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ON FUNCTIONS TAKING THE SAME VALUE ON MANY PAIRS OF POINTS

Abstract

Let $0 < p < 1$, $p \neq 1/2$, and let $I \subset \mathbb{R}$ be an interval. We say that the function $f : I \to \mathbb{R}$ has property $(\mathrm{M_p})$ if, whenever $x, y \in I$ and $f(x) \neq f(y)$, then $f\big(px + (1-p)y\big) = f\big((1-p)x + py\big)$. We prove that

(i) if f has property $(\mathrm{M_p})$ and has a point of continuity in I, then f is constant apart from a countable set;

(ii) if f is measurable and has property $(\mathrm{M_p})$, then f is constant a.e.

As a corollary we obtain that if f is a derivative and has property $(\mathrm{M_p})$, then f is constant. Then we apply this result to solve a functional equation that appears in a variant of the Matkowski-Sutô problem.

1 Introduction.

Our starting point is the functional equation

$$f\big(px + (1-p)y\big)\,[g(y) - g(x)] = \mu\,[f(x)g(y) - f(y)g(x)] \qquad (x, y \in I), \quad (1)$$

where f and g are real valued functions defined on the nonempty interval I, and the real numbers $0 < p < 1$ and $\mu \neq 0$ are given. This equation appears in

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the investigations of the Matkowski-Sutô problem [4, 5, 7, 9]. Now, changing the roles of x and y in (1) we get

$$f((1-p)x+py)\,[g(x)-g(y)] = \mu\,[f(y)g(x)-f(x)g(y)] \tag{2}$$

for every $x, y \in I$. Taking the sum of (1) and (2) we obtain

$$\bigl[f\bigl(px+(1-p)y\bigr) - f\bigl((1-p)x+py\bigr)\bigr]\,[g(y)-g(x)] = 0 \qquad (x, y \in I). \tag{3}$$

If $g(x) = g(y) \neq 0$, then (1) gives $f(x) = f(y)$. On the other hand, if $g(x) \neq g(y)$, then (3) implies $f\bigl(px+(1-p)y\bigr) = f\bigl((1-p)x+py\bigr)$. Therefore, assuming $g \neq 0$, it follows from (1) that the function f has the following property labeled by $(\mathrm{M_p})$:

if $x, y \in I$ and $f(x) \neq f(y)$, then $f(px+(1-p)y) = f((1-p)x+py)$.

In this paper we shall prove that if a function f has property $(\mathrm{M_p})$ with a $p \in (0,1)$, $p \neq 1/2$, and if f satisfies some regularity properties as well, then f has to be constant apart from a small set. Then we shall apply these results to the functional equation (1).

In the sequel by the term interval we shall always mean a nonempty open subinterval of $\mathbb{R}$. The Lebesgue measure on $\mathbb{R}$ will be denoted by λ.

2 Some Preliminary Results.

Our first result will be used in the investigation of measurable functions having property $(\mathrm{M_p})$, but it may have independent interest.

Lemma 1. *Let $I \subset \mathbb{R}$ be an interval, and let $A_1, A_2, \ldots, A_n$ $(n \in \mathbb{N})$ be measurable subsets of I having positive measure in each subinterval of I.*

Let $G \subset \mathbb{R}^2$ be an open set, and let the functions $f_i : G \to \mathbb{R}$ $(i = 1, 2, \ldots, n)$ be continuously differentiable. Suppose there is a point $(x_0, y_0) \in G$ such that

(i) *$f_i(x_0, y_0) \in I$ for every $i = 1, \ldots, n$,*

(ii) *the gradient vectors $v_i = \nabla f_i(x_0, y_0)$ $(i = 1, \ldots, n)$ are nonzero, and*

(iii) *the unit vectors $\pm v_i/|v_i|$ $(i = 1, \ldots, n)$ are pairwise different.*

Then there exists a point $(x, y) \in G$ such that $f_i(x, y) \in A_i$ for every $i = 1, 2, \ldots, n$.

The proof of Lemma 1 is based on the following result, which is well-known; see [8, 3.5. Corollary] for a substantial generalization. However, in order to make this paper self-contained, and since the special case we need is easily obtained, we provide a simple proof.

Lemma 2. *Let the functions $g_1, \dots, g_k$ and the partial derivatives $\frac{\partial}{\partial x} g_i$ $(i = 1, \dots, k)$ be defined and continuous in a neighbourhood of the point $(a, b) \in \mathbb{R}^2$, and suppose that $\frac{\partial}{\partial x} g_i(a, b) \neq 0$ for every $i = 1, \dots, k$. Let $A_1, \dots, A_k$ be measurable subsets of $\mathbb{R}$, and let $c_i = g_i(a, b)$ be a density point of A_i for every $i = 1, \dots, k$. Then there is a $\delta > 0$ such that the set*

$$G(y) = \{x : g_i(x, y) \in A_i \ (i = 1, \dots, k)\}$$

is nonempty for every $y \in (b - \delta, b + \delta)$.

Proof. Replacing the function $g_i(x, y)$ by $g_i(x+a, y+b) - c_i$ and the set A_i by $A_i - c_i$, we may assume that $a = b = 0$ and $c_i = 0$ for every $i = 1, \dots, k$. Then, multiplying the functions g_i and the sets A_i by suitable nonzero constants, we may also assume that $\frac{\partial}{\partial x} g_i(a, b) = 1$ for every $i = 1, \dots, k$. Put $\varepsilon = 1/(6k)$. Since 0 is a density point of A_i, we may choose an $h_0 > 0$ such that $\lambda([0, h] \cap A_i) > (1 - \varepsilon)h$ for every $0 < h < h_0$ and $i = 1, \dots, k$. By the continuity of the partial derivatives $\frac{\partial}{\partial x} g_i$, there exists a number $0 < h < h_0$ such that $1 - \varepsilon < \frac{\partial}{\partial x} g_i(x, y) < 1 + \varepsilon$ for every $x, y \in (-h, h)$ and $i = 1, \dots, k$. Then, by the continuity of the functions g_i, we can find a number $0 < \delta < h$ such that $|g_i(0, y)| < \varepsilon h$ for every $y \in (-\delta, \delta)$ and $i = 1, \dots, k$.

Let $y \in (-\delta, \delta)$ be fixed, and put $B_i = \{x \in [0, h] : g_i(x, y) \notin A_i\}$ $(i = 1, \dots, k)$. Since $\frac{\partial}{\partial x} g_i(x, y) > 1 - \varepsilon$ for every $x \in [0, h]$, the function g_i^y defined by $g_i^y(x) = g_i(x, y)$ is strictly increasing in $[0, h]$, and $\lambda\left(g_i^y(B_i)\right) \geq (1 - \varepsilon) \cdot \lambda(B_i)$. On the other hand, $\frac{\partial}{\partial x} g_i(x, y) < 1 + \varepsilon$ implies $g_i^y(h) - g_i^y(0) \leq (1 + \varepsilon)h$, and thus we have $g_i^y([0, h]) \subset [-\varepsilon h, (1 + 2\varepsilon)h]$ and

$$g_i^y(B_i) \subset [-\varepsilon h, 0] \cup ([0, h] \setminus A_i) \cup [h, (1 + 2\varepsilon)h].$$

Therefore, $\lambda\left(g_i^y(B_i)\right) \leq 4\varepsilon h$, $(1 - \varepsilon) \cdot \lambda(B_i) \leq 4\varepsilon h$ and $\lambda(B_i) \leq 4\varepsilon h/(1 - \varepsilon) < h/k$ for every $i = 1, \dots, k$. Consequently, $\bigcup_{i=1}^k B_i$ cannot cover $[0, h]$. If $x \in [0, h] \setminus \bigcup_{i=1}^k B_i$, then $g_i(x, y) \in A_i$ for every $i = 1, \dots, k$; that is, the set $G(y)$ is nonempty. Since this is true for every $y \in (-h, h)$, the lemma is proved. $\square$

Proof of Lemma 1. We may assume that each of the sets A_i is d-open; that is, every point of A_i is a density point of A_i. Indeed, otherwise we replace A_i by $A_i \setminus N_i$, where N_i is the null set consisting of the non-density points of A_i.

We prove the statement of the lemma by induction on n. As $\nabla f_1(x_0,\, y_0) \neq 0$, the range of f_1 contains a neighbourhood of the point $f_1(x_0,\, y_0) \in I$. As the set A_1 is dense in I, we have $f_1(x,\, y) \in A_1$ for a suitable $(x,\, y)$. This proves the statement for $n = 1$.

Let $n > 1$, and suppose the statement is true for $n-1$. Let A_i, f_i $(i = 1, \dots, n)$ and $(x_0,\, y_0)$ be as in the lemma. Since the unit vectors $\pm v_i/|v_i|$ are different, $\dfrac{\partial}{\partial x}\, f_i(x_0,\, y_0) \neq 0$ must hold for at least one i. We may assume that $\dfrac{\partial}{\partial x}\, f_n(x_0,\, y_0) \neq 0$.

Let $f_n(x_0,\, y_0) = z_0 \in I$. By the implicit function theorem, there is a neighbourhood U of $(y_0,\, z_0)$ and there is a continuously differentiable function $h : U \to \mathbb{R}$ such that $h(y_0,\, z_0) = x_0$, and $f_n\big(h(y,\, z),\, y\big) = z$ for every $(y,\, z) \in U$. Consider the functions $g_i(y,\, z) = f_i\big(h(y,\, z),\, y\big)$ $(i = 1, \dots, n)$ defined in U. It is easy to check that the gradient vectors $w_i = \nabla g_i(y_0,\, z_0)$ $(i = 1, \dots, n)$ are nonzero, and the unit vectors $\pm w_i/|w_i|$ $(i = 1, \dots, n)$ are different.

Since $g_n(y, z) = z$, we have $w_n = \nabla g_n(y_0, z_0) = (0, 1)$. Therefore, the first coordinate $\dfrac{\partial}{\partial y} g_i(y_0, z_0)$ of $w_i = \nabla g_i(y_0, z_0)$ is nonzero for every $i = 1, \dots, n-1$. Let

$$V = \{(y, z) \in U : \frac{\partial}{\partial y} g_i(y, z) \neq 0 \ (i = 1, \dots, n-1)\};$$

then V is a nonempty open subset of U. By the induction hypothesis, we can find a point $(y_1, z_1) \in V$ such that $g_i(y_1, z_1) \in A_i$ for every $i = 1, \dots, n-1$.

Let $G_i(z) = \{y : g_i(y, z) \in A_i\}$ $(i = 1, \dots, n-1)$, and put $G(z) = \bigcap_{i=1}^{n-1} G_i(z)$. By Lemma 2, the set $G(z)$ is nonempty in a neighbourhood of the point z_1. Now A_n is dense in I, and thus we may choose a point $z_2 \in A_n$ such that $G(z_2) \neq \emptyset$. Let $y_2 \in G(z_2)$ and $x_2 = h(y_2,\, z_2)$. Then $f_i(x_2,\, y_2) = g_i(y_2,\, z_2) \in A_i$ for every $i = 1, \dots, n-1$, and $f_n(x_2,\, y_2) = z_2 \in A_n$. This completes the proof. $\square$

Let $\mathcal{A}$ be an ideal of subsets of $\mathbb{R}$. We say that $\mathcal{A}$ is *admissible*, if satisfies the following conditions:

(i) $\mathcal{A}$ contains the finite sets.

(ii) If $H \in \mathcal{A}$, then $aH + b = \{ax + b : x \in H\} \in \mathcal{A}$ for every $a,\, b \in \mathbb{R},\ a \neq 0$.

(iii) Whenever $I_1 \subset I_2 \subset \dots$ is an increasing sequence of intervals, $H \subset \mathbb{R}$, and $H \cap I_k \in \mathcal{A}$ for every k, then $H \cap \bigcup_{k=1}^{\infty} I_k \in \mathcal{A}$.

It is clear that the ideals of all countable sets, all null sets or all sets of first category are admissible. It is also easy to check that the ideal of all scattered sets is admissible. (A set H is scattered if every nonempty subset of H has an isolated point.) This shows that an admissible ideal need not be a σ-ideal.

Lemma 3. *Let I be an interval, and suppose that $f : I \to \mathbb{R}$ has property* $(\mathrm{M_p})$, *where $0 < p < 1$ and $p \neq 1/2$. Let $\mathcal{A}$ be an admissible ideal. If $J \subset I$ is a subinterval, $K \subset \mathbb{R}$ and $f(x) \in K$ for $\mathcal{A}$-a.e. $x \in J$, then $f(x) \in K$ for $\mathcal{A}$-a.e. $x \in I$.*

Proof. We may assume that $0 < p < 1/2$. Let $a \in J$ be fixed, let

$$B = \{x \in I : x > a,\ \{\, y \in [a, x] : f(y) \notin K \,\} \in \mathcal{A}\},$$

and put $b = \sup B$. Since $J \cap (a, \infty) \subset B$, we have $a < b$. We prove that $b = \sup I$.

Suppose $b < \sup I$. For every $x < b$ there is a unique y such that $px + (1 - p)y = b$. Clearly, $y = \dfrac{b - px}{1 - p}$. Then we have $y > b$, and also $y \in I$, assuming that $x \in (b - \delta, b)$, if δ is small enough. If x runs through the interval $(b - \delta, b)$, then y runs through a right hand side neighbourhood of the point b. Then it follows from the definition of b that there exists a point $x_0 \in (b - \delta, b)$ such that $f(y_0) \notin K$, where $y_0 = \dfrac{b - px_0}{1 - p}$.

By assumption, $f(x) \in K$ for $\mathcal{A}$-a.e. $x \in (x_0, b)$. For every such x we have $f(x) \neq f(y_0)$, and thus, by property $(\mathrm{M_p})$ we obtain

$$f((1 - p)x + py_0) = f(px + (1 - p)y_0). \tag{4}$$

If $\eta > 0$ is small enough, and x runs through the interval $(x_0, x_0 + \eta)$, then the point $(1 - p)x + py_0$ runs through a subinterval of (x_0, b), since

$$x_0 < (1 - p)x_0 + py_0 < px_0 + (1 - p)y_0 = b.$$

For $\mathcal{A}$-a.e. $x \in (x_0, b)$ we have $f(x) \in K$, and thus, by (4), for $\mathcal{A}$-a.e. $x \in (x_0, x_0 + \eta)$ we have $f((1 - p)x + py_0) = f(px + (1 - p)y_0) \in K$. Since $px + (1 - p)y_0$ runs through a right hand side neighbourhood of b, this means that in a suitable right hand side neighbourhood of b the values of f belong to K, apart from a set belonging to $\mathcal{A}$ (cf. property (ii)). This, however, contradicts the definition of b, which proves that $b = \sup I$. Therefore, we have $f(x) \in K$ for $\mathcal{A}$-a.e. $x \in [a, \sup I)$. A similar argument proves that for $\mathcal{A}$-a.e. $x \in (\inf I, a]$ we have $f(x) \in K$. $\square$

3 $(\mathrm{M_p})$ Functions with Regularity Properties.

Theorem 4. *Let* $0 < p < 1,\ p \neq 1/2$, *and suppose that the function* $f : I \to \mathbb{R}$ *has property* $(\mathrm{M_p})$ *in the interval* I. *If* f *has at least one point of continuity, then* f *is constant in* I, *apart from a countable set.*

Proof. Let f be continuous at the point $a \in I$. Then, for every positive integer n there is a subinterval $J_n \subset I$ such that $f(J_n) \subset K_n = (f(a) - 1/n, f(a) + 1/n)$. Applying Lemma 3 to the ideal of countable sets we obtain that $f(x) \in K_n$ on I except at the points of a countable set. This implies that $f(x) \in \bigcap_{n=1}^{\infty} K_n = \{f(a)\}$ for every $x \in I$ except at the points of a countable set. □

Corollary 5. *Let* $0 < p < 1,\ p \neq 1/2$, *and suppose that the function* $f : I \to \mathbb{R}$ *has property* $(\mathrm{M_p})$ *in the interval* I. *If* f *has at least one point of continuity and has the Darboux property, then* f *is constant.*

Proof. By Theorem 4, f is constant in I, apart from a countable set. If f is not constant and Darboux, then the range of f contains an interval, which is impossible. □

Corollary 6. *Let* $0 < p < 1,\ p \neq 1/2$, *and suppose that the function* $f : I \to \mathbb{R}$ *has property* $(\mathrm{M_p})$ *in the interval* I. *If* f *is a derivative, then* f *is constant.*

Proof. Every derivative is Darboux and Baire 1, hence continuous at the points of a dense set. □

Theorem 7. *Let* $0 < p < 1,\ p \neq 1/2$, *and suppose that the function* $f : I \to \mathbb{R}$ *has property* $(\mathrm{M_p})$ *in the interval* I. *If* f *is measurable, then* f *is a.e. constant in* I.

Proof. Suppose f is not constant a.e. in I. Then there is a real number a such that both of the sets $B = \{x \in I : f(x) < a\}$ and $C = \{x \in I : f(x) \geq a\}$ have positive measure. We shall distinguish between two cases.

I. Suppose that both of B and C are of positive measure in every subinterval of I. Then put $A_1 = B,\ A_2 = C,\ A_3 = B,\ A_4 = C$, and $f_1(x, y) = x,\ f_2(x, y) = px + (1 - p)y,\ f_3(x, y) = (1 - p)x + py,\ f_4(x, y) = y$ for every $(x, y) \in I^2$. Since the gradient vectors $v_1 = (1, 0)$, $v_2 = (p, 1-p)$, $v_3 = (1-p, p)$ and $v_4 = (0, 1)$ are nonzero, and the unit vectors $\pm v_i/|v_i|$ $(i = 1, 2, 3, 4)$ are different, we may apply Lemma 1 to the sets A_i and functions $f_i : G = I^2 \to I$ $(i = 1, 2, 3, 4)$. We find a point $(x, y) \in I^2$ such that $x \in B,\ px + (1-p)y \in C$, $(1 - p)x + py \in B$, and $y \in C$. Then $f(x) < a < f(y)$. By property $(\mathrm{M_p})$, we have $f(px+(1-p)y) = f((1-p)x+py)$, which contradicts $px+(1-p)y \in C$ and $(1-p)x+py \in B$.

II. Now suppose that there is an interval $J \subset I$ such that at least one of the sets $B \cap J$ and $C \cap J$ is of measure zero. Then we put

$$g(x) = \begin{cases} 0 & \text{if } x \in B, \\ 1 & \text{if } x \in C. \end{cases}$$

It is clear that the function g has property ($\mathrm{M_p}$) in I. Since g is constant a.e. in J, it follows from Lemma 3 when applied to the ideal of null sets, that g is a.e. constant in I. This, however, contradicts the condition that B and C both have positive measure. □

Remark 8. In possession of the previous theorem we can give a second proof of Corollary 6.

Proof. Suppose $f : I \to \mathbb{R}$ is a derivative and has property ($\mathrm{M_p}$). Then, by Theorem 7, f is a.e. constant in I. But then f has to be constant in I. Indeed, suppose that $f(x) = b$ a.e. in I, and there is point $x_0 \in I$ such that $f(x_0) \neq b$. Let J be an interval such that $f(x_0) \in J$ and $b \notin J$. Then $f^{-1}(J)$ is nonempty and thus, by the Denjoy-Clarkson property of derivatives [1, 3, 6], $f^{-1}(J)$ is of positive measure. This, however, contradicts the fact that $f(x) = b$ a.e. □

4 Application to the Equation (1).

In one of the variants of the Matkowski-Sutô problem [4, 5] the unknown functions $f, g : I \to \mathbb{R}$ satisfy equation (1) and, in addition, have the representation $f = \phi' \circ \phi^{-1}$ and $g = \psi' \circ \phi^{-1}$, where ϕ and ψ are differentiable on an interval J, ϕ maps J onto I, and $\phi'(x) \cdot \psi'(x) \neq 0$ for every $x \in J$. It is clear that in this case $1/f$ is the derivative of the function ϕ^{-1}. Thus the following theorem gives the complete solution of (1) under this extra condition.

Theorem 9. *Let $0 < p < 1$, $p \neq 1/2$, and $\mu \neq 0, 1$. Let I be an interval, and suppose that the functions $f, g : I \to \mathbb{R} \setminus \{0\}$ satisfy the functional equation (1). If $1/f$ is a derivative on I, then f and g are both constants on I.*

Proof. As we saw in the introduction, f satisfies condition ($\mathrm{M_p}$). Then, obviously, $1/f$ also satisfies ($\mathrm{M_p}$). Since $1/f$ is a derivative, it follows from Corollary 6 that $1/f$ is a (nonzero) constant, and thus f is a nonzero constant as well. Then (1) gives $g(y) - g(x) = \mu(g(y) - g(x))$ for every $x, y \in I$. By $\mu \neq 1$ this implies $g(y) - g(x) = 0$ for every $x, y \in I$, and thus g is also constant. □

5 Further Results on Functions with the $(\mathrm{M_p})$ Property.

Let K be a proper subfield of $\mathbb{R}$, and let

$$f(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \notin K. \end{cases} \tag{5}$$

Then the function f has property $(\mathrm{M_p})$ on $\mathbb{R}$ for every $p \in K \cap (0,1)$. Indeed, if $x, y \in I$ and $f(x) \neq f(y)$, then $x \in K$ and $y \notin K$ or $x \notin K$ and $y \in K$. Thus none of the numbers $px + (1-p)y$ and $(1-p)x + py$ belongs to K, and hence $f(px + (1-p)y) = 0 = f\big((1-p)x + py\big)$.

Now, one can prove that $\mathbb{R}$ has nonmeasurable subfields[1]. If K is a nonmeasurable subfield of $\mathbb{R}$, then (5) defines a nonmeasurable function with property $(\mathrm{M_p})$ for every $p \in \mathbb{Q} \cap (0,1)$. One can also prove that there is a subfield $K \subset \mathbb{R}$ of measure zero and of the cardinality of continuum[2]. Then (5) shows that there exists a function f having the $(\mathrm{M_p})$ property for every $p \in \mathbb{Q} \cap (0,1)$ such that $f = 0$ a.e., but $f \neq 0$ on a set of cardinality of continuum.

Next we show that there exist nonconstant functions with the Darboux property and having the $(\mathrm{M_p})$ property for every $p \in \mathbb{Q} \cap (0,1)$. This is an immediate consequence of the following result.

Theorem 10. *There exists a function $f : \mathbb{R} \to \mathbb{R}$ such that*

(i) *for every $x, y \in \mathbb{R}$, if $f(x) \neq f(y)$, then f takes the same value on the numbers $px + qy$ for every $p, q \in \mathbb{Q} \setminus \{0\}$, and*

(ii) *f takes every value in every interval.*

Proof. Let H be a Hamel base of $\mathbb{R}$, and let $\{h_\alpha : \alpha < \kappa\}$ be a well-ordering of H, where $h_0 = 1$ and κ is the initial ordinal of the continuum. Let L_α denote the linear space over $\mathbb{Q}$ generated by the elements $\{h_\beta : \beta \leq \alpha\}$. (That is, L_α is the set of all linear combinations of these elements with rational coefficients.) Clearly, for every $x \in \mathbb{R}$ there is a minimal ordinal $\phi(x)$ such that $1 \leq \phi(x) < \kappa$ and $x \in L_{\phi(x)}$. We define $\phi(0) = 1$. It is easy to see that if $\phi(x) < \phi(y)$, then $\phi(px + qy) = \phi(y)$ for every $p,\, q \in \mathbb{Q} \setminus \{0\}$. Now L_α is

[1] This is probably a piece of folklore; it was also posed as Problem 1 of the 2007 Miklós Schweitzer Memorial Competition in Mathematics [11]. There are many ways to prove the statement. Take, for example, any maximal subfield of $\mathbb{R}$ not containing a fixed irrational number.

[2] See also [11]. A possible construction is the following. By a theorem of J. von Neumann [10], there is a nonempty perfect set $P \subset \mathbb{R}$ such that the elements of P are algebraically independent over $\mathbb{Q}$. If P' is a nonempty proper perfect subset of P, then the field generated by P' is a proper measurable (in fact, F_σ) subfield of $\mathbb{R}$, and then it has to be of measure zero.

periodic modulo every nonzero rational number, and thus the function ϕ takes every ordinal $1 \leq \alpha < \kappa$ in every interval. Therefore, if ψ is a bijection from $\{\alpha : 1 \leq \alpha < \kappa\}$ onto $\mathbb{R}$, then the function $f = \psi \circ \phi$ satisfies (i) and (ii). $\square$

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