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ON SERIES OF NON-NEGATIVE TERMS

Abstract

If $\sum a_k$ and $\sum b_k$ are series of non-negative terms, we provide a necessary and sufficient condition that $\sup_n\{\sum_1^n a_k / \sum_1^n b_k\} = \infty$.

The properties of infinite series all of whose terms are greater than or equal to 0, are of importance in many problems of analysis. Given two such series, $\sum a_k$ and $\sum b_k$, we shall be interested in the magnitude of the ratio of the partial sums. We give two examples of problems in which this ratio figures.

Let $\Lambda = \{\lambda_k\}$ be an increasing sequence of positive numbers such that $\sum 1/\lambda_k = \infty$. If f is a real function of a real variable and $\{I_k\} = \{[a_k, b_k]\}$ is a collection of nonoverlapping intervals in its domain, we write $f(I_k) = f(b_k) - f(a_k)$. The class of functions for which $\sup_{\{I_n\}} \sum |f(I_k)| / \lambda_k < \infty$ are known as the functions of Λ-*bounded variation,* often referred to as the Waterman class ΛBV ([1], for an overview see [2]). If Λ is as above and $\Gamma = \{\gamma_k\}$ is another such sequence, then Perlman and Waterman [3] showed that $\Lambda BV \supsetneq \Gamma BV$ if and only if

$$\sup_n \frac{\sum_1^n \frac{1}{\gamma_k}}{\sum_1^n \frac{1}{\lambda_k}} = \infty.$$

More recently, Kvernadze[4], in studying Lagrange interpolation, showed that if $\{\lambda_k\}$ is a nondecreasing sequence satisfying

$$\sup_n \frac{\sum_1^n k^{q-1/2}}{\sum_1^n \frac{1}{\lambda_k}} = \infty,$$

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where $q > -1/2$, then there is a sequence $\{\varepsilon_k\} \searrow 0$ such that

$$\sum \frac{\varepsilon_k}{\lambda_k} < \infty \text{ and } \sum \frac{\varepsilon_k}{k^{q-1/2}} = \infty.$$

Here we shall give a necessary and sufficient condition for

$$\sup_n \left\{ \frac{\sum_1^n a_k}{\sum_1^n b_k} \right\} = \infty$$

for arbitrary series of non-negative terms. Note in particular that we do not assume $\{a_k\}$ and $\{b_k\}$ to be monotone.

Theorem 1. *Let $\{a_k\}$ and $\{b_k\}$ be two sequences of non-negative numbers, $b_1 \neq 0$. Then*

$$\sup_n \sum_1^n a_k / \sum_1^n b_k = \infty \tag{1}$$

if and only if there exists a monotone-decreasing null sequence $\{\varepsilon_k\}$ such that

$$\sum_1^n \varepsilon_k b_k < \infty \text{ and } \sum_1^n \varepsilon_k a_k = \infty.$$

PROOF. If (1) holds, then there is a subsequence of the natural numbers $\{n_i\}$ such that

$$\sum_1^{n_i} a_k > 2^i \sum_1^{n_i} b_k. \tag{2}$$

It is clear that (1) implies $\sum_1^\infty a_k = \infty$, so there is a subsequence $\{n_{i_j}\}$ such that

$$\sum_1^{n_{i_{j+1}}} a_k > 2 \sum_1^{n_{i_j}} a_k. \tag{3}$$

From (2) and (3) we have

$$\sum_{n_{i_j}+1}^{n_{i_{j+1}}} a_k \geqslant \frac{1}{2} \sum_1^{n_{i_{j+1}}} a_k \geqslant 2^{i_{j+1}-1} \sum_1^{n_{i_{j+1}}} b_k. \tag{4}$$

For $n_{i_j+1} \leqslant k \leqslant n_{i_{j+1}}$, let

$$\varepsilon_k = \frac{1}{2^{i_{j+1}} \sum_{l=1}^{n_{i_{j+1}}} b_l} \tag{5}$$

and, for $k = 1, 2, \dots, n_{i_1}$, let ε_k be chosen so that $\{\varepsilon_k\}_1^\infty$ is monotone. Thus $\{\varepsilon_k\}_1^\infty$ is a monotone-decreasing null sequence. From (4) and (5) we have

$$\sum_{n_{i_1}+1}^{\infty} \varepsilon_k a_k = \sum_{j=1}^{\infty} \sum_{n_{i_j}+1}^{n_{i_{j+1}}} \varepsilon_k a_k = \sum_{j=1}^{\infty} \frac{1}{2^{i_{j+1}} \sum_1^{n_{i_{j+1}}} b_k} \sum_{n_{i_j}+1}^{n_{i_{j+1}}} a_k \geqslant \sum_1^{\infty} \frac{1}{2} = \infty,$$

and, at the same time,

$$\sum_{n_{i_1}+1}^{\infty} \varepsilon_k b_k = \sum_{j=1}^{\infty} \frac{\sum_{n_{i_j}+1}^{n_{i_{j+1}}} b_k}{2^{i_{j+1}} \sum_1^{n_{i_{j+1}}} b_k} \leqslant \sum_{j=1}^{\infty} \frac{1}{2^{i_{j+1}}} < \infty,$$

which establishes the necessity of our condition.

Now we assume that (1) is false. Then there is an $M > 0$ such that, for all n,

$$\sum_1^n a_k \leqslant M \sum_1^n b_k.$$

Letting $A_k = \sum_1^n a_j, B_k = \sum_1^n b_j$ and applying Abel's partial summation, we have

$$\begin{aligned} \sum_1^n \varepsilon_k a_k &= \sum_1^{n-1} A_k(\varepsilon_k - \varepsilon_{k+1}) + A_n \varepsilon_n \\ &\leqslant M \sum_1^{n-1} B_k(\varepsilon_k - \varepsilon_{k+1}) + M B_n \varepsilon_n = M \sum_1^n \varepsilon_k b_k, \end{aligned}$$

noting that $\varepsilon_k - \varepsilon_{k+1} \geqslant 0$ for all k. Thus we see that, under the assumption that (1) is false, $\sum \varepsilon_k a_k$ converges if $\sum \varepsilon_k b_k$ converges, which completes the proof. $\square$

Note that monotonicity of $\{a_k\}$ and $\{b_k\}$ was not required, unlike the particular case considered in [4].

Remark. In the proof of our theorem, the monotonicity of $\{\varepsilon_k\}$ was used strongly. In fact, it is essential to the validity of the theorem. Let us suppose

$$a_k = \begin{cases} \frac{1}{k+1}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases} \qquad b_k = \begin{cases} \frac{1}{k}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$$

and

$$\varepsilon_k = \begin{cases} 1/\log k, & k \text{ even} \\ 1/\log^2 k, & k \text{ odd} \end{cases}.$$

Then $\{\varepsilon_k\}$ is a positive null sequence with $\sum \varepsilon_k a_k = \infty$ and $\sum \varepsilon_k b_k < \infty$, but

$$\sup_n \sum_1^n a_k / \sum_1^n b_k = 1.$$

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