INROADS

Real Analysis Exchange
Vol. 33(2), 2007/2008, pp. 431–442

B. K. Lahiri, B-1/146 Kalyani, Kalyani - 741235, West Bengal, India.
Pratulananda Das, Department of Mathematics, Jadavpur University, Kolkata-700032, West Bengal, India. email: pratulananda@yahoo.co.in

I AND I^* - CONVERGENCE OF NETS

Abstract

In this paper we consider the idea of I - convergence of nets in a topological space and derive several basic properties. This idea extends the concept of I - convergence of sequences considered so far in various spaces.

1 Introduction.

The concept of convergence of a real sequence has been extended to statistical convergence by Fast [7] (see also Schoenberg [20]) as follows.

If $\mathbb{N}$ denotes the set of natural numbers and $K \subset \mathbb{N}$, then K_n denotes the set $\{k \in K; k \leq n\}$ and $|K_n|$ stands for the cardinality of the set K_n. The natural density of the subset K is defined by $d(K) = \lim_{n\to\infty} \frac{|K_n|}{n}$ provided the limit exists ([8], [16]).

A sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to l if for arbitrary $\varepsilon > 0$, the set $K(\varepsilon) = \{k \in \mathbb{N}; |x_k - l| \geq \varepsilon\}$ has natural density zero.

Applications of statistical convergence in mathematical analysis and the theory of numbers may be seen in [2], [3] and [15].

The concept of I-convergence of a sequence is an extension of statistical convergence which depends on the structure of ideals of subsets of the set of natural numbers.

Key Words: ideal, filter, net, I-convergence, I^*-convergence, I-cluster points, condition (DP)
Mathematical Reviews subject classification: Primary: 40A30; Secondary: 40A99
Received by the editors April 24, 2007
Communicated by: Udayan B. Darji

Definition 1 ([11], p. 34)**.** If P is a non-void set, then a family $I \subset 2^P$ is called an ideal if
(i) $\phi \in I$ and
(ii) $A, B \in I$ implies $A \cup B \in I$ and
(iii) $A \in I, B \subset A$ implies $B \in I$.
The ideal I is called non-trivial if $I \neq \{\phi\}$ and $P \notin I$.

Definition 2 ([11], p. 34)**.** A non empty family F of subsets of a non-void set P is a filter if
(i) $\phi \notin F$ and
(ii) $A, B \in F$ implies $A \cap B \in F$ and
(iii) $A \in F, A \subset B$ implies $B \in F$.
Clearly $I \subset 2^P$ is a non-trivial ideal of P if and only if $F = F(I) = \{A \subset P; P \setminus A \in I\}$ is a filter on P, called the filter associated with I. A non-trivial ideal I is called admissible if I contains all the singleton sets.

Several examples of non-trivial admissible ideals have been considered in [9]. We are now in a position to define I-convergence of a real sequence.

Definition 3 ([1])**.** A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to converge to x with respect to an ideal I of the set of natural numbers or I-convergent to x if for any $\varepsilon > 0$, $A(\varepsilon) = \{n \in \mathbb{N}; |x_n - x| \geq \varepsilon\} \in I$. In this case we write $I - \lim_{n\to\infty} x_n = x$.

I-convergence includes ordinary convergence and statistical convergence when I is the ideal of all finite subsets of the set of natural numbers and all subsets of the set of natural numbers of natural density zero respectively.

In recent times several papers on I-convergence including substantial contributions by Šalát et. al. have been published ([1], [4], [6], [9], [10], [12], [13], [19]). Another concept closely related to that of I-convergence is I^*-convergence. The concept arises from the following result on statistical convergence [18]: a real sequence $\{x_n\}_{n\in\mathbb{N}}$ is statistically convergent to ξ if and only if there exists a set $M = \{m_1 < m_2 < m_3 < \dots\} \subset \mathbb{N}$ such that $d(M) = 1$ and $\lim_{k\to\infty} x_{m_k} = \xi$. In this area also Šalát et. al. [9] made remarkable contributions.

The concept of I-convergence of sequences has been extended recently from the real number space to a metric space [9], to a normed linear space [19], to a finite dimensional space [17], and to a topological space [13]. In this paper we intend to broaden the idea of I-convergence in a separate direction. We consider nets in a topological space instead of sequences and examine how far the concept of I-convergence of nets reasonably ensures all the basic properties. Similarly we introduce the idea of I^*-convergence of nets and study when I-convergence and I^*-convergence of nets coincide.

2 Definitions and Notation.

The following two definitions are widely known.

Definition 4. Let D be a non-void set and let $\geq$ be a binary relation on D such that $\geq$ is reflexive, transitive, and for any two elements $m, n \in D$, there is an element $p \in D$ such that $p \geq m$ and $p \geq n$. The pair $(D, \geq)$ is called a directed set.

Definition 5. Let $(D, \geq)$ be a directed set and let X be a non-void set. A mapping $s : D \to X$ is called a net in X denoted by $\{s_n; n \in D\}$ or simply by $\{s_n\}$ when the set D is clear.

Throughout the paper (X, τ) will denote a topological space (which will be written sometimes simply as X) and I will denote a non-trivial ideal of a directed set D. Also the symbol $\mathbb{N}$ is reserved for the set of natural numbers.

For $n \in D$ let $M_n = \{k \in D; k \geq n\}$. Then the collection $F_0 = \{A \subset D; A \supset M_n \text{ for some } n\}$ forms a filter in D. Let $I_0 = \{A \subset D; D \setminus A \in F_0\}$. Then I_0 is also a non-trivial ideal in D.

Definition 6. A non-trivial ideal I of D will be called D-admissible if $M_n \in F(I)$ for all $n \in D$.

We now define the I-convergence of a net where I is an ideal of D.

Definition 7. A net $\{s_n\}$ in X is said to be I-convergent to $x_0 \in X$ if for any open set U containing x_0, $\{n \in D; s_n \notin U\} \in I$.

Symbolically we write $I - \lim s_n = x_0$ and we say that x_0 is the I-limit of the net $\{s_n\}$.

Example 1. Let $X = \{a, b, c, d\}$ and $\tau = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Let $D = \mathbf{U}_a = \{\{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$ with the binary relation '$\geq$' defined by $U \geq V$ if $U \subset V$ for any $U, V \in D$. Then D is a directed set. Take $I = \{\{\{a\}, \{a, c\}\}, \{\{a\}\}, \{\{a, c\}\}, \phi\}$. Clearly I is a non-trivial ideal of D. Define a net $s : D \to X$ by $s_U = \begin{cases} d & \text{if } U = \{a\} \text{ or } \{a, c\} \\ a & \text{otherwise.} \end{cases}$

Then for any open set Q containing a, $\{U \in D; s_U \notin Q\}$ is either the void set ϕ or $\{\{a\}, \{a, c\}\}$. But both belong to I and so $\{s_U\}$ is I-convergent to a. But $\{s_U\}$ is not convergent to a because for any open set Q containing a, there does not exist a $U \in D$ such that $s_V \in Q$ for all $V \in D$ such that $V \geq U$.

Note 1. If I is D-admissible, then net convergence in the topology τ implies I-convergence and the converse holds if $I = I_0$. In other words, I_0-convergence implies net convergence, which fact will be used several times in the paper. Also if $D = \mathbb{N}$ with the natural ordering, then the concepts of D-admissibility and admissibility coincide and in this case I_0 is the ideal of all finite subsets of $\mathbb{N}$.

From this stage onwards we shall assume that the ideals are always non-trivial and that all ideals are from some directed set, which will be evident from the context. For convenience, an ideal I of D will be written sometimes as I_D to indicate the directed set D of which I is an ideal.

3 Basic Properties.

Theorem 1. *If X is Hausdorff, then an I-convergent net has a unique I-limit.*

Proof. If possible, suppose that an I-convergent net $\{s_n\}$ has two distinct I-limits, say x_0 and y_0. Because X is Hausdorff, there exist $U, V \in \tau$ such that $x_0 \in U$, $y_0 \in V$ and $U \cap V = \phi$. Because $\{k; s_k \notin U\} \in I$ and $\{k; s_k \notin V\} \in I$, we have $\{k; s_k \in (U \cup V)^c\} \subset \{k; s_k \in U^c\} \cup \{k; s_k \in V^c\} \in I$ where ^{c} stands for the complement in X. Since I is non-trivial, there exists $k_0 \in D$ with $k_0 \notin \{k; s_k \in (U \cap V)^c\}$. This implies $s_{k_0} \in U \cap V$, a contradiction. □

A sort of converse of Theorem 1 is given below.

Theorem 2. *If every I-convergent net in X has a unique I-limit for every D-admissible ideal I, then X is Hausdorff.*

Proof. If possible let X be not Hausdorff. Then there exist at least two points $x, y \in X$, $x \neq y$ such that whenever $x \in U$, $y \in V$, $U, V \in \tau$ we have $U \cap V \neq \phi$. Let $\mathbf{U}_x$ and $\mathbf{U}_y$ be the families of all neighborhoods of x and y respectively. Let $D = \mathbf{U}_x \times \mathbf{U}_y$ with the binary relation '$\geq$' defined by $(U, V) \geq (U_1, V_1)$ if $U \subset U_1$ and $V \subset V_1$. Then D is a directed set. Let I be any D-admissible ideal. For any $U \in \mathbf{U}_x$ and $V \in \mathbf{U}_y$ there exists a point $z_{(U,V)} \in U \cap V$. We now consider the net $s : D \to X$ defined by $s_{(U,V)} = z_{(U,V)}$. Then it is easy to check that the net converges to both x and y and since I is D-admissible, the net is I-convergent to both x and y, a contradiction. □

In the following theorem we examine the relation between a limit point of a set (in the usual topology of X) and I-limit of a certain net.

Theorem 3. *Let $A \subset X$ and $x_0 \in X$. If there is a net in $A \setminus \{x_0\}$ which is I-convergent to x_0, then x_0 is a limit point of A (in the usual topology of X). Conversely if x_0 is a limit point of A, then there is a net in $A \setminus \{x_0\}$ which is I-convergent to x_0 for some D-admissible ideal I.*

PROOF. Suppose that there is a net $\{s_n\}$ in $A \setminus \{x_0\}$ which is I-convergent to x_0. Let U be any arbitrary open set containing x_0. Since $I - \lim s_n = x_0$, $\{n; s_n \notin U\} \in I$ and so $\{n; s_n \in U\} \notin I$ (since I is non-trivial). Hence $\{n; s_n \in U\} \neq \phi$. Let $n_0 \in \{n; s_n \in U\}$. Then $s_{n_0} \in U \cap (A \setminus \{x_0\})$. Thus x_0 is a limit point of A.

Conversely if x_0 is a limit point of A, then for arbitrary neighborhood U of x_0, $U \cap (A \setminus \{x_0\}) \neq \phi$. Then taking the directed set D as $\mathbf{U}_{x_0}$ as the family of all neighborhoods U of x_0 with $U_1 \geq U$ if $U_1 \subset U$ and defining the net $s : D \to X$ by $s_U \in U \cap (A \setminus \{x_0\})$ for any $U \in \mathbf{U}_{x_0}$, we see that the net $\{s_U\}$ converges to x_0 and so is I-convergent to x_0 if I is D-admissible. This proves the theorem. □

We observe in the following theorem that a continuous mapping may be characterized through I-convergence of nets.

Theorem 4. *A continuous mapping $g : X \to X$ preserves I-convergence of a net. Conversely if $g : X \to X$ preserves I-convergence of nets for any D-admissible ideal I, then g is continuous.*

PROOF. Let $I - \lim s_n = x$. Let V be any open set containing $g(x)$. There exists an open set U containing x such that $g(U) \subset V$. Since

$$\{n; g(s_n) \notin V\} \subset \{n; s_n \notin U\}$$

and $\{n; s_n \notin U\} \in I$, we have $\{n; g(s_n) \notin V\} \in I$. This shows that $I - \lim g(s_n) = g(x)$.

Conversely, suppose g is not continuous at $x \in X$. Then there is an open set V containing $g(x)$ such that for any open neighborhood U of x, $g(U) \nsubseteq V$. Then there is an $s_U \in U$ such that $g(s_U) \notin V$. Consider the net $\{s_U; U \in D = \mathbf{U}_x\}$ where $\mathbf{U}_x$ is the family of all neighborhoods of x and D is directed as in Theorem 3. Then $\{s_U\}$ is convergent to x and so is I-convergent to x for any D-admissible ideal I. But since $\{U; g(s_U) \notin V\} = D \notin I$ (since I is non-trivial), $\{g(s_U)\}$ is not I-convergent to $g(x)$. This contradiction shows that g is continuous. □

4 I-Cluster Points.

We recall the following known definition of a subnet.

Definition 8. A net $\{t_\alpha; \alpha \in E\}$, where E is a directed set, is said to be a subnet of the net $\{s_n; n \in D\}$ if there is a mapping $i; E \to D$ such that
(i.) $t = s \circ i$ and
(ii.) for any m in D there is an element $\alpha_0 \in E$ with the property that $i(\alpha) \geq m$ for all α in E with $\alpha \geq \alpha_0$.

Note 2. Let $\{t_\alpha; \alpha \in E\}$ be a subnet of $\{s_n; n \in D\}$. We consider the collection $\{A \subset D; i^{-1}(A) \in I_E\}$. Then this collection is an ideal of D which will be denoted by $I_E(i)$. So $I_E(i) = \{A \subset D; i^{-1}(A) \in I_E\}$. Let I_E be E-admissible and $n \in D$. Let $M_n = \{k \in D; k \geq n\}$ as before. From above (ii.) it follows that corresponding to $n \in D$ there is $\alpha_0 \in E$ such that $i(\alpha) \geq n$ for all $\alpha \in E$, $\alpha \geq \alpha_0$. Thus $i(M_{\alpha_0}) \subset M_n$ where $M_{\alpha_0} = \{\alpha \in E; \alpha \geq \alpha_0\}$. Hence $M_{\alpha_0} \subset i^{-1}(M_n)$. Since I_E is E-admissible, $M_{\alpha_0} \in F(I_E)$ and so $i^{-1}(M_n) \in F(I_E)$ which in turn implies $M_n \in F(I_E(i))$. this shows that $I_E(i)$ is D-admissible.

We now introduce the following definitions, where Definition 9 is needed to show that an I-cluster point of a net is necessarily an I-limit of the net under certain restrictions on the net.

Definition 9. $y \in X$ is called I-cluster point of a net $\{s_n\}$ if for every open set U containing y, $\{n; s_n \in U\} \notin I$.

Definition 10. A net $\{s_n\}$ is called I-maximal if for any set $A \subset X$, either $\{n; s_n \notin A\} \in I$ or $\{n; s_n \notin X \setminus A\} \in I$.

Theorem 5. *Let the net $\{s_n\}$ be I-maximal. If $x_0 \in X$ is an I-cluster point of $\{s_n\}$, then $\{s_n\}$ is I-convergent to x_0.*

Proof. Let U be any open set containing x_0. Since $\{s_n\}$ is I-maximal, $\{n; s_n \notin U\} \in I$ or $\{n; s_n \notin X \setminus U\} \in I$. If possible, let $\{n; s_n \notin X \setminus U\} \in I$. Then $\{n; s_n \in U\} \in I$. But as x_0 is an I-cluster point of $\{s_n\}$, $\{n; s_n \in U\} \notin I$, a contradiction. Hence $\{n; s_n \notin U\} \in I$ and this proves the theorem. □

Theorem 6. *Let $\{s_n; n \in D\}$ be a net. If x_0 is an I_D-cluster point of $\{s_n\}$ for some D-admissible ideal I_D, then there is a subnet $\{t_\alpha; \alpha \in E\}$ of $\{s_n\}$ which is I_E-convergent to x_0 provided the ideal I_E is E-admissible.*

PROOF. Let $\mathbf{U}_{x_0}$ denote the family of all open neighborhoods of x_0. Let $E = \{(U,n); U \in \mathbf{U}_{x_0}, n \in D\}$. For (U,n) and (V,p) in E, define $(U,n) \geq (V,p)$ if $U \subset V$ and $n \geq p$. Then E is a directed set.

Let $(U,m) \in E$. Since x_0 is an I_D-cluster point of $\{s_n\}$, $\{n; s_n \in U\} \notin I_D$. Then $\{n; s_n \in X \setminus U\} \notin F(I_D)$. Since I_D is D-admissible, $M_m = \{k \in D; k \geq m\} \in F(I_D)$ for each $m \in D$. So $M_m \nsubseteq \{n; s_n \in X \setminus U\}$. Then there exists a $p \in M_m$ (and so $p \geq m$) such that $s_p \notin X \setminus U$ i.e. $s_p \in U$. Write $p = p_{(U,m)}$. Now define $i : E \to D$ by $i(U,m) = p_{(U,m)}$ and $t : E \to X$ by $t(U,m) = s_{p_{(U,m)}}$. Then it is easy to check that the subnet $\{t_\alpha; \alpha \in E\}$ is convergent to x_0 and so is I_E-convergent to x_0 because the ideal I_E is E-admissible. This proves the theorem. □

The following theorem represents something similar to the converse of Theorem 6.

Theorem 7. *If a subnet $\{t_\alpha; \alpha \in E\}$ of a net $\{s_n; n \in D\}$ is I_E-convergent to x_0 for some ideal I_E, then x_0 is an I_D-cluster point of $\{s_n\}$ for some ideal I_D.*

PROOF. Since $\{t_\alpha; \alpha \in E\}$ is a subnet of the net $\{s_n; n \in D\}$, there is a mapping $i : E \to D$ with $t = s \circ i$. Let $I_D = I_E(i)$ (see Note 2). We will show that x_0 is an I_D-cluster point of $\{s_n\}$. Let U be an open set containing x_0. If possible, let $\{n \in D; s_n \in U\} \in I_D$. Then let $B = i^{-1}(\{n \in D; s_n \in U\}) \in I_E$. Now since $\{t_\alpha\}$ is I_E-convergent to x_0, $\{\alpha \in E; t_\alpha \notin U\} \in I_E$. Then let $A = \{\alpha \in E; t_\alpha \in U\} \in F(I_E)$. But $\alpha \in A \Longrightarrow t_\alpha \in U \Longrightarrow (s \circ i)(\alpha) \in U \Longrightarrow s_{i(\alpha)} \in U \Longrightarrow i(\alpha) \in \{n \in D; s_n \in U\} \Longrightarrow \alpha \in B$. Thus $A \subset B$ and so $B = i^{-1}(\{n \in D; s_n \in U\}) \in F(I_E)$, which contradicts the fact that $B \in I_E$ (since I_E is non-trivial). Hence $\{n \in D; s_n \in U\} \notin I_D$ and so x_0 is an I_D-cluster point of $\{s_n\}$. This proves the theorem. □

We characterize I-cluster points through the following theorem in terms of a certain subset of X.

Theorem 8. *Let $\{s_n; n \in D\}$ be a net in X. Then $x_0 \in X$ is an I-cluster point of $\{s_n\}$ if and only if $x_0 \in \overline{A_T}$ for every $T \in F(I)$ where $A_T = \{x \in X; x = s_t$ for $t \in T\}$. Here bar denotes the closure in X.*

PROOF. First suppose that x_0 is an I-cluster point of $\{s_n\}$. Let U be an open set containing x_0. Then $\{n; s_n \in U\} \notin I$. Hence $\{n; s_n \in X \setminus U\} \notin F(I)$. Let $T \in F(I)$. Clearly $T \nsubseteq \{n; s_n \in X \setminus U\}$ and so there is a $t \in T$ such that $t \notin \{n; s_n \in X \setminus U\}$. But then $s_t \notin X \setminus U$ and so $s_t \in U$. Therefore $U \cap A_T \neq \phi$. Since this is true for any open set U containing x_0, $x_0 \in \overline{A_T}$.

Conversely, let $x_0 \in \overline{A_T}$ for every $T \in F(I)$. Let U be an open set containing x_0. If possible, let $\{n; s_n \in U\} \in I$. Then let $T_0 = \{n; s_n \in X \setminus U\} \in F(I)$. But then $A_{T_0} \cap U = \phi$ which contradicts the fact that $x_0 \in \overline{A_{T_0}}$. This shows that $\{n; s_n \in U\} \notin I$ and the proof is complete. □

Let $I(C_s)$ denote the collection of all I-cluster points of a net $s = \{s_n\}$. The following theorem gives a set-theoretic character of the set $I(C_s)$.

Theorem 9. *(i.) $I(C_s)$ is closed for any net $s = \{s_n; n \in D\}$ and any ideal I of D.*

(ii.) Suppose X is completely separable and let I be a given ideal of a directed set D. Let there exist a pairwise disjoint sequence $\{R_p\}$ of sets such that $R_p \subset D$, $R_p \notin I$ for all p. Then for any non-void closed set $F \subset X$, there is a net $s = \{s_n; n \in D\}$ in X such that $F = I(C_s)$.

Proof. The proof of $(i.)$ is omitted as being similar to that of Theorem 10 (i) in [13]. For $(ii.)$, since X is completely separable, F is separable and so let $A = \{a_1, a_2, \dots\} \subset F$ be a countable set with $\overline{A} = F$. For $n \in R_i$, let $s_n = a_i$. For $n \in D$ such that $n \notin R_i$ for any i (if there is any), we take $s_n = a$, a fixed element from F for any such n. Let $s = \{s_n; n \in D\}$ and $y \in I(C_s)$. If $y = a$ or a_i for some i, then $y \in F$. So let $y \neq a$ or a_i for any i.

Let U be any open set containing y. Then from definition, $\{n; s_n \in U\} \notin I$ and so $\{n; s_n \in U\} \neq \phi$. This implies either $a \in U$ or $a_i \in U$ for some i and so $U \cap F \neq \phi$. Thus y is a limit point of F and so $y \in F$. So $I(C_s) \subset F$.

To prove the reverse inclusion, let $z \in F$ and let U be any open set containing z. Then there is $a_i \in A$ such that $a_i \in U$. Thus $\{n; s_n \in U\} \supset R_i$ and so $\{n; s_n \in U\} \notin I$. This implies $z \in I(C_s)$. □

5 I^*-Convergence and the Condition (DP).

In this section we introduce the concept of I^*-convergence of a net and examine its equality with I-convergence. One may be referred to ([9], [13]) which deal with this concept for sequences.

Definition 11. A net $\{s_n; n \in D\}$ is said to be $\underline{I^*\text{-convergent}}$ to $x \in X$ if there exists a set $M \in F(I)$ (i.e. $D \setminus M \in I$) such that M itself is a directed set and the net $\{s_n; n \in M\}$ is convergent to x. In this case we write $I^* - \lim s_n = x$ and x is called the I^*-limit of $\{s_n\}$.

Example 2. We consider Example 1 with the same D, I and binary relation. Since $M = D \setminus \{\{a\}, \{a,c\}\} = \{\{a,b\}, \{a,b,c\}, X\} \in F(I)$ which is clearly a directed set as per the binary relation of Ex. 1, the net of Ex. 1 is I^*-convergent to a. However if we define a net $s : D \to X$ by

$$s_U = \begin{cases} d \text{ if } U = \{a\} \text{ or } \{a,c\} \text{ or } \{a,b,c\} \\ a \text{ otherwise.} \end{cases}$$

then the net $\{s_U; U \in D\}$ is still I^*-convergent to a but is neither I-convergent to a nor convergent to a in the usual sense. Note that I is not D-admissible.

Theorem 10. *If I is D-admissible, then $I^* - \lim s_n = x$ implies $I - \lim s_n = x$. If, in addition, X is Hausdorff, then $I^* - \lim s_n$ is unique irrespective of $M \in F(I)$.*

PROOF. There exists a set $M \in F(I)$ such that $\{s_n; n \in M\}$ converges to x. Then for any open set U containing x, there is a $p \in M$ such that $s_n \in U$ for all $n \in M$, $n \geq p$. Thus $s_n \in U$ for all $n \in M \cap M_p \in F(I)$ where, as defined earlier, $M_p = \{k \in D; k \geq p\}$. Clearly $\{n \in D; s_n \notin U\} \subset D \setminus (M \cap M_p) \in I$ and so $I - \lim s_n = x$. This proves the theorem, in light of Theorem 1. □

The following theorem may act as a converse.

Theorem 11. *If X has no limit point, then I and I^* convergence coincides for every D-admissible ideal I.*

PROOF. Let $I - \lim s_n = x_0$. We should show that $I^* - \lim s_n = x_0$. Since X has no limit point, $U = \{x_0\}$ is open. So $\{n; s_n \notin U\} \in I$. Hence $\{n; s_n \in U\} = \{n; s_n = x_0\} = M$ (say) $\in F(I)$. The proof will be complete if we show that M is directed with respect to the binary relation induced from $(D, \geq)$. It is obvious that $\geq$ is reflexive and transitive in M. Let $n_1, n_2 \in M$. Then there is a $p \in D$ such that $p \geq n_1, n_2$. Now since I is D-admissible, $M_p = \{k \in D; k \geq p\} \in F(I)$. Then $M \cap M_p \in F(I)$ and so $M \cap M_p \neq \phi$. Thus there exists a $k \in M$ such that $k \geq p \geq n_1, n_2$. Clearly $\{s_n; n \in M\}$ converges to x_0 and this proves the theorem. □

Note 3. In [13] we observed that if a first axiom Hausdorff space X has a limit point x, then there exists an admissible ideal I of the set of natural numbers and a sequence $\{y_n\}$ in X such that $I - \lim y_n = x$ but $I^* - \lim y_n$ does not exist. Here we may also infer that under the same conditions I-convergence of a net may not imply its I^*-convergence even if I is D-admissible. However it is known (Theorem 10) that I^*-convergence always implies I-convergence for any D-admissible ideal I. In the following we study the converse part under some condition which becomes necessary as well as sufficient on certain restrictions of the space. A similar condition has been widely used in the cited papers on various situations by considering sequences of elements.

Definition 12 (cf. [4], [7], [9], [13])**.** A D-admissible ideal I is said to satisfy the condition (DP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belonging to I there exists a countable family of sets $\{B_1, B_2, \dots\}$ from D such that for each j, $A_j \triangle B_j \subset D \setminus M_{p_j}$ for some $p_j \in D$ and $B = \bigcup B_j \in I$.

Note that $B_j \in I$ for all $j \in \mathbb{N}$. Here $\triangle$ stands for the symmetric difference.

Theorem 12. *Let I be a D-admissible ideal of a directed set $(D, \geq)$.*

(i.) If I satisfies the condition (DP) and (X, τ) is a first axiom space, then for any net $\{s_n; n \in D\}$ in X, $I - \lim s_n = x$ implies $I^* - \lim s_n = x$.

(ii.) Conversely if (X, τ) is a first axiom Hausdorff space containing at least one limit point and for each $x \in X$ and any net $\{s_n; n \in D\}$, $I - \lim s_n = x$ implies $I^* - \lim s_n = x$, then I satisfies the condition (DP).

Proof. The proof of the theorem is patterned after Theorem 8 of [13] with necessary modifications.

(i.) Let $I - \lim s_n = x$. Then for any open set U containing x, $\{n; s_n \notin U\} \in I$. Let $B_r(x)$ be a monotonically decreasing local base at x. Let $A_1 = \{n \in D; s_n \notin B_1(x)\}$ and for $m \geq 2$, $A_m = \{n; s_n \notin B_m(x) \text{ but } s_n \in B_{m-1}(x)\}$. Then $\{A_1, A_2, \dots\}$ is a sequence of sets in I with $A_i \cap A_j = \phi$ for $i \neq j$. By the condition (DP) there exists a countable family of sets $\{B_1, B_2, \dots\}$ in I such that $A_j \triangle B_j \subset D \setminus M_{p_j}$ for some $p_j \in D$ and $B = \cup B_j \in I$. Let $M = D \setminus B$. Then $M \in F(I)$ and so is itself directed with respect to the relation $\geq$ (see Theorem 11). We will show that $\{s_n; n \in M\}$ converges to x.

For this, let U be any open set containing x. Then there is $l \in N$ such that $B_r(x) \subset U$ for all $r \geq l$. Now $\{n; s_n \notin U\} \subset \cup_{j=1}^{l} A_j$. Also $A_j \triangle B_j \subset D \setminus M_{p_j}$ for some $p_j \in D$ for $j = 1, 2, \dots l$. Choose $n_0 \in D$ such that $n_0 \geq p_1, p_2, \dots p_l$. Then

$$\bigcup_{j=1}^{l} B_j \cap M_{n_0} = \bigcup_{j=1}^{l} A_j \cap M_{n_0}.$$

Since I is D-admissible, $M_{n_0} \in F(I)$ and so $M \cap M_{n_0} \in F(I)$. This implies $M \cap M_{n_0} \neq \phi$. So there is a $p \in M$ such that $p \geq n_0$. Now for $q \in M$, $q \geq p$ gives $q \notin B$ and this implies from the above relation that $q \notin \cup_{j=1}^{l} A_j$ (since

$q \geq p \geq n_0$). So $s_q \in B_l(x) \subset U$. This shows that $\{s_n; n \in M\}$ converges to x and so $I^* - \lim s_n = x$.

(ii.) Suppose that $x \in X$ is a limit point of X. Let $B_r(x)$ be a monotonically decreasing open base at x. We can find a sequence $\{x_r\}$ of distinct elements in X such that $x_r \in B_r(x) \setminus B_{r+1}(x)$, $x_r \neq x$ for all r, and $x_r \to x$. Let $\{A_j\}$ be a mutually disjoint countable family of non-void sets from I. Define a net $\{s_n\}$ by $s_n = x_j$ if $n \in A_j$ and $s_n = x$ if $n \notin A_j$ for any j. Let U be any open set containing x. Then there is a $m \in N$ such that $B_r(x) \subset U$ for all $r \geq m$. Now

$$\{n \in D; s_n \notin U\} \subset A_1 \cup A_2 \cup \cdots \cup A_{m-1}$$

and so belongs to I and thus $I - \lim s_n = x$. By our assumption $I^* - \lim s_n = x$. Hence there exists a set $H \in I$ such that $M = D \setminus H \in F(I)$ and $\{s_n; n \in M\}$ converges to x. Let $B_j = A_j \cap H$ for all $j \in N$. Then $B_j \in I$ for all $j \in N$ and also $\cup B_j \subset H \in I$ and thus $\cup B_j \in I$. Take $j \in N$. Now there must exist a $p \in D$ such that A_j is disjoint from $M \cap M_p$. For otherwise, for any $n \in M$ choose $n_1 \geq n$. Now $M \cap M_{n_1} \in F(I)$ and so is non-void and also $A_j \cap (M \cap M_{n_1}) \neq \phi$. This implies that there exists a $q \in A_j \cap (M \cap M_{n_1})$ which implies $q \in M$, $q \geq n_1 \geq n$ and $s_q = x_j$. Since $x_j \neq x$, there exists an open set V containing x such that $x_j \notin V$ (since X is Hausdorff) and it follows from above that there does not exist any $n \in M$ such that $m \geq n$, $m \in M \Longrightarrow s_m \in V$. This contradicts the fact that $\{s_n; n \in M\}$ converges to x. Thus we have $A_j \subset B_j \cup (M \setminus M_p)$. Then $A_j \triangle B_j = A_j \setminus B_j \subset M \setminus M_p \subset D \setminus M_p$. Since this is true for all $j \in N$, I satisfies the condition (DP). This proves the theorem. □

References

[1] V.Baláž, J. Červeňanský, P. Kostyrko and T. Šalát, *I-convergence and I-continuity of real functions*, Acta Math. (Nitra), **5** (2002), 43–50.

[2] R. C. Buck, *The measure theoretic approach to density*, Amer. J. Math., **68** (1946), 560–580.

[3] R. C. Buck, *Generalized asymptotic density*, Amer. J. Math., **75** (1953), 335–346.

[4] J. Cincura, M. Sleziak, T. Šalát, V. Toma, *Sets of statistical cluster points and I-cluster points*, Real Anal. Exchange, **30** (2005), 565–580.

[5] J. S. Connor, *The statistical and strong P-Cesaro convergence of sequences*, Analysis, **8** (1988), 47–63.

[6] K. Demirci, *I-limit superior and limit inferior*, Mathematical Communications, **6** (2001), 165–172.

[7] H. Fast, *Sur la convergence statistique*, Colloq. Math., **2** (1951), 241–244.

[8] H. Halberstem, K. F. Roth, *Sequences*, Springer, New York, 1993.

[9] P. Kostyrko, T. Šalát, W.Wilczyński, *I-convergence*, Real Anal. Exchange, **26(2)** (2000/2001), 669–685.

[10] P. Kostyrko, M. Mačaj, T. Šalát, M. Sleziak, *I-convergence and extremal I-limit points*, Math. Slovaca, **55(4)** (2005), 443–464.

[11] K. Kuratowski, *Topologie I*, PWN, Warszawa, 1961.

[12] B. K. Lahiri, Pratulananda Das, *Further remarks on I-limit superior and I-limit inferior*, Mathematical Communications, **8** (2003), 151–156.

[13] B. K. Lahiri, Pratulananda Das, *I and I^*-convergence in topological spaces*, Math. Bohemica, **130(2)** (2005), 153–160.

[14] M. Mačaj, T. Šalát, *Statistical convergence of subsequences of a given sequence*, Math. Bohemica, **126** (2001), 191–208.

[15] D. S. Mitrinovic, J. Sandor, B. Crstici, *Handbook of Number Theory*, Kluwer Acad. Publ., Dordrecht - Boston - London, 1996.

[16] I. Niven, H. S. Zuckerman, *An Introduction to the Theory of Numbers*, 4th Ed., John Wiley, New York, 1980.

[17] S. Pehlivan, A. Guncan, M. A. Mamedov, *Statistical cluster points of sequences in finite dimensional spaces*, Czechoslovak Math. J., **54** (129) (2004), 95–102.

[18] T. Šalát, *On statistically convergent sequences of real numbers*, Math. Slovaca, **30** (1980), 139–150.

[19] T Šalát, B. C. Tripathy, M. Ziman, *A note on I - convergence field*, Italian J. Pure Appl. Math., **17** (2005), 45–54.

[20] I. J. Schoenberg, *The integrability of certain functions and related summability methods*, Amer. Math. Monthly, **66** (1959), 361–375.