that*

$$\forall F \in \Gamma, \quad \|F - T(F,x)\|_1 \leqslant \varepsilon.$$

- *The set* Γ *is uniformly strongly equi-integrable on* $[a,b]$ *if for every* $\varepsilon > 0$ *there is a real* $\eta > 0$ *such that*

$$\forall F \in \Gamma, \quad \|F - T(F,x)\|_1 \leqslant \varepsilon.$$

whenever x *is a subdivision with* $\tau(x) \leqslant \eta$.

Notice that each of the previous properties is satisfied if and only if it is satisfied for a sequence (ε_n) with null limit, for instance $\varepsilon_n = 2^{-n}$.

Lemma 10. *Let* (F_n) *be a sequence of* $E([a,b],X)$ *such that there exists a subdivision* $x = (x_i)_{0 \leqslant i \leqslant p}$ *of* $[a,b]$*, adapted at every term of* (F_n)*. If* (F_n) *is bounded in the* $\| \ \|_1$ *norm, we can extract a subsequence which converges in* $(E([a,b],X), \| \ \|_1)$*.*

PROOF. For all n, and for each $i \in \{1, \dots, p\}$ we define a function $f_n \in F_n$ by

$$f_n(t) = f_n^i = \begin{cases} \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} F_n & \text{if } t \in]x_{i-1}, x_i[, \\ 0 & \text{if } t \in \{x_0, \dots, x_p\}. \end{cases}$$

Let $M = \sup_{n \in \mathbb{N}} \|F_n\|_1$ and $\alpha = \operatorname{Max}_{1 \leqslant i \leqslant p} \dfrac{1}{x_i - x_{i-1}}$. Then,

$$\forall n \in \mathbb{N}, \forall i \in \{1, \dots, p\}, \quad \left\|f_n^i\right\| \leqslant M\alpha.$$

Hence, $(f_n^1, \dots, f_n^p)_{n \in \mathbb{N}}$ is a bounded sequence of X^p, and there exists a strictly increasing sequence $\varphi : \mathbb{N} \to \mathbb{N}$ such that $(f_{\varphi(n)}^1, \dots, f_{\varphi(n)}^p)_{n \in \mathbb{N}}$ converges. The lemma is now follows easily. □

Recall that a part Γ of a vector space is precompact if its closure $\overline{\Gamma}$ is compact. The following is a characterization of compacts parts of $L^1([a,b],X)$.

Theorem 11. *Let* Γ *be a part of* $L^1(\Omega)$*. The following properties are equivalent:*

(i) Γ *is precompact;*

(ii) Γ *is bounded and uniformly strongly equi-integrable;*

(iii) Γ *is bounded and strongly equi-integrable;*

(iv) Γ *is bounded and uniformly equi-integrable;*

(v) Γ *is bounded and equi-integrable.*

PROOF. PROOF OF $(i) \Rightarrow (ii)$

As a precompact part Γ of $L^1([a,b],X)$ is bounded, we need only prove the uniform strong equi-integrability.

For a fixed $\varepsilon > 0$, there exist sets of functions $G_1, \dots, G_p \in L^1([a,b],X)$ such that $\Gamma \subset \bigcup_{k=1}^{p} B\Big(G_k, \frac{\varepsilon}{3}\Big)$. From the remark following Theorem 5 we conclude that there exists $\eta > 0$ such that

$$\forall k \in \{0, \dots, p\}, \quad \|G_k - T(G_k, x)\|_1 \leqslant \frac{\varepsilon}{3}$$

for all subdivision x of $[a,b]$ with $\tau(x) \leqslant \eta$.

For $F \in \Gamma$, there exists $k \in \{1, \dots, p\}$ such that $\|F - G_k\|_1 \leqslant \frac{\varepsilon}{3}$. Then for every $i \in \{1, \dots, n\}$ and almost every $t \in]x_{i-1}, x_i[$, we have

$$\begin{aligned} \|T(G_k, x)(t) - T(F, x)(t)\| &= \frac{1}{x_i - x_{i-1}} \left\| \int_{x_{i-1}}^{x_i} G_k - F \right\| \\ &\leqslant \frac{1}{x_i - x_{i-1}} \int_{x_{i-1}}^{x_i} \|G_k - F\|. \end{aligned}$$

We deduce $\|T(G_k, x) - T(F, x)\|_1 \leqslant \|G_k - F\|_1$, and that

$$\begin{aligned} \|F - T(F, x)\|_1 &\leqslant \|F - G_k\|_1 + \|G_k - T(G_k, x)\|_1 + \|T(G_k, x) - T(F, x)\|_1 \\ &\leqslant \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

The proofs of $(ii) \Rightarrow (iii) \Rightarrow (v)$ and $(ii) \Rightarrow (iv) \Rightarrow (v)$ are obvious.

PROOF OF $(v) \Rightarrow (i)$

Let Γ be a bounded and equi-integrable part of $L^1([a,b],X)$. Let us denote $M = \operatorname{Sup}_{F \in \Gamma} \|F\|_1$, and let (F_n) be a sequence of Γ.

For all $p \in \mathbb{N}$, there exists a subdivision x^p such that, for all $n \in \mathbb{N}$, there exists $F_n^p \in E([a,b],X)$ adapted to the subdivision x^p and satisfying $\|F_n - F_n^p\|_1 \leqslant 2^{-p}$.

The end of the proof uses an application of Cantor diagonal process similar to that used to prove Ascoli's theorem . From Lemma 10 we can build a strictly increasing application $\varphi_0 : \mathbb{N} \to \mathbb{N}$ such that

$$\forall (n,m) \in \mathbb{N}^2, \quad \left\| F^0_{\varphi_0(n)} - F^0_{\varphi_0(m)} \right\|_1 \leqslant 1.$$

Then for every p, we build a strictly increasing application $\varphi_p : \mathbb{N} \to \mathbb{N}$ such that

$$\forall (n,m) \in \mathbb{N}^2, \quad \left\| F^p_{\varphi_0 \circ \ldots \circ \varphi_p(n)} - F^p_{\varphi_0 \circ \ldots \circ \varphi_p(m)} \right\|_1 \leqslant 2^{-p}.$$

Let $\varphi : \mathbb{N} \to \mathbb{N}$ the a strictly increasing application defined by $\varphi(p) = \varphi_0 \circ \ldots \circ \varphi_p(p)$. Then for every $p < q$ we have

$$\begin{aligned} \left\| F_{\varphi(p)} - F_{\varphi(q)} \right\|_1 &\leqslant \left\| F_{\varphi(p)} - F^p_{\varphi(p)} \right\|_1 + \left\| F^p_{\varphi(p)} - F^p_{\varphi(q)} \right\|_1 + \left\| F^p_{\varphi(q)} - F_{\varphi(q)} \right\|_1 \\ &\leqslant 3\, 2^{-p}. \end{aligned}$$

The sequence $(F_{\varphi(p)})$ is a Cauchy sequence, and $\overline{\Gamma}$ is a compact part of $L^1([a,b])$. □

In a "gauge" formalism, we have the following theorem.

Theorem 12. *A part Γ of $L^1([a,b],X)$ is precompact if and only if Γ is bounded and, for all $\varepsilon > 0$, there exists a gauge δ ε-adapted to the elements of Γ (in the class meaning).*

PROOF. If Γ is precompact, Γ is bounded and uniformly equi-integrable. Thus, for all $\varepsilon > 0$, there exists a suitable *constant* gauge δ_ε from Theorem 8.

Conversely, the result follows from the implication $(v) \Rightarrow (i)$ of Theorem 11. □

Notice the simplicity of those theorems, each isomorphic to the Ascoli Theorem. They can be compared with the Fréchet-Kolmogorov Theorem ([2, p 72 and corollary IV-26], or [1, p. 31]) which characterizes the precompact parts of $L^1([a,b],X)$.

We conclude this section by using Theorem 11 to prove a convergence in mean result for sequence of functions. First recall a classical theorem in Lebesgue's theory.

Theorem 13. *Let (f_n) be a sequence of $\mathcal{L}^1([a,b],X)$ and $f \in \mathcal{L}^1([a,b],X)$ such that $\lim\limits_{n\to+\infty} \|f_n - f\|_1 = 0$.*

Then there is a subsequence of (f_n), which converges to f almost everywhere, with each term dominated by the same integrable function.

The proof can be found in [2] or [3] (in this last reference, the domination is not mentioned, but appears clearly in the proof).

Theorem 14. *Let (f_n) be a bounded sequence of $\mathcal{L}^1([a,b],X)$ which converges almost everywhere to $f \in \mathcal{L}^1([a,b],X)$. Then $\lim\limits_{n\to+\infty} \int_a^b \|f_n - f\| = 0$ if and only if for every $\varepsilon > 0$ there exists a subdivision x such that*

$$\int_a^b \|f_n - T(f_n, x)\| \leqslant \varepsilon, \text{ for all } n \in \mathbb{N}$$

.

Proof. We denote by F the class of f and F_n the class of f_n.

If we have $\lim\limits_{n\to+\infty}\displaystyle\int_a^b \|f_n - f\| = 0$, then (F_n) converges to F in $L^1([a,b],X)$. Thus, the union of terms of (F_n) with F is a compact set, that is, it is strongly equi-integrable.

Conversely, suppose the set $\{F_n,\ n \in \mathbb{N}\}$ is bounded and strongly equi-integrable; that is, it is precompact. Let $(F_{\varphi(n)})$ be a subsequence of (F_n) which converges to a limit, G. If $g \in G$, it follows from Theorem 13, that there is a subsequence $(f_{\varphi\circ\psi(n)})$ which converges to g almost everywhere on $[a,b]$. So we have $f = g$ almost everywhere, and $F = G$. Accordingly, (F_n) converges to F, and the result follows. □

5 Application 2: Integrable Functions are Lebesgue Integrable

In this section, X will be a finite dimensional space.

It is easy to establish that Lebesgue integrable functions are integrable. Just like in [8], this is a consequence of Dominated Convergence Theorem in the generalized Riemann theories.

The converse implication is more difficult, and the hard point is the measurability of an integrable function. The standard proof uses a delicate theorem about the almost everywhere derivability of the indefinite integral (see [7, p. 145] for instance). The same results can be obtained using Theorem 3 and some standard theorems of Lebesgue theory.

Theorem 15. *Let X be a finite dimensional vector space. If $f : [a,b] \to X$ is integrable, then f is Lebesgue integrable.*

Proof. Let F be the class of f and (F_n) be a sequence of $E([a,b],X)$ such that $\|F - F_n\|_1 \leqslant 2^{-n}$ for every $n \in \mathbb{N}$.

The generalized Riemann integral and Lebesgue integral coincide for the step functions and thus, (F_n) is a Cauchy sequence in L^1 (of Lebesgue's theory). As L^1 is complete, the sequence (F_n) converges to a limit F^* in L^1.

Let $f_n \in F_n$ for every $n \in \mathbb{N}$ and apply Theorem 13 in the Lebesgue theory. In this way, we extract a subsequence from (f_n) which converges to an element f^* of F almost everywhere, with a Lebesgue integrable function which dominates all terms of this subsequence.

We deduce that f^* is integrable and the Dominated Convergence Theorem (in the generalized Riemann integral) leads to $\int_a^b \|f - f^*\| = 0$. This result implies $f = f^*$ almost everywhere [4] and the conclusion that f is measurable and Lebesgue integrable now follows. □

6 Application 3: Integrable Functions are Bochner Integrable

In this section, X will denote a general Banach space and μ, Lebesgue measure on $[a, b]$.

Definition 16. *A function $f : [a, b] \to X$ is called simple if there exists $z_1, z_2, \ldots, z_n \in X$ and $E_1, E_2, \ldots, E_p$, some measurable parts of $[a, b]$, such that $f = \sum_{i=1}^{p} z_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of E_i.*

A function $f : [a, b] \to X$ is measurable if it is an almost everywhere pointwise limit of simple functions.

Definition 17. *A measurable function $f : [a, b] \to X$ is called Bochner-integrable if there exists a sequence of simple functions (f_n) such that*

$$\lim_{n\to+\infty} \int_a^b \|f_n - f\| d\mu = 0.$$

In this case, the integral of f on $[a, b]$ is defined by $\int_a^b f = \lim_{n\to+\infty} \int_a^b f_n$ and is independent of the defining sequence (f_n).

The equivalence of the generalized Riemann theory used in this paper and the Bochner theory was proved in [4]. Once more, it is easy to prove that a Bochner integrable function is integrable, but the converse is harder. In [4] we use a difficult Bochner characterization theorem, namely, a measurable function $f : [a, b] \to X$ is Bochner-integrable if and only if $\|f\|$ is Lebesgue integrable. Theorem 3 will allow us to avoid this difficulty.

Theorem 18. *If $f : [a, b] \to X$ is integrable, then f is Bochner integrable on $[a, b]$.*

PROOF. It is easy to see using the triangle inequality that the integrability of $g : [a, b] \to X$ implies the integrability of $\|g\| : [a, b] \to \mathbb{R}$ on $[a, b]$. Thus, $\|g\| : [a, b] \to \mathbb{R}$ is Lebesgue integrable.

Let $f : [a, b] \to X$ be an integrable function. It follows from the Theorem 3 that there exists a sequence of step functions (f_n) such that $(\int_a^b \|f_n - f\|)_{n\in\mathbb{N}}$ converges to 0. As the functions $\|f_n - f\| : [a, b] \to \mathbb{R}$ are Lebesgue integrable, it follows from Theorem 13 that there exists a subsequence (g_n) of (f_n) which converges to f almost everywhere. Accordingly, f is measurable and $\lim_{n\to+\infty} \int_a^b \|f - g_n\| d\mu = 0$. That is, the function f is Bochner integrable. □

7 Conclusion

The approximation theorems proposed in this paper can be obtained in the Lebesgue theory. Nevertheless, their natural framework is in the general Riemann theories and it is in this context that compacity theorems become clear.

The simplicity and the importance of Theorems 11 and 12 show the interest of introducing the notion of equi-integrability. This notion, expressed without reference to gauge, becomes a relevant tool in the Lebesgue theory as well.

In the measure theory framework, the compacity results proved in this paper can be generalized in two directions [5].

- We can substitute a metrizable locally compact space with a Radon measure for $[a, b]$ with the Lebesgue measure. In this case, the notion of partition in measurable sets must be used instead of subdivision.
- Approximation results and characterization of compact subset of L^p, $1 \leqslant p < +\infty$, can be obtained.

Finally, the usefulness of this approach is validated by the extension of some results concerning compact imbedding theorems for Sobolev spaces [6].

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