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ON MARCZEWSKI-BURSTIN LIKE CHARACTERIZATIONS OF CERTAIN σ-ALGEBRAS AND σ-IDEALS

Abstract

Consider a σ-ideal, σ-algebra pair $\mathcal{I} \subseteq \mathcal{A}$ on a Polish space X which has no isolated points, such that $\mathcal{A}$ contains all the Borel subsets of X while $\mathcal{I}$ contains all the countable subsets of X, but none of the perfect subsets of X. We show that if $(\mathcal{I}, \mathcal{A})$ admits a simultaneous MB-like characterization consisting of Borel sets, then $(\mathcal{I}, \mathcal{A})$ is $((s_0), (s))$, the σ-ideal, σ-algebra pair of Marczewski null, Marczewski measurable sets. We deduce some results about uniformly completely Ramsey sets.

Let X be a Polish space with no isolated points, and let $\mathcal{I} \subseteq \mathcal{A}$ be a σ-ideal, σ-algebra pair on X. $\mathcal{A}$ is said to have a *"Marczewski-Burstin-like"* (or "*MB-like*") characterization based on a collection $\mathcal{G}$ of Borel subsets of X provided $\emptyset \notin \mathcal{G}$ and

$$M \in \mathcal{A} \iff \forall P \in \mathcal{G},\ \exists Q \in \mathcal{G},\ Q \subseteq P \ni Q \subseteq M \text{ or } Q \cap M = \emptyset. \tag{1}$$

$\mathcal{I}$ is said to have an "*MB-like*" characterization based on $\mathcal{G}$ provided $\emptyset \notin \mathcal{G}$ and

$$M \in \mathcal{I} \iff \forall P \in \mathcal{G},\ \exists Q \in \mathcal{G},\ Q \subseteq P \ni Q \cap M = \emptyset. \tag{2}$$

If both (1) and (2) hold with the same class $\mathcal{G}$, we say that the pair $(\mathcal{I}, \mathcal{A})$ has a *simultaneous MB-like* characterization based on $\mathcal{G}$. We say that the pair $(\mathcal{I}, \mathcal{A})$ satisfies *condition (C)* if $\mathcal{A}$ contains all the Borel subsets of X, while $\mathcal{I}$ contains all the countable subsets of X, but none of the perfect subsets of X.

It was seen in Corollary 2 of [1] that the σ-ideal, σ-algebra pairs (AFC, B_r), of the always first category sets, sets with the Baire property in the restricted sense, and (U_0, U), of the universally null sets, universally measurable sets do

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not have a simultaneous MB-like characterization based on any class of Borel sets. This was a consequence of Corollary 1 of [1] which says that if $\mathcal{I} \subseteq \mathcal{A}$ is a σ-ideal, σ-algebra pair on X that satisfies condition (C) and admits a simultaneous MB-like characterization based on a collection $\mathcal{G}$ of Borel sets, then $(\mathcal{I}, \mathcal{A})$ satisfies the *Marczewski hull condition*, which says:

$$\forall Z \subseteq X, \exists M \in \mathcal{A} \ni Z \subseteq M \text{ and } \forall N \in \mathcal{A} \ni Z \subseteq N,\ M \setminus N \in \mathcal{I}. \qquad (3)$$

Since the publication of [1], the current author discovered the following theorem, which greatly clarifies and simplifies the proof of Corollary 2 of [1].

Theorem 1. *Let $(\mathcal{I}, \mathcal{A})$ be a σ-ideal, σ-algebra pair satisfying condition C. If $(\mathcal{I}, \mathcal{A})$ admits a simultaneous MB-like characterization based on a class of Borel sets $\mathcal{G}$, then $(\mathcal{I}, \mathcal{A}) = (S_0(X), S(X))$.*

PROOF. We first note that the class $\mathcal{G}$ cannot contain any countable Borel sets because if there were a countable $P \in \mathcal{G}$, P would belong to $\mathcal{I}$. But there could then exist no $Q \subseteq P$, $Q \in \mathcal{G}$ for which $Q \cap P = \emptyset$.

Let $M \in S(X)$ and let $P \in \mathcal{G}$. Since P is an uncountable Borel set, it contains a perfect set P'. So there exists a perfect set $Q' \subseteq P'$ such that $Q' \subseteq M$ or $Q' \cap M = \emptyset$. By Lemma 1 of [1], there exists a set $Q \in \mathcal{G}$ such that $Q \subseteq Q' \subseteq P$. If $Q' \subseteq M$, then so is Q. If $Q' \cap M = \emptyset$, then $Q \cap M = \emptyset$. So $M \in \mathcal{A}$.

The proof in the other direction is similar. So $\mathcal{A} = S(X)$. Similarly we get $\mathcal{I} = S_0(X)$. This finishes the proof. $\square$

It is known that the pair (CR_0, CR) of completely Ramsey null, completely Ramsey sets has a simultaneous MB-like characterization based on a certain class of Borel sets. On the other hand, since the pair (UCR_0, UCR) of uniformly completely Ramsey null, uniformly completely Ramsey sets satisfies condition (C), and under (CH), there exists an (s_0) set which is not in UCR (cf. Th. 14 of [2]), we conclude from Theorem 1,

Corollary 2. *Under (CH), the pair (UCR_0, UCR) has no simultaneous MB-like characterization based on a collection of Borel sets.*

The existence of an (s_0) set which is not UCR actually answers the question of whether the (UCR_0, UCR) pair satisfies either the Marczewski hull condition or the countable chain condition (ccc). Recall that the *countable chain condition* states that every collection of pairwise disjoint sets from $\mathcal{A} \setminus \mathcal{I}$ is countable. It is known that the pair (CR_0, CR) satisfies the Marczewski hull condition, but not the (ccc). We have the following result whose proof follows Walsh's remarks in [3],

Corollary 3. *Under (CH), the pair* (UCR_0, UCR) *satisfies neither the Marczewski hull nor the countable chain conditions.*

Proof. Under (CH) (cf. Th. 14 of [2]), we know that there exists a set Z which is (s_0), but not UCR. Let $M \in UCR$ be such that $Z \subseteq M$. Then, since a set is in UCR_0 if and only if it is in UCR and is totally imperfect (Thm. 7 of [2]), $M \in UCR \setminus UCR_0$ and contains a perfect set P disjoint from Z. By letting $N := P^c$, we see that the Marczewski hull condition (3) is not satisfied by the pair (UCR_0, UCR). Since the (ccc) implies the Marczewski hull condition, we see that the (ccc) is not satisfied either. $\square$

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