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APPROXIMATION OF CONTINUOUS FUNCTIONS BY LIPSCHITZ FUNCTIONS

Abstract

Georganopoulos (see [1]) has shown that a continuous function $f : X \to B$, where X is a compact metric space and B a convex subset of a real normed space Y, is the uniform limit of a sequence of Lipschitz maps from X to B.

In this note we obtain a similar result, namely we show that a continuous function $f : X \to \mathbb{R}$, where X is a metric space, is a uniform limit of a sequence of locally Lipschitz maps from X to $\mathbb{R}$.

When X is compact and $B = Y = \mathbb{R}$, we get the Georganopoulos' result.

1 Introduction

Definition 1. A map f of a metric space (X, d) into a metric space $(Y, d^{'})$ is said to be Lipschitz, if there is a constant $M \geq 0$ such that $\mathrm{d}^{'}(f(x), f(y)) \leq M \cdot d(x, y)$ for all x, y in X.

If every $x \in X$ has a neighborhood U such that $f_{|U}$ is Lipschitz, f is said to be locally Lipschitz (abbreviated LIP).

Definition 2. A LIP-partition of unity, subordinated to an open cover $(U_j)_{j\in J}$ of a metric space X, is a family $(\varphi_j)_{j\in J}$ of LIP maps $\varphi_j : X \to [0, 1]$ such that:

i) the supports $spt\ \varphi_j = \overline{\varphi_j^{-1}(0, 1]}$ form a locally finite family,

ii) $\mathrm{spt}\, \varphi_j \subseteq U_j$ for all $j \in J$ and

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iii) $\sum_{j\in J}\varphi_j(x)=1$ for all $x\in X$.

Theorem 1. *(LIP- partition of unity, see [2]) Let $(U_j)_{j\in J}$ be an open cover of a metric space X. Then, there is a LIP-partition of unity, subordinated to this cover.*

2 The Result

Theorem 2. *Let X be a metric space. Then, given a continuous function $f: X\to\mathbb{R}$, there exists a sequence of LIP functions $(f_n)_{n\geq 1}$, $f_n: X\to\mathbb{R}$, such that $f_n \overset{u}{\to} f$*

PROOF. For each $r\in\mathbb{Q}$, let us consider the set

$$U_r^1=\{y\in X\mid f(y)-1<r\}\cap\{y\in X\mid r<f(y)+1\}.$$

Due to the fact that f is continuous, U_r^1 is open for all $r\in\mathbb{Q}$. As $(U_r^1)_{r\in\mathbb{Q}}$ is an open cover of X, we can consider, according to Theorem 1, $(\varphi_r^1)_{r\in\mathbb{Q}}$ a LIP-partition of unity subordinated to this cover. Then, let us consider $f_1: X\to\mathbb{R}$ given by

$$f_1(y)=\sum_{r\in\mathbb{Q}} r\cdot\varphi_r^1(y).$$

It is clear that f_1 is LIP because φ_r^1 are LIP.

For $y\in X$, if $y\in spt\varphi_{r_i}^1\subseteq U_{r_i}^1$, $i\in\{1,...,n\}$ and $y\notin spt\varphi_r^1$ for $r\notin\{r_1,...,r_n\}$, then

$$\begin{aligned} f(y)-1 &= \sum_{i=1}^{n}\varphi_{r_i}^1(y)\cdot(f(y)-1) < \sum_{i=1}^{n} r_i\cdot\varphi_{r_i}^1(y) \\ &= f_1(y) < \sum_{i=1}^{n}\varphi_{r_i}^1(y)\cdot(f(y)+1) = f(y)+1, \end{aligned}$$

i.e.

$$f(y)-1<f_1(y)<f(y)+1$$

for all $y\in X$.

Now, for each $r\in\mathbb{Q}$, let us consider the set

$$U_r^2=\{y\in X\mid \max\{f(y)-\tfrac{1}{2},f_1(y)-\tfrac{1}{2}\}<r\}\cap$$

$$\cap\{y\in X\mid r<\min\{f(y)+\tfrac{1}{2},f_1(y)+\tfrac{1}{2}\}.$$

Due to the fact that $f(y)-\frac{1}{2}$, $f_1(y)-\frac{1}{2}$, $f(y)+\frac{1}{2}$ and $f_1(y)+\frac{1}{2}$ are continuous, U_r^2 is open for all $r\in\mathbb{Q}$. As $(U_r^2)_{r\in\mathbb{Q}}$ is an open cover of X, we can consider, according to Theorem 1, $(\varphi_r^2)_{r\in\mathbb{Q}}$ a LIP-partition of unity subordinated to this cover. Let us consider $f_2 : X \to \mathbb{R}$ given by

$$f_2(y) = \sum_{r\in\mathbb{Q}} r \cdot \varphi_r^2(y).$$

Exactly as above we have that f_2 is LIP and

$$\max\{f(y) - \frac{1}{2}, f_1(y) - \frac{1}{2}\} < f_2(y) < \min\{f(y) + \frac{1}{2}, f_1(y) + \frac{1}{2}\}.$$

Continuing these process we obtain for each $n \in N$ a LIP function $f_n : X \to \mathbb{R}$ such that

$$\max\{f(y) - \frac{1}{2^{n-1}} \ , \ f_{n-1}(y) - \frac{1}{2^{n-1}}\} < f_n(y) <$$
$$< \min\{f(y) + \frac{1}{2^{n-1}}, f_{n-1}(y) + \frac{1}{2^{n-1}}\}.$$

It is clear that

$$|f_n(y) - f_{n-1}(y)| < \frac{1}{2^{n-1}}.$$

Consequently, for each $n, p \in \mathbb{N}$ and for each $y \in X$ we have

$$|f_{n+p}(y) - f_n(y)| < \frac{1}{2^{n-1}} \cdot (1 - \frac{1}{2^p}) < \frac{1}{2^{n-1}}.$$

Hence $(f_n(y))_{n\in\mathbb{N}}$ is uniformly Cauchy, so $(f_n(y))_{n\in\mathbb{N}}$ is uniformly convergent. Moreover for each $n \in \mathbb{N}$ and for each $y \in X$ we have

$$f(y) - \frac{1}{2^{n-1}} < f_n(y) < f(y) + \frac{1}{2^{n-1}}.$$

This implies that $\lim_{n\to\infty} f_n(y) = f(y)$ uniformly. □

Remark. There exists continuous functions that cannot be the uniform limit of a sequence of Lipschitz functions. So, in this sense, our result is the best one can obtain.

For example, the continuous function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$, has the above property. Indeed, let us suppose that there exists a sequence of Lipschitz functions $f_n : \mathbb{R} \to \mathbb{R}$ such that $f_n \to f$ uniformly. Then there exists an integer n_0 so that

$$|f_{n_0}(x) - f(x)| < \frac{1}{2} \text{ for every } x \in \mathbb{R}.$$

Since f_{n_0} is Lipschitz there exists a constant M_{n_0} such that

$$|f_{n_0}(x) - f_{n_0}(y)| < M_{n_0} |x - y| \ \text{ for every } x, y \in \mathbb{R}.$$

Hence, for every $x, y \in \mathbb{R}$, we have

$$|f(x) - f(y)| \leq |f_{n_0}(x) - f(x)| + |f_{n_0}(x) - f_{n_0}(y)| + |f_{n_0}(y) - f(y)|$$

so, we obtain

$$|f(x) - f(y)| < 1 + M_{n_0} |x - y| .$$

According to the mean value theorem, for every $x, y \in \mathbb{R}$, there exists ζ between x and y such that

$$f(x) - f(y) = f^{'}(\zeta)(x - y).$$

Hence

$$|x - y| (e^{\zeta} - M_{n_0}) < 1.$$

Choosing $y = n + 1$ and $x = n$, where n is an arbitrary integer, we have

$$e^n - M_{n_0} < 1 \text{ for each } n \in \mathbb{N}.$$

The last relation obviously is not true.

References

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