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POROSITY IN SPACES OF DARBOUX-LIKE FUNCTIONS

Abstract

It is known that the six Darboux-like function spaces of continuous, extendable, almost continuous, connectivity, Darboux, and peripherally continuous functions $f : \mathbb{R} \to \mathbb{R}$, with the metric of uniform convergence, form a strictly increasing chain of subspaces. We denote these spaces by C, Ext, AC, Conn, D, and PC, respectively. We show that C and D are porous and AC and Conn are not porous in their successive spaces of this chain.

The porosity of special sets in spaces of Darboux-like functions has been studied, for example, in [10] and [11]. For functions $f : \mathbb{R} \to \mathbb{R}$, it is known that C $\subset$ Ext $\subset$ AC $\subset$ Conn $\subset$ D $\subset$ PC [12]. Each function space we study will have on it the metric d of uniform convergence defined by $d(f, g) = \min\{1, \sup\{|f(x) - g(x)| : x \in \mathbb{R}\}\}$. In [2, thm 7, p.445], Bruckner and Ceder show that if $f \in$ cl(D) and the graph of f is dense in $\mathbb{R}^2$, then $f \in$ cl(Conn) and Conn is dense in each open ball in cl(D) with radius ≤ 1 and centered at f. Therefore Conn is somewhere dense in D, and it follows that Conn is not porous at some point of D. However, we show Conn is a boundary set in D. We also show C is porous in Ext, D is porous in PC, but AC is not porous in Conn. Whether or not Extis porous in AC is left as an open problem.

A subset K of $\mathbb{R}^2$ is said to be *bilaterally dense* (resp. *bilaterally c-dense) in itself* if for each $z \in K$, each open square which has a vertical side bisected by z contains infinitely many (resp. c-many) points of K.

For a function $f : \mathbb{R} \to \mathbb{R}$ we define:

1. $f \in$ PC if the graph of f is bilaterally dense in itself.

2. $f \in$ D if $f(J)$ is connected for each connected set $J \subset \mathbb{R}$.

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3. $f \in$ Conn if the graph of the restriction $f \restriction J$ is connected for each connected set $J \subset \mathbb{R}$.

4. $f \in$ AC if each open neighborhood in $\mathbb{R}^2$ of the graph of f contains the graph of a continuous function $g : \mathbb{R} \to \mathbb{R}$.

5. $f \in$ Ext if there is a function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F(x, 0) = f(x)$ for all $x \in \mathbb{R}$ and the graph of $F \restriction J$ is connected for each connected set $J \subset \mathbb{R}^2$.

For $\epsilon > 0$, $S_\epsilon(f) = \{(x, y) : x \in \mathbb{R} \text{ and } |y - f(x)| < \epsilon\}$ denotes the ϵ-strip about f. For a subset K of $\mathbb{R}^2$, $\Pi_1(K)$ denotes the x-projection of K and $K_x = K \cap \Pi_1^{-1}(x) = K \cap (\{x\} \times \mathbb{R})$. Suppose A and B are intervals in $\mathbb{R}$. A *blocking set* in $A \times B$ is a closed subset K of $A \times B$ which meets every continuous function from A into B and which misses some function from A into B. A function $f : A \to B$ is almost continuous relative to $A \times B$ if and only if it meets every blocking set in $A \times B$. Each blocking set in $A \times B$ contains a minimal blocking set K, and $\Pi_1(K)$ is a nondegenerate connected set and K is a perfect set [8, thm 1, p. 182], [7, lem 3, p.126].

In a metric space (X, d), $B(x, r)$ denotes the open ball centered at x with radius $r > 0$. Let $M \subset X$, $x \in X$, and $r > 0$. Then $\gamma(x, r, M)$ denotes the supremum of the set of all $s > 0$ for which there exists $z \in X$ such that $B(z, s) \subset B(x, r) \setminus M$. M is *porous at* x if $\limsup_{r \to 0^+} \dfrac{\gamma(x, r, M)}{r} > 0$. M is *porous* in X if M is porous at each $x \in X$. A porous set M turns out to be a boundary set in X.

Let $\mathcal{A} \subset \mathcal{B}$ be consecutive spaces in the above chain of Darboux-like spaces. $\mathcal{B} \setminus \mathrm{cl}(\mathcal{A}) \neq \emptyset$ because of [4, thm 9.10, p. 517] and because $y = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ belongs to Ext $\setminus$ cl(C) $=$ Ext $\setminus$ C and the characteristic function $\chi_{\mathbb{Q}}$ of the set $\mathbb{Q}$ of rational numbers belongs to PC $\setminus$ cl(D). $\mathcal{A}$ is porous at each member of the open set $\mathcal{B} \setminus \mathrm{cl}(\mathcal{A})$. So to verify whether $\mathcal{A}$ is porous in $\mathcal{B}$, it suffices to check porosity at just the functions f in $\mathcal{B}$ that are uniform limits of sequences $\langle f_n \rangle$ in $\mathcal{A}$.

Theorem 1. *C is porous in Ext.*

PROOF. According to the last observation, it suffices to show C is porous at $f \in$ Ext when f is a uniform limit of a sequence in C. But then $f \in$ C. Let $0 < r \leq 1$ and $x_0 \in \mathbb{R}$. There exists $\delta > 0$ such that $f([x_0 - \delta, x_0 + \delta]) \subset (f(x_0) - \frac{r}{8}, f(x_0) + \frac{r}{8})$. As in the proof of Theorem 2 in [11], define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \notin (x_0 - \delta, x_0 + \delta) \\ l_1(x) & \text{if } x \in [x_0 - \delta, x_0] \\ \frac{r}{8} \sin \frac{1}{x - x_0} + f(x_0) & \text{if } x \in (x_0, x_0 + \frac{\delta}{2}) \\ l_2(x) & \text{if } x \in [x_0 + \frac{\delta}{2}, x_0 + \delta] \end{cases}$$

where l_1 and l_2 are linear functions such that $l_1(x_0 - \delta) = f(x_0 - \delta)$, $l_1(x_0) = f(x_0)$, $l_2(x_0 + \frac{\delta}{2}) = \frac{r}{8} \sin \frac{2}{\delta} + f(x_0)$, and $l_2(x_0 + \delta) = f(x_0 + \delta)$. Then $g \in \mathrm{Ext}$ and $d(g, f) < \frac{r}{4}$. Therefore $B(g, \frac{r}{16}) \subset B(f, r)$ and $B(g, \frac{r}{16}) \cap \mathrm{C} = \emptyset$. Since $\gamma(f, r, C) \geq \frac{r}{16}$, $\limsup_{r \to 0^+} \frac{\gamma(f,r,C)}{r} \geq \frac{1}{16} > 0$. This shows C is porous at f. □

The next two results are analogous to Theorems 6 and 7 in [2, p.445]. The proof of the second result depends on the part of Natkaniec's Theorem 1 in [9, p. 40] which states the following: Define a condition for any function $f : [0, 1] \to \mathbb{R}$ this way: (α) for sufficiently small $\epsilon > 0$ and for every blocking set K in $[0, 1] \times \mathbb{R}$, either $\mathrm{card}(\mathrm{dom}(K \cap S_\epsilon(f))) = c$ or $(f(x) - \epsilon, f(x) + \epsilon) \subset K_x$ for some $x \in [0, 1]$. Then $(\alpha) \to f \in \mathrm{cl(AC)}$.

He does not prove this part in [9], but he gave the following proof in a preprint of an earlier version of [9] without the Continuum Hypothesis.

Suppose the collection $\{K_\alpha : \alpha \in A\}$ of all blocking sets of $[0, 1] \times \mathbb{R}$ is well ordered so that for each $\alpha \in A$, $\mathrm{card}(\{K_\beta : \beta < \alpha\}) < c$. For sufficiently small $\epsilon > 0$, one can use transfinite induction to choose for each $\alpha \in A$ a point $(x_\alpha, y_\alpha) \in S_\epsilon(f) \cap K_\alpha$ such that if $\mathrm{card}(\mathrm{dom}(K_\alpha \cap S_\epsilon(f))) = c$, then $x_\alpha \neq x_\beta$ for all $x \in [0, 1]$. But if $\mathrm{card}(\mathrm{dom}(K_\alpha \cap S_\epsilon(f))) < c$ and $(f(x) - \epsilon, f(x) + \epsilon) \subset (K_\alpha)_x$ for some $\beta < \alpha$, then choose $x_\alpha = x$ and either $y_\alpha = y_\beta$ whenever $x_\beta = x$ for some $\beta < \alpha$ or $y_\alpha = f(x)$ otherwise. The function $g : [0, 1] \to \mathbb{R}$ defined by $g(x) = \begin{cases} y_\alpha & \text{for } x = x_\alpha \text{ and } \alpha \in A \\ f(x) & \text{otherwise} \end{cases}$ is almost continuous and $g \subset S_\epsilon(f)$.

We can replace $[0, 1]$ with $\mathbb{R}$ and we only have to check condition (α) holds for minimal blocking sets.

Theorem 2. *If $g \in \mathrm{cl}(Conn)$ has a point x_0 of continuity, then there exist balls in $\mathrm{cl}(Conn)$ arbitrarily close to g and containing no members of AC.*

Proof. Let $0 < \epsilon < 1$. There is a $\delta > 0$ such that whenever $|x - x_0| < \delta$, then $|g(x) - g(x_0)| < \frac{\epsilon}{4}$. In [5], Jastrzebski gives an example of a function h from $[0, 1]$ onto $[-1, 1]$ such that $h \in \mathrm{Conn} \setminus \mathrm{cl}(AC)$. Let f_0 be a scaled-down copy of h to $[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}] \times [g(x_0) - \frac{\epsilon}{4}, g(x_0) + \frac{\epsilon}{4}]$ instead of $[0, 1] \times [-1, 1]$. Since $g \in \mathrm{cl(Conn)}$, we can extend the domain of f_0 to all of $\mathbb{R}$ in such a way that $f_0 \in \mathrm{Conn}$ and $|f_0(x) - g(x)| < \epsilon$ for all x. Therefore there is a ball in cl(Conn) centered at f_0 and missing AC because $h \notin \mathrm{cl(AC)}$ implies $f_0 \notin \mathrm{cl(AC)}$. □

Theorem 3. *If $f \in \mathrm{cl}(Conn)$ is dense in $\mathbb{R}^2$, then AC is dense in each open ball in $\mathrm{cl}(Conn)$ of radius ≤ 1 with center f.*

PROOF. If $g \in \mathrm{cl}(\mathrm{Conn})$ is dense in R^2, then we initially show $g \in \mathrm{cl}(\mathrm{AC})$ according to [9] by verifying that g obeys (α). Suppose K is a minimal blocking set in $\mathbb{R}^2$. Let $S_\epsilon(K)$ denote the set obtained by replacing each point of K by an open vertical interval of length 2ϵ centered at the point. Because $\Pi_1(S_\epsilon(K)) = \Pi_1(K)$ is a nondegenerate interval [7], then by the Baire Category Theorem, $S_\epsilon(K)$ contains a rectangle B with a vertical side of length ϵ. Since $g \in \mathrm{cl}(\mathrm{Conn})$ is dense in $\mathbb{R}^2$, then $\mathrm{card}(g \cap B) = c$ and so $\mathrm{card}(g \cap S_\epsilon(K)) = c$. That is, $\mathrm{card}(\mathrm{dom}(S_\epsilon(g) \cap K) = c$ and therefore g obeys (α).

Next, if $f_0 \in \mathrm{cl}(\mathrm{Conn})$ and $d(f_0, f) < 1$, then f_0 must be dense in $\mathbb{R}^2$ and so, as shown first, $f_0 \in \mathrm{cl}(\mathrm{AC})$. This shows AC is dense in every open ball in $\mathrm{cl}(\mathrm{Conn})$ of radius ≤ 1 and with center f. □

The next result follows immediately from Theorem 3.

Theorem 4. *AC is not porous in Conn.*

Theorem 5. *Conn is not porous in D [2], but Conn is a boundary set in D.*

PROOF. We must show that for each $f \in D$ and $r > 0$, there exists $g \in B(f, r) \setminus \mathrm{Conn}$. According to the proof of Theorem 6 in [2], for the case when $f \in$ D and has a point of continuity, there exist balls in D arbitrarily close to f and missing Conn. This implies D $\setminus$ Conn has points arbitrarily close to f. (According to their proof, Conn is actually porous at this f.) Now consider the case when $f \in$ D and has no point of continuity. Then the graph of f is somewhere dense in $\mathbb{R}^2$ [6], [1]. Let L be a closed line segment having positive slope and lying in a circular open neighborhood $U \subset \mathrm{cl}(f)$ with radius $\leq \frac{r}{2}$. A function $g : \mathbb{R} \to \mathbb{R}$ belonging to D can be obtained from f by shifting vertically any points of $f \cap L$ off L to points in U. Then $g \in B(f, r) \setminus \mathrm{Conn}$. Together both cases show Conn is a boundary set in D. □

Theorem 6. *D is porous in PC.*

PROOF. Let $f \in$ PC. We may suppose $f \in \mathrm{cl}(\mathrm{D})$. Let $\mathcal{U}$ denote the class of all functions $f : R \to \mathbb{R}$ such that for every interval $J \subset \mathbb{R}$ and every set A of cardinality less than c, $f(J \setminus A)$ is dense in $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$. According to [3, thm 4.3, p. 71], $\mathcal{U} = \mathrm{cl}(\mathrm{D})$. Then $f \in \mathcal{U} \subset$ PC. First suppose f is not a constant function. For each sufficiently small $r > 0$ with $r \leq 1$, there exists an interval $J = [a, b]$ such that $\frac{r}{4} < |f(a) - f(b)| < \frac{r}{2}$. For argument's sake, suppose $f(a) < f(b)$. Let $B = (a, b) \cap f^{-1}((f(a), f(b)))$. It follows that $f \restriction B$ is bilaterally c-dense in itself.

We show $B = E \cup F$, where E and F are disjoint bilaterally dense in itself sets and $f \restriction E$ and $f \restriction F$ are each dense in $f \restriction B$. According to Theorem 3.2 in [3, pp. 67–68], since $f \in \mathcal{U}$, for each open interval N, $f^{-1}(N)$ is empty or c-dense in itself. Each such nonempty $f^{-1}(N)$, like $f^{-1}((f(a), f(b)))$, is actually bilaterally c-dense in itself. Let P be a countable dense subset of the graph of $f \restriction B$, and let $E = \Pi_1(P)$. E is bilaterally dense in itself and $f \restriction E = P$ is dense in $f \restriction B$. Since $f((a,b) \setminus E)$ is dense in $[\inf_{x \in J} f(x), \sup_{x \in J} f(x)]$, the set $F = B \setminus E$ is bilaterally c-dense in itself and $f \restriction F$ is dense in $f \restriction B$. So $B = E \cup F$.

Define $g : \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(a) & \text{if } x \in E \\ f(b) & \text{if } x \in F \\ f(x) & \text{if } x \in \mathbb{R} \setminus B. \end{cases}$$

Then $g \in PC \setminus \mathcal{U}$, and $g \in B(f, \frac{r}{2})$ because $|f(a) - f(b)| < \frac{r}{2}$. $B(g, \frac{r}{8}) \subset B(f, r) \setminus \mathrm{D}$ because $|f(a) - f(b)| > \frac{r}{4}$. Since $\gamma(f, r, D) \geq \frac{r}{8}$, it follows that $\limsup_{r \to 0^+} \frac{\gamma(f,r,D)}{r} \geq \frac{1}{8} > 0$ and so D is porous at f. When f is a constant function with value k and $0 < r < 1$, define

$$g(x) = \begin{cases} k + \frac{r}{2} & \text{if } x \in \mathbb{Q} \\ k & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then $B(g, \frac{r}{4}) \subset B(f, r) \setminus D$, and it follows that D is porous at f. □

References

[1] J. B. Brown, *Nowhere dense Darboux graphs*, Colloq. Math. **20** (1969), 243–247.

[2] A. M. Bruckner and J. G. Ceder, *On jumping functions by connected sets*, Czech. Math. J. **22** (1972), 435–448.

[3] A. M. Bruckner, J. G. Ceder, and M. Weiss, *Uniform limits of Darboux functions*, Colloq. Math. , **15** (1966), 65–77.

[4] R. G. Gibson and T. Natkaniec, *Darboux like functions*, Real Anal. Exchange **22** (1996-97), 492–533.

[5] J. Jastrzebski, *An answer to a question of R. G. Gibson and F. Roush*, Real Anal. Exchange **15** (1989-90), 340–341.

[6] F. B. Jones and E. S. Thomas, Jr. , *Connected G_δ graphs*, Duke Math. J. **33** (1966), 341–345.

[7] K. R. Kellum, *Sums and limits of almost continuous functions*, Colloq. Math. **31** (1974), 125–128.

[8] K. R. Kellum and B. D. Garrett, *Almost continuous real functions*, Proc. Amer. Math. Soc. **33** (1972), 181–184.

[9] T. Natkaniec, *On some problems concerning almost continuity*, Real Anal. Exchange **17** (1991-92), 40–41.

[10] R. J. Pawlak, *On some properties of the spaces of almost continuous functions*, Internat. J. Math. Math. Sci. **19** (1996), 19–23.

[11] P. J. Pawlak, J. Kucner, and B. Swiatek, *On small subsets of the space of Darboux functions*, Real Anal. Exchange, **25** (1999-2000), 343–357.

[12] J. Stallings, *Fixed point theorems for connectivity maps*, Fund. Math. **47** (1959), 249–263.