RESEARCH

Real Analysis Exchange
Vol. 33(2), 2007/2008, pp. 317–322

Maciej Malicki, Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green St., IL 61801.
email: mamalicki@gmail.com

POLISHABLE SUBSPACES OF INFINITE-DIMENSIONAL SEPARABLE BANACH SPACES

Abstract

We show that there exist Polishable subspaces of arbitrarily high Borel class in every infinite-dimensional separable Banach space.

1 Introduction.

In this paper we study Polishable subspaces of infinite-dimensional separable Banach spaces, analogous to Polishable subgroups of Polish groups (that is, topological groups whose topology is separable and completely metrizable). It is known [1] that there exist Polishable subgroups of arbitrarily high Borel class in every non-discrete abelian Polish group. A natural question arises whether the same is true of infinite-dimensional separable Banach spaces. We answer this question in the positive by constructing a family of Polishable subspaces in l_1, in a manner similar to the one presented in [4]. The general statement follows from the fact that l_1 can be continuously embedded in every Banach space.

2 Some Background, Notation and Definitions.

All Banach spaces considered in this paper are assumed to be separable. By a linear subspace of a Banach space X we mean not only closed subspaces but all subsets of X closed under addition and scalar multiplication. A linear

Key Words: separable Banach spaces, Polishable subspaces
Mathematical Reviews subject classification: 46B99, 03E15, 22A99
Received by the editors January 26, 2007
Communicated by: Udayan B. Darji

subspace Y of a Banach space X is called Polishable if there exists a Polish topology on Y making it into a Banach space and whose Borel subsets coincide with all intersections of Borel subsets of X with Y. Equivalently, Y is Polishable if there exists a one-to-one continuous linear mapping from some Banach space Y' onto Y. This implies, by the Lusin-Souslin theorem [2, p.89], that Polishable subspaces are always Borel.

An important fact about Polish topology of a Polishable subspace is that it is unique, which essentially follows from Pettis theorem, [2, p.61]. See [3], [5], [6] for more information about the notion of Polishability.

Borel subsets of a Polish space X are those obtained from open subsets of X by the operations of complementation and countable union. We use the following standard notation (see [2]) for the hierarchy of Borel sets: $\mathbf{\Sigma}^0_1 =$ open, $\mathbf{\Pi}^0_1 =$ closed,

$$\mathbf{\Sigma}^0_\alpha = \big\{ \bigcup_{n\in\mathbb{N}} A_n : A_n \text{ is in } \mathbf{\Pi}^0_{\alpha_n} \text{ for } \alpha_n < \alpha \big\},$$

and $\mathbf{\Pi}^0_\alpha =$ the complements of $\mathbf{\Sigma}^0_\alpha$, where $1 < \alpha < \omega_1$. Even though this notation does not make it explicit where the Borel sets in question originate, it will not cause any confusion as X will always be clear from the context.

A mapping f from a Polish space X into a Polish space Y is said to be of Baire class 1 if $f^{-1}(U)$ is in $\mathbf{\Sigma}^0_2$ for every open $U \subseteq Y$. Then f is of Baire class α, $\alpha < \omega_1$, if it is the pointwise limit of a sequence of mappings f_n, where all f_n are of Baire class smaller than α. We say that f is strictly of Baire class α if it is of Baire class α and not of Baire class γ for any $\gamma < \alpha$.

A classical theorem of Lebesgue, Hausdorff and Banach ([2, p.190]) says that f is of Baire class α if and only if the pullbacks of all open sets are in $\mathbf{\Sigma}^0_{\alpha+1}$.

For a countable family $(A_m, \|\cdot\|_m)$ of Banach spaces, let the direct sum of A_m, with norm $\|(x_m)\|_\Sigma = \sum_m \|(x_m)\|_m$, be $\sum A_m$. Finally, $\mathbb{N}$ and $\mathbb{Q}$ stand for the natural and rational numbers, respectively.

3 Main Result.

Lemma 1. *Assume that $(A_m, \|\cdot\|_m)$ are infinite-dimensional separable Banach spaces and $g_m : A_m \to \mathbb{R}$ are continuous linear functionals, for $m \in \mathbb{N}$. Then the space*

$$B(A_m) = \{(x_m) \in \sum A_m : \lim_m g_m(x_m) \textit{ exists}\}$$

endowed with the norm $\|x\|_B = \|x\|_\Sigma + \sup_m |g_m(x_m)|$ *is a separable Banach space. Furthermore, the linear functional*

$$g(x) = \lim_m g_m(x_m)$$

is continuous on $(B(A_m), \|\cdot\|_B)$.

PROOF. Checking that $\|\cdot\|_B$ is a norm is straightforward.

First we show that $B(A_m)$ is separable. Fix countable dense sets $D_m \subseteq A_m$, and, for every $p \in \mathbb{Q}$, $n \in \mathbb{N}$, let $(d_m^{p,n})$ be an element of $B(A_m)$ such that

$$| \sup_m g_m(d_m^{p,n}) - p | < 1/n$$

and

$$\|(d_m^{p,n})\|_\Sigma < 1/n,$$

provided that one exists. Otherwise, set $(d_m^{p,n}) = 0$.

We show that the countable set E consisting of the elements of $B(A_m)$ of the form $(e_m^{p,n,M})$ is dense in $B(A_m)$, where

$$e_m^{p,n,M} = \begin{cases} d & \text{for some } d \in D_m \text{ if } m < M \\ d_m^{p,n} & \text{otherwise,} \end{cases}$$

n, M range over $\mathbb{N}$, and p ranges over $\mathbb{Q}$.

Fix $(x_m) \in B(A_m)$ and $n > 0$. Then there exists $p \in \mathbb{Q}$ and a natural M such that, for all $m \geq M$, we have the following:

(i) $\sum_{m=M}^{\infty} \|x_m\|_m < 1/n$;

(ii) $|g_m(x_m) - p| < 1/n$.

By continuity of the mappings g_m, we can pick $d_m \in D_m$ such that, for every $m < M$,

$$\sum_{m=1}^{M} \|x_m - d_m\|_m < 1/n,$$

and $|g_m(x_m) - g_m(d_m)| < 1/n$.

Then it is easy to check that $\left\|(x_m) - (e_m^{p,n,M})\right\|_B < 3/n$.

Now we show that $B(A_m)$ is complete with respect to $\|\cdot\|_B$. Let $\{x^n\}$ be a $\|\cdot\|_B$-Cauchy sequence. Then $\{f_n\}$ defined by $f_n = \lim_m g_m(x_m^n)$ is also Cauchy, so it converges to some f. Since $\{x^n\}$ is Cauchy in the norm $\|\cdot\|_\Sigma$, it converges in this norm to some $x = (x_m)$ (possibly not in $B(A_m)$). We show

that actually $\lim_m g_m(x_m) = f$, that is $x \in B(A_m)$, and (x^n) converges to x in $\|\cdot\|_B$. Suppose that this does not hold. Then there exist $\epsilon > 0$ and infinitely many $m \in \mathbb{N}$ such that
$$|g_m(x_m) - f| > \epsilon. \tag{1}$$
Now, fix a natural N such that for all $n, m > N$ we have that $|g_m(x_m^n) - f| < \epsilon/2$. It is possible to find such an N by the definition of $\|\cdot\|_B$ and convergence of $\{f_n\}$. Because x^n converge to x in $\|\cdot\|_\Sigma$, for any fixed m_0 there is N_0 such that for any $n > N_0$
$$\left|g_{m_0}(x_{m_0}^n) - g_{m_0}(x_{m_0})\right| < \epsilon/2.$$
But this means that $|g_m(x_m) - f| \leq \epsilon$ for all $m > M$, which contradicts (1). □

Remark. If A_m are linear subspaces of l_1, then $B(A_m)$ can be identified with a linear subspace of l_1 via a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$. Note that in this case $\|\cdot\|_B$ is stronger than the l_1-norm, provided that the norms $\|\cdot\|_m$ are stronger than the l_1-norm. Thus $id : (B(A_m), \|\cdot\|_B) \to l_1$ is a continuous mapping.

Now we are ready to prove the following:

Theorem 2. *For every $\alpha < \omega_1$ there exist a subspace B_α of l_1, a norm $\|\cdot\|_\alpha$ on B_α and a mapping g_α, with the following properties holding for every $1 \leq \alpha \leq \omega_1$:*

1. *$\|\cdot\|_\alpha$ is stronger than the l_1-norm and makes $(B_\alpha, \|\cdot\|_\alpha)$ into a separable Banach space;*
2. *$g_\alpha : (B_\alpha, \|\cdot\|_\alpha) \to \mathbb{R}$ is linear and continuous;*
3. *Denote by id_α the continuous identity embedding of $(B_\alpha, \|\cdot\|_\alpha)$ into l_1. For every mapping $\phi : X \to [0,1]$ of Baire class α, where X is a Polish space, there exists a continuous $\psi : X \to l_1$ such that $\psi(X) \subseteq B_\alpha$ and $\phi = g_\alpha \circ id_\alpha^{-1} \circ \psi$;*
4. *$B_\alpha \in \mathbf{\Sigma}^0_{\alpha+1} \setminus \mathbf{\Pi}^0_{\alpha-1}$ for all ordinals $2 < \alpha < \omega_1$ of the form $\alpha = \alpha' + 3$.*

PROOF. We proceed by induction on α. For $\alpha = 0$ let $B_0 = \mathbb{R}$, $g_0(x) = x$, $\|x\|_0 = |x|$.

At each successor step apply Lemma 1 to $A_m = B_\alpha$, $g_m = 2^m g_\alpha$, to get $B_{\alpha+1} = B(A_m)$, $\|\cdot\|_{\alpha+1} = \|\cdot\|_B$, and $g_{\alpha+1} = g$. By the above Remark we can assume that $B_{\alpha+1}$ is a subspace of l_1 and $\|\cdot\|_{\alpha+1}$ is stronger than the l_1-norm.

We need to verify Points (3) and (4) of the theorem. Let $\phi : X \to [0,1]$ be a mapping of Baire class $\alpha+1$. Then there are $\phi_m : X \to [0,1]$ of Baire class α and continuous $\psi'_m : X \to B_\alpha$ such that $\phi(x) = \lim_m \phi_m(x)$, where $\phi_m = g_\alpha \circ id_\alpha^{-1} \circ \psi'_m$.

Let $\psi = (2^{-m}\psi'_m)$. Clearly, ψ is continuous and we have the following:

$$\begin{aligned} g_{\alpha+1} \circ \psi(x) &= \lim_m g_m \circ \psi_m(x) = \lim_m 2^m g_\alpha \circ 2^{-m}\psi'_m(x) \\ &= \lim_m g_\alpha \circ \psi'_m(x) = \lim_m \phi_m(x) = \phi(x), \end{aligned}$$

which proves (3).

The fact that $B_{\alpha+1} \in \mathbf{\Sigma}^0_{\alpha+2}$ is a consequence of $id_{\alpha+1}^{-1}$ being of Baire class $\alpha+1$. This is clear if we write $id_{\alpha+1}^{-1}$ in the form $id_{\alpha+1}^{-1}(x) = \lim_m i_m(x)$, where $i_m(x) = (x_0, \ldots, x_m, 0, 0, 0, \ldots)$ and all i_m are of Baire class α by the induction hypothesis.

If in addition $\alpha = \alpha' + 1$, we also have that $B_{\alpha+1} \notin \mathbf{\Pi}^0_\alpha$. This results from Theorem 3.1(i) of [6], along with some of the claims from its proof. We will state them and leave to the reader easy computations that lead to the desired result. The main point is that 3) implies that $id_{\alpha+1}^{-1}$ is strictly of Baire class $\alpha+1$, since there exist mappings $\psi : X \to [0,1]$ that are strictly of Baire class $\alpha+1$. Thus there is a set $U \subseteq B_{\alpha+1}$ which is open in the Polish topology on $B_{\alpha+1}$ defined by $\|\cdot\|_{\alpha+1}$ and $U \notin \mathbf{\Sigma}^0_{\alpha+1}$, so in particular $U \notin \mathbf{\Pi}^0_\alpha$.

Below, for a Polish group G and a Polishable subgroup $H \leq G$, $\mathrm{bor}(H,G)$ is defined by letting $\mathrm{bor}(H,G) = \min\{\gamma < \omega_1 : H \text{ is a } \mathbf{\Pi}^0_\gamma \text{ subset of } G\}$ while $\mathrm{pol}(H,G)$ is a rank value of H as a Polishable group. Its precise definition is not necessary for the present purposes and actually would only obscure our point (see [6] for details).

Theorem 3. *Let G be a Polish group and H be a Polishable subgroup of G.*

(i) If $\mathrm{pol}(H,G)$ is a successor, then $\mathrm{bor}(H,G) = 1 + \mathrm{pol}(H,G) + 1$;

(ii) If $\mathrm{pol}(H,G)$ is 0 or limit, then $\mathrm{bor}(H,G) = 1 + \mathrm{pol}(H,G)$.

In the proof of Theorem 3 the authors show that if $\xi = \mathrm{pol}(H,G)$ is a successor, then $\tau \subseteq \mathbf{\Sigma}^0_{1+\xi} \mid H$, where τ denotes the unique Polish topology on H and $\mathbf{\Sigma}^0_{1+\xi} \mid H$ stands for the family of intersections of $\mathbf{\Sigma}^0_{1+\xi}$ subsets of G with H.

Since $B_{\alpha+1}$ can be viewed as a Polishable subgroup of l_1 regarded as a Polish group, the above mentioned results can be applied in the present situation. Assume that $\mathrm{bor}(B_{\alpha+1}, l_1) = \xi'$ is finite, the other cases being similar. If $\xi' \leq \alpha$ $\xi' = \xi + 2$, then by Theorem 3, $\mathrm{pol}(B_{\alpha+1}, l_1)$ is a successor and $\mathrm{pol}(B_{\alpha+1}, l_1) = \xi$. It follows that $\tau \subseteq (\mathbf{\Sigma}^0_{\xi+1} \mid B_{\alpha+1}) \subseteq \mathbf{\Pi}^0_{\xi+2} = \mathbf{\Pi}^0_{\xi'}$, which cannot be true since $U \notin \mathbf{\Pi}^0_\alpha$.

To deal with a limit step, we fix (α_m), a strictly increasing sequence of ordinals converging to α and apply Lemma 1 to $A_m = B(l_{\alpha_m})$, $g_m = 2^m g_{\alpha_m}$. Since every mapping $\phi : X \to [0,1]$ of Baire class α is a limit of functions ϕ_m of Baire class α_m, we are done by the same argument as in the case of a successor step. □

Fix now an arbitrary infinite-dimensional separable Banach space X. It is well known that there exists a continuous linear embedding of l_1 into X, say γ, so that $\gamma \circ id_\alpha$ witness that $\gamma(B_\alpha)$ are Polishable subspaces of X for each $\alpha < \omega_1$. By a standard observation the family of all $\gamma(B_\alpha)$ is unbounded in the hierarchy of Borel subsets of X Therefore we get the following Corollary.

Corollary 4. *Let X be an infinite-dimensional separable Banach space. For every $\alpha < \omega_1$ there exists a Polishable subspace Y of X such that $Y \notin \mathbf{\Pi}^0_\alpha$.*

We would like to finish with two questions. First of all, for a given Banach space X, does there exist a single Banach space X' whose images in X under linear embeddings are unbounded in terms of the Borel hierarchy? Another problem that we find interesting concerns a lower bound of Borel classes of Polishable subspaces of Banach spaces: for a given $\alpha < \omega_1$, is there a Banach space X all of whose nontrivial Polishable subspaces have Borel class higher than α? Note that our construction does not yield such a bound since we do not know in general what the possible Borel classes of copies of l_1 in Banach spaces are.

References

[1] G. Hjorth, *Subgroups of Abelian Polish groups,* preprint.

[2] A. Kechris, *Classical Descriptive Set Theory,* Grad. Texts in Math., **156** (1995), Springer, Berlin.

[3] A. Kechris, *Lectures on definable group actions and equivalence relations,* unpublished notes.

[4] J. Saint-Raymond, *Espaces a modele separable,* Ann. Inst. Fourier (Grenoble), **26** (1976), 211–225.

[5] S. Solecki, *Polish group topologies,* S. Cooper, J. Truss (Eds.), *Sets and Proofs,* London Math. Soc. Lecture Note Ser., **258** (1999), 339–364.

[6] S. Solecki, I. Farah, *Borel subgroups of Polish groups,* Adv. Math., **199** (2006), 499–541.