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Ehrhard Behrends, Mathematisches Institut, Freie Universität Berlin, Arnimallee 6, 14195 Berlin, Germany. email: behrends@math.fu-berlin.de

Stefan Geschke, Mathematisches Institut, Freie Universität Berlin, Arnimallee 3, 14195 Berlin, Germany, and Department of Mathematics, Boise State University, 1910 University Drive, Boise, ID 83725–1555. email: geschke@math.fu-berlin.de

Tomasz Natkaniec, Instytut Matematyki, Uniwersytet Gdański, Wita Stwosza 57, 80–952 Gdańsk, Poland. email: mattn@math.univ.gda.pl

FUNCTIONS FOR WHICH ALL POINTS ARE LOCAL EXTREMA

Abstract

Let X be a connected separable linear order, a connected separable metric space, or a connected, locally connected complete metric space. We show that every continuous function $f : X \to \mathbb{R}$ with the property that every $x \in X$ is a local maximum or minimum of f is in fact constant. We provide an example of a compact connected linear order X and a continuous function $f : X \to \mathbb{R}$ that is not constant and yet every point of X is a local minimum or maximum of f.

The following question was recently asked by M. R. Wojcik [1]:

Question 1. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that every point in $[0, 1]$ is a local maximum or minimum of f. Is it true that f has to be constant?

The answer is clearly yes if f is assumed to be differentiable but the question is about continuous functions. The answer to Question 1 remains yes, as shown by a number of people independently. However, we are not aware of any published proof of this fact. In this note we give two elementary proofs showing that a continuous function from $[0, 1]$ to $\mathbb{R}$ for which every point in $[0, 1]$ is a local minimum or maximum indeed must be constant. The first proof only uses the most basic topological properties of $\mathbb{R}$. We get the following theorem:

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Theorem 2. *Let X be a connected separable metric space. Then every continuous function $f : X \to \mathbb{R}$ for which every $x \in X$ is a local minimum or maximum is constant.*

The second proof uses the linear order on the reals.

Theorem 3. *Let X be a connected separable linearly ordered space. Then every continuous function $f : X \to \mathbb{R}$ for which every $x \in X$ is a local minimum or maximum is constant.*

Note however, that Theorem 3 is weaker than it looks at first sight. Every connected separable linear order is actually isomorphic to some interval of the real line. Refer to Remark 4.

PROOF OF THEOREM 2. Let $f : X \to \mathbb{R}$ be continuous and such that f has a local extremum at every $x \in X$. Since X is a separable metric space the topology on X has a countable base $\{B_n : n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$ let $D_n^{\min} = \{x \in B_n : \forall y \in B_n(f(x) \leq f(y))\}$ and $D_n^{\max} = \{x \in B_n : \forall y \in B_n(f(x) \geq f(y))\}$. Clearly f is constant on each $D_n^{\min}$ and each $D_n^{\max}$.

If $x \in X$ then by our assumptions on f, x is a local minimum or maximum of f. Assume it is a local minimum. Then there is some $n \in \mathbb{N}$ such that $x \in B_n$ and $f(t) \geq f(x)$ for all $t \in B_n$. In particular, $x \in D_n^{\min}$. Similarly, if x is a local maximum then $x \in D_n^{\max}$ for some $n \in \mathbb{N}$. In summary we have

$$X = \cup_{n\in\mathbb{N}}(D_n^{\min} \cup D_n^{\max}).$$

It follows that $f[X]$ is countable. Since X is connected, so is $f[X]$. But the only nonempty countable and connected subsets of the real line are the singletons. It follows that f is constant. $\square$

PROOF OF THEOREM 3. Let $<$ denote the order on X. Suppose that f is not constant. For simplicity assume that there are $x, y \in X$ such that $x < y$ and $f(x) < f(y)$. Since X is connected, $[x, y]$ is connected. Since f is continuous, $f[[x, y]]$ is connected. It follows that $[f(x), f(y)] \subseteq f[[x, y]]$.

Since X is separable, every family of pairwise disjoint open intervals in X is countable. It follows that there is some $z \in (f(x), f(y))$ such that $f^{-1}(z)$ does not contain a nonempty open interval. Since X is connected, every bounded subset of X has a supremum in X. Let $a = \sup\{b \in (x, y) : \forall c \in (x, b)(f(c) \leq z)\}$. By the continuity of f, $f(a) = z$. By the definition of a and by the connectedness of X, for every $b > a$ there is $c \in (a, b)$ such that $f(c) > z$. Since a is a local extremum of f, it follows that a is a local minimum.

This implies that there is $b < a$ such that $c \in (b, a)$ for all $f(c) \geq z$. But by the definition of a, $f(c) = z$ for all $c \in (b, a)$. Hence $f^{-1}(z)$ contains a non-empty open interval after all, contradicting the choice of z. $\square$

Remark 4. A closer analysis of the proofs of Theorem 2 and Theorem 3 shows that in both cases the separability assumption can be weakened as:

(a) Let X be a connected topological space that has a base of its topology of size $< |\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and every $x \in X$ is a local extremum of f then f is constant. This holds in particular if X is a connected metric space such that every family of pairwise disjoint open sets is of size $<|\mathbb{R}|$.

(b) Let X be a connected linear order such that every family of pairwise disjoint open intervals is of size $< |\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and every $x \in X$ is a local extremum of f then f is constant.

A question that arises naturally is this:

Question 5. Let X be a connected topological space such that every family of pairwise disjoint open sets is of size $<|\mathbb{R}|$. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of f, does f have to be constant?

Remark 4 tells us where we should look if we want to find a connected space X and a continuous function $f : X \to \mathbb{R}$ that is not constant but such that every $x \in X$ is a local minimum or maximum.

Example 6. Let I denote the closed unit interval. Consider the set $X = I \times I$ ordered lexicographically; i.e. for $a, b, c, d \in I$ let $(a, b) < (c, d)$ if $a < c$ or if $a = c$ and $b < d$. The linear order X can be considered as obtained from I by replacing every point of I by a copy of I.

It is easily checked that X is a connected linear order. It is even compact. The projection $f : X \to \mathbb{R}; (a, b) \mapsto a$ is continuous and obviously not constant. However, every $x \in X$ is a local extremum of f.

It it worth pointing out that X is not metrizable, which follows from the fact that X is compact but not separable. This brings up the following question:

Question 7. Is there an example of a connected metric space X with a continuous function $f : X \to \mathbb{R}$ that is not constant but such that every point in X is a local minimum or maximum of f?

We can provide a partial answer to this question:

Theorem 8. *Suppose X is a connected, locally connected complete metric space. If $f : X \to \mathbb{R}$ is a continuous function and every $x \in X$ is a local extremum of f then f is constant.*

The proof of this theorem is based on the following lemma:

Lemma 9. *Let X be a metric space that is Baire; i.e. in which no nonempty open set is the union of countably many nowhere dense sets. If $f : X \to \mathbb{R}$ is continuous and such that every $x \in X$ is a local extremum of f, then $V = \cup_{y\in\mathbb{R}} \operatorname{int}(f^{-1}(y))$ is dense in X.*

PROOF. Since X is a metric space, by Bing's Metrization Theorem it has a σ-discrete base $\mathcal{B}$. Let $\mathcal{B} = \cup_{n\in\mathbb{N}}\mathcal{B}_n$ with each $\mathcal{B}_n$ discrete. For each $x \in X$ fix $B_x \in \mathcal{B}$ such that $f(x) \leq f(x')$ for all $x' \in B_x$, if x is a local minimum of f, or $f(x) \geq f(x')$ for all $x' \in B_x$, if x is a local maximum. For every $n \in \mathbb{N}$ let $X_n^{\min}$ denote the set of all $x \in X$ that are local minima of f with $B_x \in \mathcal{B}_n$. Similarly, let $X_n^{\max}$ denote the set of all $x \in X$ that are local maxima with $B_x \in \mathcal{B}_n$.

Now let $G \subseteq X$ be nonempty and open. Since X is Baire, there is $n \in \mathbb{N}$ such that $X_n^{\min}$ or $X_n^{\max}$ is dense in some nonempty open set $G_0 \subseteq G$. Assume that $X_n^{\min}$ is dense in G_0 and fix $x \in X_n^{\min} \cap G_0$. Then $H = G_0 \cap B_x$ is nonempty and open, and $X_n^{\min} \cap H$ is dense in H. Since $\mathcal{B}_n$ is discrete, for every $x' \in B_x \cap X_n^{\min}$ we have $B_x = B_{x'}$ and thus $f(x) = f(x')$. It follows that f is constant on $H \cap X_n^{\min}$. Since f is continuous, f is constant on all of H. Therefore $H \subseteq V$ and hence $G \cap V \neq \emptyset$. $\square$

PROOF OF THEOREM 8. Suppose f is not constant. For every $y \in \mathbb{R}$ let $V_y = \operatorname{int}(f^{-1}(y))$. Let $V = \cup_{y\in\mathbb{R}}V_y$ and $F = X\setminus V$. Note that $F = \cup_{y\in\mathbb{R}} \operatorname{bd}(f^{-1}(y))$. Since X is connected and f is not constant, for every $y \in f[X]$ we have $\operatorname{bd}(f^{-1}(y)) \neq \emptyset$. In particular $F \neq \emptyset$.

Since F is closed in X, F is a complete metric space. By Lemma 9 the space F has a nonempty open subset on which f is constant. In other words, there is an open subset U of X such that $U \cap F \neq \emptyset$ and f is constant on $U \cap F$. Since X is locally connected, we may assume that U is connected.

Since $U \not\subseteq V$, f is not constant on U. Let $y \in f[U]$ be different from the unique value of f on $U \cap F$. Now $\operatorname{bd}(V_y) \subseteq F$. Since $y \notin f[U \cap F]$, $\operatorname{bd}(V_y) \cap U = \emptyset$. But this implies that $V_y \cap U = \operatorname{int}(V_y) \cap U$ is a proper clopen subset of U, contradicting the assumption that U is connected. $\square$

References

[1] M. R. Wojcik, problem session, 34^{th} Winter School in Abstract Analysis, Lhota nad Rohanovem, Czech Republik (2006).