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Mikhail G. Plotnikov,* Department of Mathematics, Vologda State Milk Industry Academy, Vologda 160555, Russia. email: mgplotnikov@mail.ru

RECOVERY OF THE COEFFICIENTS OF MULTIPLE HAAR AND WALSH SERIES

Abstract

A family of multidimensional generalized Perron type integrals is constructed. It is shown that these integrals solve the problem of recovering, by generalized Fourier formulae, the coefficients of multiple Haar and Walsh series of some class. This class includes in particular series convergent ρ-regularly everywhere except some countable set $E \subset G^d$. It is shown that some properties of rectangularly convergent multiple Haar and Walsh series do not hold for the ρ-regular convergence.

1 Introduction.

The problem of recovering the coefficients of orthogonal series from their sums is the generalization of the uniqueness problem for these series. It is known (see [21]) that the series $\sum_{k=2}^{\infty} \sin kx/\ln k$ converges everywhere but its sum is not Lebesgue integrable and this series fails to be the Fourier-Lebesgue series. Analogous situation takes place for many other orthogonal systems. Therefore the coefficients problem requires integration processes more general than the Lebesgue one.

In our work the problem of recovering the coefficients of multiple Haar and Walsh series is considered. Generalized integrals which solve this problem are defined in terms of the dyadic base of differentiation. The first integral solving the coefficients problem for one-dimensional Haar series was constructed

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in [12]. This integral solves analogous problem also for one-dimensional Walsh series. In [13] a Perron-type integral recovering the coefficients of Haar series with convergent subsequences of partial sums was introduced. A multidimensional generalized integral solving the coefficients problem for rectangular convergent multiple Haar and Walsh series was considered in [14]. In our paper more general ρ-regular rectangular convergence of multiple series is considered. In this case integrals permitting to recover, by Fourier formulae, the coefficients of multiple Haar (but not Walsh) series were introduced in [8, 10]. But the constructions of this integrals are rather complicated. In the present paper we consider a simpler setting where the Walsh and Haar functions are defined on the dyadic group G. We show that in this case the coefficients problem for both multiple Haar and Walsh series can be solved by more simple integrals which are defined here.

2 Preliminaries.

Recall (see [1, 5, 11, 17]) that *the dyadic group* G is a set of sequences $t = \{t_i\}_{i=0}^{\infty}$ where $t_i = 0$ or 1. *Addition* in G is defined as the coordinatewise addition ($mod\, 2$). The mapping $\phi(t) = \sum_{i=0}^{\infty} t_i 2^{-i-1}$ establishes the one-one correspondence between G and the so-called *modified segment* J^*. The modified segment $J^* = [0,1]^*$ can be interpreted as the closed segment $[0,1]$ in which the dyadic rational points are counted twice: the 'left' point $p/2^k - 0$ corresponds to the infinite dyadic expansion and the 'right' point $p/2^k + 0$ corresponds to the finite expansion. The topology in G is defined by the system of neighborhoods $V_k = \{t = \{t_i\} : t_i = a_i,\, i \le k-1\}$. The corresponding neighborhoods in J^* are the segments $[p/2^k + 0, (p+1)/2^k - 0]$. Since G and J^* are isomorphic, we shall identify them.

The *Walsh-Paley functions* on G (see [4, 5, 7, 11]) are defined by

$$\omega_n(t) = \prod_{i=0}^{\infty} (-1)^{t_i \varepsilon_i^{(n)}}$$

where

$$t = \{t_i\} \in G, \qquad n = \sum_{i=0}^{\infty} 2^i \varepsilon_i^{(n)} \ \ (\varepsilon_i^{(n)} \in \{0,1\}).$$

Now we define the *Haar functions* on J^* (see [2, 18]). Put $\chi_0(t) \equiv 1$. If $n = 2^k + i$ ($k = 0, 1, \ldots,\ i = 0, \ldots, 2^k - 1$), we put

$$\chi_n(x) = \begin{cases} 2^{k/2}, & \text{if } x \in \left[\frac{2i}{2^{k+1}} + 0, \frac{2i+1}{2^{k+1}} - 0\right], \\ -2^{k/2}, & \text{if } x \in \left[\frac{2i+1}{2^{k+1}} + 0, \frac{2i+2}{2^{k+1}} - 0\right], \\ 0, & \text{if } x \in [0,1]^* \setminus \left[\frac{2i}{2^{k+1}} + 0, \frac{2i+2}{2^{k+1}} - 0\right]. \end{cases}$$

Fix natural $d \geq 1$. Consider intervals

$$\left[\frac{p_1}{2^{k_1}}+0,\ \frac{p_1+1}{2^{k_1}}-0\right] \times \ldots \times \left[\frac{p_d}{2^{k_d}}+0,\ \frac{p_d+1}{2^{k_d}}-0\right] \subset (J^*)^d \tag{1}$$

where $k_s = 0, 1, \ldots$, $p_s = 0, \ldots, 2^{k_s}-1$. We call those intervals *dyadic intervals of rank* $\mathbf{k} = (k_1, \ldots, k_d)$. The *dyadic cube of rank* k is the dyadic interval of the form (1) with $k_1 = \ldots = k_d = k$. If Δ is a dyadic interval of rank $\mathbf{k}$, then $|\Delta|$ denotes its measure; i.e., $2^{-(k_1+\ldots+k_d)}$. The *parameter of regularity of a dyadic interval* of the form (1) is defined as $\min_{i,j}\{2^{k_i}/2^{k_j}\}$. Analogously the *parameter of regularity of a vector* $\mathbf{a} = (a_1, \ldots, a_d)$ is defined as $\min_{i,j}\{a_i/a_j\}$. We write $reg(\Delta)$ (resp. $reg(\mathbf{a})$) for the parameter of regularity of dyadic interval Δ (resp. of vector $\mathbf{a}$). Below, the sum of vectors and multiplication of a vector by a number are understood in the usual sense. Further we denote by $\mathbf{1}$ the d-dimensional vector $(1, \ldots, 1)$.

Consider a point $t \in J^*$. We say that the sequence $\{\Delta_k\}$ of one-dimensional dyadic intervals is the *basic sequence convergent to* t if $t \in \Delta_k$ for all k and rank of Δ_k equals k. Then the d-multiple sequence $\{\Delta_{\mathbf{k}} = \Delta_{k_1,\ldots,k_d}\}$ of d-dimensional dyadic intervals is the *basic sequence convergent to* $\mathbf{t} = (t^1, \ldots, t^d) \in (J^*)^d$ if

$$\Delta_{\mathbf{k}} = \Delta_{k_1} \times \ldots \times \Delta_{k_d} \tag{2}$$

where $\{\Delta_{k_i} : k_i = 0, 1, \ldots\}$ is the one-dimensional basic sequence convergent to t^i $(i = 1, \ldots, d)$.

The *d-dimensional Walsh* (resp. *Haar*) *series* is defined by

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}}\omega_{\mathbf{n}}(\mathbf{t}) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_d=0}^{\infty} b_{n_1,\ldots,n_d} \prod_{i=1}^{d} \omega_{n_i}(t^i) \tag{3}$$

$$\left(\text{resp.}\ \sum_{\mathbf{n}=\mathbf{0}}^{\infty} a_{\mathbf{n}}\chi_{\mathbf{n}}(\mathbf{t}) = \sum_{n_1=0}^{\infty} \ldots \sum_{n_d=0}^{\infty} a_{n_1,\ldots,n_d} \prod_{i=1}^{d} \chi_{n_i}(t^i)\ \right) \tag{4}$$

where $a_{\mathbf{n}}$ and $b_{\mathbf{n}}$ are real numbers. Let $\mathbf{N} = (N_1, \ldots, N_d)$, then the $\mathbf{N}$*th rectangular partial sum* $S_{\mathbf{N}}$ of series (3) (resp. (4)) at a point $\mathbf{t}$ is

$$S_{\mathbf{N}}(\mathbf{t}) = \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_d=0}^{N_d-1} b_{\mathbf{n}}\omega_{\mathbf{n}}(\mathbf{t}) \quad \left(\text{resp.}\ S_{\mathbf{N}}(\mathbf{t}) = \sum_{n_1=0}^{N_1-1} \ldots \sum_{n_d=0}^{N_d-1} a_{\mathbf{n}}\chi_{\mathbf{n}}(\mathbf{t})\ \right).$$

The series (3) (or (4)) *rectangularly converges* to sum $S(\mathbf{t})$ at point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \ \text{ as } \ \min_i\{N_i\} \to \infty.$$

Let $\rho \in (0,1]$; then the series (3) (or (4)) *ρ-regularly converges* to sum $S(\mathbf{t})$ at a point $\mathbf{t}$ if

$$S_{\mathbf{N}}(\mathbf{t}) \to S(\mathbf{t}) \text{ as } \min_i\{N_i\} \to \infty \text{ and } reg(\mathbf{N}) \geq \rho.$$

It is obvious that if the series (3) (or (4)) rectangularly converges to sum $S(\mathbf{t})$ at a point $\mathbf{t}$, then for every $\rho \in (0,1]$ this series ρ-regularly converges to $S(\mathbf{t})$ at $\mathbf{t}$.

3 Continuity of Set Functions.

Let $\mathcal{B}$ denote the family of all dyadic intervals (1). In this section we consider some properties of *$\mathcal{B}$-interval functions* $\tau : \mathcal{B} \to \mathbb{R}$. Recall some definitions (see [6]). A $\mathcal{B}$-interval function τ is called *$\mathcal{B}$-superadditive* (resp. *$\mathcal{B}$-subadditive*) if for every finite collection $\{\Delta_i\}_{i=1}^p$ of pairwise disjoint dyadic intervals such that $\bigcup_{i=1}^p \Delta_i \in \mathcal{B}$ we have

$$\sum_{i=1}^p \tau(\Delta_i) \leq \tau\left(\bigcup_{i=1}^p \Delta_i\right) \text{ (resp. } \sum_{i=1}^p \tau(\Delta_i) \geq \tau\left(\bigcup_{i=1}^p \Delta_i\right) \text{)}$$

By $\overline{\mathcal{A}}_{\mathcal{B}}$ (resp. $\underline{\mathcal{A}}_{\mathcal{B}}$) we denote the set of all $\mathcal{B}$-superadditive (resp. $\mathcal{B}$-subadditive) functions. A $\mathcal{B}$-interval function τ is called *$\mathcal{B}$-additive* if $\tau \in \overline{\mathcal{A}}_{\mathcal{B}} \cap \underline{\mathcal{A}}_{\mathcal{B}}$. By $\mathcal{A}_{\mathcal{B}}$ we denote the set of all $\mathcal{B}$-additive functions.

Consider different types of continuity of $\mathcal{B}$-interval functions. A $\mathcal{B}$-interval function τ is called *continuous in the sense of Saks* if

$$\lim \tau(\Delta) \to 0 \text{ as } |\Delta| \to 0. \tag{5}$$

A $\mathcal{B}$-interval function τ is *strongly continuous at a point* $\mathbf{t} \in G^d$ if

$$\lim \tau(\Delta) \to 0 \text{ as } |\Delta| \to 0, \ \ \mathbf{t} \in \Delta. \tag{6}$$

Let $\rho \in (0,1]$; then we say that a function τ is *ρ-continuous at a point* $\mathbf{t} \in G^d$ if

$$\lim \tau(\Delta) \to 0 \text{ as } |\Delta| \to 0, \ \ reg(\Delta) \geq \rho, \ \ \mathbf{t} \in \Delta. \tag{7}$$

Put

$$\Sigma_d = \{\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_d) : \sigma_i = 0 \text{ or } 1 \text{ for all } i = 1, \ldots, d\}; \ \ |\boldsymbol{\sigma}| = \sum_{i=1}^d \sigma_i.$$

Let $\{\Delta_{\mathbf{k}}\}$ be the basic sequence of the form (2) convergent to a point $\mathbf{t} \in G^d$ (see Section 2). Put

$$\Delta_{k_i}^0 = \Delta_{k_i+1}, \quad \Delta_{k_i}^1 = \Delta_{k_i} \setminus \Delta_{k_i+1}; \quad \text{if } \boldsymbol{\sigma} \in \Sigma_d \text{ and } \mathbf{k} = (k_1, \ldots, k_d),$$

$$\text{then } \Delta_{\mathbf{k}}^{\boldsymbol{\sigma}} = \Delta_{k_1}^{\sigma_1} \times \ldots \times \Delta_{k_d}^{\sigma_d}.$$

We say that a $\mathcal{B}$-interval function τ is Σ_d-*continuous at* $\mathbf{t}$ if

$$\lim_{k\to\infty} \sum_{\boldsymbol{\sigma}\in\Sigma_d} \left(-\frac{1}{2}\right)^{|\boldsymbol{\sigma}|} \tau(\Delta_{k\cdot\mathbf{1}-\boldsymbol{\sigma}}) = 0. \tag{8}$$

We say that a function τ is Σ_d^*-*continuous at* $\mathbf{t}$ if

$$\lim_{k\to\infty} \sum_{\boldsymbol{\sigma}\in\Sigma_d} (-1)^{|\boldsymbol{\sigma}|} \tau(\Delta_{k\cdot\mathbf{1}}^{\boldsymbol{\sigma}}) = 0.$$

It is obvious that for all $\rho \in (0, 1]$ and $\mathbf{t} \in G^d$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7). It can be proved that if $\rho \le 1/2$, then (7) $\Rightarrow$ (8) at every point $\mathbf{t} \in G^d$. The next fact follows from [10, Section 2].

Proposition 1. *Let a $\mathcal{B}$-interval function $\tau \in \mathcal{A}_{\mathcal{B}}$; then τ is Σ_d-continuous at a point $\mathbf{t}$ if and only if this function is Σ_d^*-continuous at $\mathbf{t}$.*

We need also the following statement (see [10, Section 3, Lemma 1]).

Proposition 2. *Let $\tau \in \overline{\mathcal{A}}_{\mathcal{B}}$ and let Δ be a dyadic cube such that $\tau(\Delta) < 0$. Suppose that the function τ is Σ_d^*-continuous at every point $\mathbf{t} \in G^d$. Then there are disjoint dyadic cubes Δ^1, $\Delta^2 \subset \Delta$ such that $\tau(\Delta^i) < 0$ $(i = 1, 2)$.*

Repeatedly using the previous proposition we get

Proposition 3. *Suppose a $\mathcal{B}$-interval function τ and a dyadic cube Δ satisfy the conditions of Proposition 2. Then there is a double sequence $\{\Delta_k^i : k = 0, 1, \ldots, i = 1, \ldots, 2^k\}$ of dyadic cubes with the following properties:*

- $\Delta_k^i \bigcap \Delta_k^j = \emptyset$ *for all k and $i \ne j$;*
- $\Delta_{k+1}^{2i-1} \subset \Delta_k^i$ *and $\Delta_{k+1}^{2i} \subset \Delta_k^i$ for every k and i;*
- $\tau(\Delta_k^i) < 0$ *for all k and i.*

4 Perron-Type Integral.

In this section we construct a Perron-type integral, using the results of the previous section. In the next section we use this integral to solve the coefficients problem.

Recall some definitions (see [6]). Let τ be a $\mathcal{B}$-interval function and $\rho \in (0,1]$. *Upper* (*resp. lower*) *dyadic ρ-regular derivative* of function τ at a point $\mathbf{t} \in G^d$ is defined by

$$\overline{D}^{\rho}_{d}\tau(\mathbf{t}) \overset{def}{=} \overline{\lim}\frac{\tau(\Delta)}{|\Delta|} \quad (\text{resp. } \underline{D}^{\rho}_{d}\tau(\mathbf{t}) \overset{def}{=} \underline{\lim}\frac{\tau(\Delta)}{|\Delta|}\ )$$

$$\text{as } \ |\Delta| \to 0, \ \ reg(\Delta) \geq \rho, \ \ \mathbf{t} \in \Delta.$$

If $\overline{D}^{\rho}_{d}\tau(\mathbf{t}) = \underline{D}^{\rho}_{d}\tau(\mathbf{t}) = d \neq \pm\infty$, we say that $D^{\rho}_{d}\tau(\mathbf{t}) \overset{def}{=} d$ is the *dyadic ρ-regular derivative* of function τ at the point $\mathbf{t} \in G^d$.

The following statement is the *'monotonicity theorem'* for $\mathcal{B}$-interval functions.

Theorem 1. *Let τ be a $\mathcal{B}$-interval function, $\tau \in \overline{\mathcal{A}}_{\mathcal{B}}$. Suppose that the function τ satisfies*

$$\overline{D}^{1}_{d}\tau(\mathbf{t}) \geq 0 \tag{9}$$

at every point $\mathbf{t} \in G^d$ except possibly a countable set L. Let the function τ be Σ^*_d-continuous at every point $\mathbf{t} \in G^d$. Then $\tau(\Delta) \geq 0$ for every dyadic interval Δ.

Proof. Assume that $\tau(\Delta_0) < 0$ for some dyadic interval Δ_0. Since $\tau \in \overline{\mathcal{A}}_{\mathcal{B}}$, then $\tau(\Delta_1) < 0$ for some dyadic cube $\Delta_1 \subset \Delta_0$ of rank k_1. Choose $\varepsilon > 0$ such that $\tau_1(\Delta_1) < 0$ where

$$\tau_1(\Delta) \overset{def}{=} \tau(\Delta) + \varepsilon|\Delta|$$

It is obvious that $\tau_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and τ_1 is Σ^*_d-continuous at every point $\mathbf{t} \in G^d$. Let S_k denote the union of all dyadic cubes $\Delta \subset \Delta_1$ of rank k such that $\tau_1(\Delta) < 0$. Consider the set $S = \bigcap_{k=k_1}^{\infty} S_k$. By the Proposition 3 the set S contains a perfect subset S_1. Note that for every $\mathbf{t} \in S_1$ there is a sequence $\{\Delta_k(\mathbf{t})\}_{k=k_1}^{\infty}$ of dyadic cubes such that for all $k \geq k_1$ rank $\Delta_k(\mathbf{t})$ equals k, $\mathbf{t} \in \Delta_k$, and $\tau_1(\Delta_k(\mathbf{t})) < 0$. Then for $\mathbf{t} \in S_1$,

$$\overline{D}^{1}_{d}\tau(\mathbf{t}) = \overline{\lim_{k\to\infty}}\,\frac{\tau_1(\Delta_k(\mathbf{t})) - \varepsilon|\Delta_k(\mathbf{t})|}{|\Delta_k(\mathbf{t})|}$$

$$= \overline{\lim_{k\to\infty}}\,\frac{\tau_1(\Delta_k(\mathbf{t}))}{|\Delta_k(\mathbf{t})|} - \lim_{k\to\infty}\frac{\varepsilon|\Delta_k(\mathbf{t})|}{|\Delta_k(\mathbf{t})|} \leq 0 - \varepsilon = -\varepsilon$$

and this leads to a contradiction with the formula (9) and completes the proof. □

Theorem 1 opens a way for constructing a family of Perron-type integrals. Further we use the next designation. Let for every point $\mathbf{t} \in G^d$ except possibly a countable set L an increasing sequence of natural numbers $\{k_j = k_j(\mathbf{t})\}$ be chosen. Then $\mathbf{a}$ denotes the family

$$\{k_j = k_j(\mathbf{t}) : \, \mathbf{t} \in G^d \setminus L, \, j = 1, 2, \dots\}.$$

Definition 1. Let $L \subset G^d$ be a countable set. Suppose that a family $\mathbf{a}$ be chosen. Then we say that a finite function f defined on $G^d \setminus L$ is $PHW(\mathbf{a})$-integrable if for every $\varepsilon > 0$ there exist $\mathcal{B}$-interval functions $F_1 \in \overline{\mathcal{A}}_{\mathcal{B}}$ and $F_2 \in \underline{\mathcal{A}}_{\mathcal{B}}$ with the following properties:
(A) F_1 and F_2 are Σ_d^*-continuous at every point $\mathbf{t} \in G^d$;
(B) if $\mathbf{t}$ is any point of $G^d \setminus L$ and $\{\Delta_n\}$ is the basic sequence of dyadic cubes convergent to $\mathbf{t}$, then $\underline{\lim}_{j\to\infty} F_1(\Delta_{k_j})/|\Delta_{k_j}| \geq f(\mathbf{t}) \geq \overline{\lim}_{j\to\infty} F_2(\Delta_{k_j})/|\Delta_{k_j}|$;
(C) $F_1(G^d) - F_2(G^d) < \varepsilon$.

For every dyadic interval Δ we define $PHW(\mathbf{a})$-integral (Perron-Haar-Walsh integral) of function f on Δ as $(PHW(\mathbf{a})) \int_\Delta f(\mathbf{t})\, d\mathbf{t} = \inf_{F_1} F_1(\Delta) = \sup_{F_2} F_2(\Delta)$.

Let functions F_1 and F_2 satisfy the conditions (A), (B) and (C) from the foregoing definition; then we have for every $\mathbf{t} \in G^d \setminus L$:

$$\begin{aligned} \overline{D}_d^1(F_1 - F_2)(\mathbf{t}) &\geq \overline{\lim_{j\to\infty}} \frac{F_1(\Delta_{k_j}) - F_2(\Delta_{k_j})}{|\Delta_{k_j}|} \geq \underline{\lim}_{j\to\infty} \frac{F_1(\Delta_{k_j}) - F_2(\Delta_{k_j})}{|\Delta_{k_j}|} \\ &\geq \underline{\lim}_{j\to\infty} \frac{F_1(\Delta_{k_j})}{|\Delta_{k_j}|} - \overline{\lim_{j\to\infty}} \frac{F_2(\Delta_{k_j})}{|\Delta_{k_j}|} \geq f(\mathbf{t}) - f(\mathbf{t}) = 0. \end{aligned} \tag{10}$$

By (10) and by Theorem 1 the concept of $PHW(\mathbf{a})$-integral is well defined. It can be proved that $PHW(\mathbf{a})$-integral has some standard properties. In particular the $\mathcal{B}$-interval function $F(\Delta) \stackrel{def}{=} (PHW(\mathbf{a})) \int_\Delta f(\mathbf{t})\, d\mathbf{t}$ is a $\mathcal{B}$-additive function.

5 Recovering the Coefficients of Multiple Series.

We prove in this section the principal theorems of the paper. We use 'formal integration' of series as a tool of studying the properties of Haar and Walsh series. If $\mathbf{k} = (k_1, \dots, k_d)$, then we denote by $2^{\mathbf{k}}$ the vector $(2^{k_1}, \dots, 2^{k_d})$. For series (3) or (4) we define a $\mathcal{B}$-interval function $\psi(\Delta)$ via

$$\psi(\Delta) = S_{2^{\mathbf{k}}}(\mathbf{t})|\Delta| \tag{11}$$

where Δ denotes the dyadic interval of rank $\mathbf{k}$ such that $\mathbf{t} \in \Delta$. Note that $\psi(\Delta)$ does not depend on $\mathbf{t} \in \Delta$. It is known that ψ is a $\mathcal{B}$-additive function. The function ψ is often called *quasi-measure* (see [11, 19, 20]). The continuity of function ψ is closely related to the convergence of the associated series. The next statement follows from [15, 16].

Proposition 4. *Suppose the series* (3) (*resp.* (4)) *rectangularly converges everywhere to a finite sum. Then the function* ψ *defined for this series by* (11) *is continuous in the sense of Saks* (*resp. is strongly continuous at every point* $\mathbf{t} \in G^d$).

We show in the next section that the last result does not hold for ρ-regular convergence. We need the following multidimensional analogue of the well-known *Arutunyan-Talalyan condition* (see [3]) for the series (4):

$$a_{\mathbf{n}}\chi_{\mathbf{n}}(\mathbf{t}) = \bar{\bar{o}}_{\mathbf{t}}(n_1 \cdot \ldots \cdot n_d) \ \text{ as } \min_i\{n_i\} \to \infty,\ reg(\mathbf{n}) \geq 1/2. \tag{12}$$

We need also the following condition for the series (3):

$$b_{\mathbf{n}} = \bar{\bar{o}}(1) \ \text{ as } \min_i\{n_i\} \to \infty,\ reg(\mathbf{n}) \geq 1/2. \tag{13}$$

Proposition 5. (*See* [*10, Section* 3]). *Suppose the series of the form* (4) *satisfies* (12) *at some point* $\mathbf{t} \in G^d$*; then the function* ψ *defined for this series by* (11) *is* Σ_d^**-continuous at the point* $\mathbf{t}$.

Proposition 6. *Assume that the partial sums* $S_{N_1,\dots,N_d}(\mathbf{t})$ *of series* (4) *at some point* $\mathbf{t} \in G^d$ *satisfy the next condition:*

$$S_{N_1,\dots,N_d}(\mathbf{t}) = \bar{\bar{o}}_{\mathbf{t}}(N_1 \cdot \ldots \cdot N_d) \ \text{ as } \min_i\{N_i\} \to \infty,\ reg(\mathbf{N}) \geq 1/2. \tag{14}$$

Then the function ψ *defined for this series by* (11) *is* Σ_d^**-continuous at the point* $\mathbf{t}$.

PROOF. Let $\{\Delta_{\mathbf{k}}\}$ be the basic sequence convergent to $\mathbf{t}$. Suppose that the partial sums of series (4) at a point $\mathbf{t} \in G^d$ satisfy the condition (14). Then for every $\boldsymbol{\sigma} \in \Sigma_d$ the expression $S_{2^{k\cdot\mathbf{1}-\boldsymbol{\sigma}}}(\mathbf{t})$ is $\bar{\bar{o}}(2^{kd})$ as $k \to \infty$. Therefore,

$$\sum_{\boldsymbol{\sigma} \in \Sigma_d} (-1)^{|\boldsymbol{\sigma}|} S_{2^{k\cdot\mathbf{1}-\boldsymbol{\sigma}}}(\mathbf{t}) = \bar{\bar{o}}(2^{kd}) \ \text{ as } k \to \infty. \tag{15}$$

On the other hand we have

$$\begin{aligned}\sum_{\boldsymbol{\sigma}\in\Sigma_d}(-1)^{|\boldsymbol{\sigma}|}S_{2^{k\cdot\mathbf{1}-\boldsymbol{\sigma}}}(\mathbf{t}) &\overset{\text{by }(11)}{=} \sum_{\boldsymbol{\sigma}\in\Sigma_d}(-1)^{|\boldsymbol{\sigma}|}\frac{\psi(\Delta_{k\cdot\mathbf{1}-\boldsymbol{\sigma}})}{|\Delta_{k\cdot\mathbf{1}-\boldsymbol{\sigma}}|}\\ = \sum_{\boldsymbol{\sigma}\in\Sigma_d}\left(-\frac{1}{2}\right)^{|\boldsymbol{\sigma}|}\frac{\psi(\Delta_{k\cdot\mathbf{1}-\boldsymbol{\sigma}})}{|\Delta_{k\cdot\mathbf{1}}|} &= 2^{kd}\sum_{\boldsymbol{\sigma}\in\Sigma_d}\left(-\frac{1}{2}\right)^{|\boldsymbol{\sigma}|}\psi(\Delta_{k\cdot\mathbf{1}-\boldsymbol{\sigma}}).\end{aligned}\tag{16}$$

Using (15) and (16), we obtain

$$\sum_{\boldsymbol{\sigma}\in\Sigma_d}\left(-\frac{1}{2}\right)^{|\boldsymbol{\sigma}|}\psi(\Delta_{k\cdot\mathbf{1}-\boldsymbol{\sigma}}) = \bar{\bar{o}}(1) \ \text{ as } k\to\infty. \tag{17}$$

Therefore the function ψ is Σ_d-continuous at the point $\mathbf{t}$. Using Proposition 1, we get that the function ψ is Σ_d^*-continuous at the point $\mathbf{t}$. This concludes the proof. □

Corollary 1. *Assume that the series* (4) *1/2-regularly converges to a finite sum at some point* $\mathbf{t}\in G^d$. *Then the function* ψ *defined for this series by* (11) *is* Σ_d^**-continuous at the point* $\mathbf{t}$.

Proposition 7. (*See* [9, *Lemma* 2]). *Assume that the series* (3) *satisfies* (13); *then the function* ψ *defined for this series by* (11) *is* Σ_d^**-continuous at* ***every point*** $\mathbf{t}\in G^d$.

Proposition 8. (*See* [9, *Remark* 1]). *Assume that the series* (3) *1/2-regularly converges to a finite sum* ***at some point*** $\mathbf{t}_0\in G^d$. *Then the function* ψ *defined for this series by* (11) *is* Σ_d^**-continuous* ***at every point*** $\mathbf{t}\in G^d$.

To solve the coefficients problem we recall some observation (see [14, Proposition 4]).

Proposition 9. *Let some integration process* $\mathcal{I}$ *be given which produces an integral additive on* $\mathcal{B}$. *Let the* $\mathcal{B}$*-interval function* ψ *be defined by* (11) *for a given series* (4) *or* (3). *Then this series is the Fourier series of an* $\mathcal{I}$*-integrable function* f *if and only if* $\psi(\Delta)=(\mathcal{I})\int_\Delta f$ *for any dyadic interval* Δ.

We present now the main results.

Theorem 2. *Let at every point* $\mathbf{t}\in G^d$, *except possibly a countable set* L, *an increasing sequence of natural numbers* $\mathbf{a}=\{k_j=k_j(\mathbf{t})\}$ *be chosen. Assume that for the series* (4) *the following conditions hold:*

(a) *at every point* $\mathbf{t} \in G^d \setminus L$ *the subsequence* $S_{2^{k_j(\mathbf{t})}\cdot\mathbf{1}}(\mathbf{t})$ *of the cubic partial sums of this series is convergent to a finite sum* $f(\mathbf{t})$*;*

(b) *at every point* $\mathbf{t} \in G^d$ *this series satisfies the condition* (12)*.*

Then the function $f(\mathbf{t})$ *is* $(PHW(\mathbf{a}))$*-integrable and the series is its Fourier-Haar series in terms of* $(PHW(\mathbf{a}))$*-integral.*

PROOF. Let ψ be a $\mathcal{B}$-interval function defined for this series by (11). Put $F_1(\Delta) = F_2(\Delta) \equiv \psi(\Delta)$. Since $\psi \in \mathcal{A}_{\mathcal{B}}$, then $F_1 \in \mathcal{A}_{\mathcal{B}}$ and $F_2 \in \mathcal{A}_{\mathcal{B}}$. Combining (a) and (11), we obtain that the functions F_1 and F_2 satisfy the condition (B) of Definition 1. Using (b) and Proposition 5, we get the condition (A) of Definition 1. Finally, the condition (C) from that definition is obvious. Hence the function $f(\mathbf{t})$ is $(PHW(\mathbf{a}))$-integrable. We have for every dyadic interval Δ

$$(PHW(\mathbf{a})) \int\limits_{\Delta} f = \sup_{F_2} F_2(\Delta) \geq \psi(\Delta) \geq \inf_{F_1} F_1(\Delta) = (PHW(\mathbf{a})) \int\limits_{\Delta} f.$$

Therefore $\psi(\Delta) = (PHW(\mathbf{a})) \int_{\Delta} f(\mathbf{t})\ d\mathbf{t}$. Combining the last formula and Proposition 9, we obtain that our series is $(PHW(\mathbf{a}))$-Fourier-Haar series. This completes the proof. □

In a similar way, using Corollary 1 and Proposition 7, the next theorems for multiple Haar and Walsh series can be proved.

Theorem 3. *Suppose that a multiple Haar series* $1/2$*-regularly converges to a finite sum* $f(\mathbf{t})$ *everywhere on* G^d*. Then for every choice of a family* $\mathbf{a} = \{k_j = k_j(\mathbf{t})\}$*, the function* $f(\mathbf{t})$ *is* $(PHW(\mathbf{a}))$*-integrable and the series is* $(PHW(\mathbf{a}))$*-Fourier-Haar series of the function* $f(\mathbf{t})$*.*

Theorem 4. *Let at every point* $\mathbf{t} \in G^d$*, except possibly a countable set* L*, an increasing sequence of natural numbers* $\{k_j = k_j(\mathbf{t})\}$ *be chosen. Assume that for the series* (3) *the following conditions hold:*

(a) *at every point* $\mathbf{t} \in G^d \setminus L$ *the subsequence* $S_{2^{k_j(\mathbf{t})}\cdot\mathbf{1}}(\mathbf{t})$ *of the cubic partial sums of this series is convergent to a finite sum* $f(\mathbf{t})$*;*

(b) *this series satisfies the condition* (13)*.*

Then the function $f(\mathbf{t})$ *is* $(PHW(\mathbf{a}))$*-integrable and the series is* $(PHW(\mathbf{a}))$*-Fourier-Walsh series of the function* $f(\mathbf{t})$*.*

Theorems similar to Theorems 2 and 4 were proved in [15] but in that paper stronger conditions were imposed on the coefficients of the series. Those conditions hold for the rectangular convergence of the series (4) and (3). In the last section we shall show that those conditions do not hold in the case of ρ-regular convergence.

The next statement follows from Theorem 2.

Corollary 2. *Let a number $\rho \in (0,1]$ be fixed. Suppose that a multiple Haar series ρ-regularly converges to a finite sum $f(\mathbf{t})$ at every point $\mathbf{t} \in G^d$ except possibly a countable set L. Assume that this series satisfies the condition* (12) *at each point $\mathbf{t} \in G^d$. Then for every choice a family $\mathbf{a} = \{k_j = k_j(\mathbf{t})\}$, the function $f(\mathbf{t})$ is $(PHW(\mathbf{a}))$-integrable and the series is $(PHW(\mathbf{a}))$-Fourier-Haar series of the function $f(\mathbf{t})$.*

Using Theorem 4 and Proposition 8, we get the next two propositions.

Corollary 3. *Suppose that a d-multiple Walsh series cubically* (*i.e.,* 1*-regularly*) *converges to a finite sum $f(\mathbf{t})$ at every point $\mathbf{t} \in G^d$ except possibly a countable set L. Assume that this series converges also* 1/2*-regularly to a finite sum* ***at least at one point*** *$\mathbf{t}_0 \in G^d$. Then for every choice of a family $\mathbf{a} = \{k_j = k_j(\mathbf{t})\}$ the function $f(\mathbf{t})$ is $(PHW(\mathbf{a}))$-integrable and the given series is $(PHW(\mathbf{a}))$-Fourier-Walsh series of the function $f(\mathbf{t})$.*

Corollary 4. *Suppose that the d-multiple Walsh series* 1/2*-regularly converges to a finite sum $f(\mathbf{t})$ at every point $\mathbf{t} \in G^d$ except possibly a countable set L. Then for every choice of a family $\mathbf{a} = \{k_j = k_j(\mathbf{t})\}$ the function $f(\mathbf{t})$ is $(PHW(\mathbf{a}))$-integrable and the given series is $(PHW(\mathbf{a}))$-Fourier-Walsh series of the function $f(\mathbf{t})$.*

Corollary 4 implies the following fact concerning sets of uniqueness. Recall that a set L is called the *set of uniqueness* (or in short: a U-set) for a system $\{\varphi_{\mathbf{n}}\}$ if from the convergence of a series $\sum_{\mathbf{n}} c_{\mathbf{n}}\varphi_{\mathbf{n}}$ to zero outside the set L it follows that $c_{\mathbf{n}} = 0$ for all $\mathbf{n}$. From the definition of $(PHW(\mathbf{a}))$-integrals it follows that for any choice of a family $\mathbf{a} = \{k_j = k_j(\mathbf{t})\}$

$$(PHW(\mathbf{a}))\int\limits_{G^d} 0\,d\mathbf{t} = 0. \tag{18}$$

Using formula (18) and Corollary 4, we obtain the following statement for d-multiple Walsh series.

Theorem 5. (*See* [*9*]). *Let a number $\rho \in (0,1/2]$ be chosen. Then any finite or countable set $L \subset G^d$ is a U-set for the multiple Walsh system with ρ-regular convergence.*

6 Some Examples of Double Haar and Walsh Series.

For the sake of simplicity we restrict ourselves here to considering the **two-dimensional** case. We shall show that some properties of rectangularly convergent double Haar and Walsh series do not hold for the ρ-regular convergence. We impose on the series of the form (4), with $d = 2$, a strong analogue of the Arutunyan-Talalyan condition:

$$a_{n_1,n_2}\chi_{n_1,n_2}(t^1,t^2) = \bar{\bar{o}}_{(t^1,t^2)}(n_1n_2) \text{ as } n_1+n_2 \to \infty. \tag{19}$$

For the double Walsh series we consider the next condition:

$$b_{n_1,n_2} = \bar{\bar{o}}(1) \text{ as } n_1+n_2 \to \infty. \tag{20}$$

The condition (19) (resp. (20)) in the two-dimensional case is stronger than the condition (12) (resp. (13)). The next two statements were proved in [15, 16].

Proposition 10. *Suppose that the series* (4) ***rectangularly*** *converges to a finite sum everywhere on G^d. Then this series satisfies the condition* (19) *at every point $\mathbf{t} \in G^d$.*

Proposition 11. *Suppose that the series* (3) ***rectangularly*** *converges to a finite sum at every point of a 'cross' $(\{a\} \times [0,1]) \bigcup ([0,1] \times \{b\})$. Then this series satisfies the condition* (20).

In the case of ρ-regular convergence the conditions (19) and (20) can fail to hold even if the appropriate series converges everywhere.

Theorem 6. *For every $\rho \in (0,1]$ there exists a double Walsh series which is ρ-regularly convergent to a finite sum everywhere on G^d, but which does not satisfy the condition* (20).

PROOF. Choose natural M so that $2^{-M} < \rho \le 2^{-M+1}$. Take **any** sequence $\{C_n\}$ of real numbers such that $\overline{\lim}_{n\to\infty}|C_n| > 0$. Consider $n = M, M+1, \ldots$; then the double sequence

$$b_{n_1,n_2} = \begin{cases} C_n, & \text{if } 0 \le n_1 \le 2^{n-M}-1,\ n_2 = 2^n; \\ -C_n, & \text{if } 2^{n-M} \le n_1 \le 2^{n-M+1}-1,\ n_2 = 2^n; \\ 0 & \text{in all other cases.} \end{cases} \tag{21}$$

does not satisfy the condition (20). To show that the series

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} b_{n_1,n_2}\omega_{n_1,n_2}(t^1,t^2) \tag{22}$$

ρ-regularly converges to a finite sum everywhere on G^d, we consider partial sums $S_{N_1,N_2}(t^1,t^2)$ of the series (22) with $\min_{i,j}\{N_i/N_j\} \geq \rho$. By definition, $b_{n_1,n_2} = 0$ if $n_1 > n_2$. Hence it is sufficient to consider partial sums $S_{N_1,N_2}(t^1,t^2)$ with $N_1 \leq N_2$. Put for $j = 1, 2$

$$N_j = 2^{K_j} + i_j, \quad K_j = 1, \dots, \quad 1 \leq i_j \leq 2^{K_j}.$$

Since $N_1 \leq N_2$ and $\min\{N_1/N_2, N_2/N_1\} \geq \rho > 2^{-M}$ we get

$$K_2 - M \leq K_1 \leq K_2. \tag{23}$$

Then we have, for sufficiently large N_1 and N_2,

$$S_{N_1,N_2}(t^1,t^2) = \sum_{s=M}^{K_2}\sum_{u=0}^{2^{s-M+1}-1} b_{u,2^s}\omega_{u,2^s}(t^1,t^2)$$

$$\overset{\text{by (21) and (23)}}{=} \sum_{s=M}^{K_2} C_s\omega_{2^s}(t^2)\Bigg(\sum_{u=0}^{2^{s-M}-1}\omega_u(t^1) - \sum_{u=2^{s-M}}^{2^{s-M+1}-1}\omega_u(t^1)\Bigg). \tag{24}$$

Consider the difference in braces in formula (24). If $t^1 = 0$ this expression is obviously equal zero. In this case for all $t^2 \in G$ we have $S_{N_1,N_2}(t^1,t^2) = 0$ for sufficiently large N_1 and N_2 satisfying the condition (23). Let $t^1 \neq 0$; then $t^1 \in [1/2^{T+1} + 0, 1/2^T - 0]$ for some $T \in \{0, 1, \dots\}$. We have

$$\sum_{u=0}^{2^{s-M}-1}\omega_u(t^1) - \sum_{u=2^{s-M}}^{2^{s-M+1}-1}\omega_u(t^1) = D_{2^{s-M}}(t^1) - (D_{2^{s-M+1}}(t^1) - D_{2^{s-M}}(t^1))$$

$$= 2D_{2^{s-M}}(t^1) - D_{2^{s-M+1}}(t^1) \tag{25}$$

where D_n is the nth *Dirichlet kernel* for Walsh system. It follows from the properties of the Dirichlet kernels for Walsh system (see, for example, [5, Chapter 1]) that

$$D_{2^k}(t^1) = 0 \ \text{ if } t^1 \in [1/2^{T+1} + 0, 1/2^T - 0],\ k \geq T + 1. \tag{26}$$

Using (25) and (26), we get

$$\sum_{u=0}^{2^{s-M}-1} \omega_u(t^1) - \sum_{u=2^{s-M}}^{2^{s-M+1}-1} \omega_u(t^1) = 0 \tag{27}$$
$$\text{if } t^1 \in [1/2^{T+1}+0, 1/2^T-0], \text{ and } s \geq M+T+1.$$

From the formulae (24) and (27) we have for sufficiently large N_1 and N_2, satisfying the condition (23), and for all $t^2 \in G$

$$S_{N_1,N_2}(t^1,t^2) = \sum_{s=M}^{M+T} C_s \omega_{2^s}(t^2) \Bigg(\sum_{u=0}^{2^{s-M}-1} \omega_u(t^1) - \sum_{u=2^{s-M}}^{2^{s-M+1}-1} \omega_u(t^1) \Bigg). \tag{28}$$

The expression in the right part of (28) is independent of N_1 and N_2. Consequently, if $2^{-M} < \rho \leq 2^{-M+1}$ and $t^1 \in [1/2^{T+1}+0, 1/2^T-0]$, then the series (22) converges ρ-regularly to a finite sum. This completes the proof. $\square$

Theorem 7. *Suppose that the sequence $\{C_n\}$ in the proof of Theorem 6 satisfies the condition*

$$\overline{\lim_{n\to\infty}} |C_n|/2^n > 0. \tag{29}$$

Let ψ be the $\mathcal{B}$-interval function defined for the series (22) by formula (11). Then this function is ρ-continuous at every point $(t^1,t^2) \in G^2$, but not $\rho/4$-continuous at point $(0,0)$.

PROOF. It follows immediately from (11) and from Theorem 6 that the function ψ is ρ-continuous at every point $(t^1,t^2) \in G^2$. To prove the second statement of the theorem we consider the partial sum $S_{2^K,2^{K+M+1}}(0,0)$ of the series (22) (see the proof of the last theorem). Using formulae (21), (23), and (24), we get

$$\begin{aligned} S_{2^K,2^{K+M+1}}(0,0) = &\sum_{s=M}^{K+M-1} C_s \omega_{2^s}(0) \Bigg(\sum_{u=0}^{2^{s-M}-1} \omega_u(0) - \sum_{u=2^{s-M}}^{2^{s-M+1}-1} \omega_u(0) \Bigg) \\ &+ C_{K+M} \sum_{u=0}^{2^K-1} \omega_u(0) = 0 + C_{K+M} 2^K = C_{K+M} 2^K. \end{aligned} \tag{30}$$

Let $\Delta_{K,K+M+1}$ denote the dyadic interval of the rank $(K, K+M+1)$ containing the point $(0,0)$. Note that $reg(\Delta_{K,K+M+1}) = 2^{-M-1} \geq \rho/4$. Combining (11), (29), and (30), we get:

$$|\psi(\Delta_{K,K+M+1})| = |S_{2^K,2^{K+M+1}}(0,0)||\Delta_{K,K+M+1}|$$

$$= |C_{K+M}|2^K 2^{-(2K+M+1)} = |C_{K+M}|2^{-(K+M+1)} \nrightarrow 0 \text{ as } K \to \infty.$$

This completes the proof. □

Corollary 5. *For every $\rho \in (0,1]$ there is a double Walsh series ρ-regularly convergent to a finite sum everywhere on G^d, but the $\mathcal{B}$-interval function ψ defined for this series by* (11) *is not continuous in the sense of Saks.*

Theorem 6, Propositions 4 and 11, and the above Corollary show a difference between rectangularly convergent multiple Walsh series and ρ-regularly convergent ones.

Let $2^{k-1} \le n \le 2^k - 1$; then we say that the number k is the *rank of a Walsh function* $\omega_n(t)$. Let

$$\omega_{\mathbf{n}}(\mathbf{t}) = \omega_{n_1}(t_1) \cdot \ldots \cdot \omega_{n_d}(t_d)$$

be a d-dimensional Walsh function, then we say that the vector $\mathbf{k} = (k_1, \ldots, k_d)$ is the *rank of a Walsh function* $\omega_{\mathbf{n}}(\mathbf{t})$ if for each $i = 1, \ldots d$ the number k_i is the rank of $\omega_{n_i}(t_i)$. Analogously a *rank of a Haar function* (one-dimensional or d-dimensional) is defined. Now we recall that any (one-dimensional) Walsh function is a linear combination of a finite number of Haar functions **of the same rank** (see [2, Chapter 1]). Conversely, any Haar function is a linear combination of finite numbers of Walsh functions of the same rank. Thus to every series of the form (3), there corresponds some series of the form (4), and all the partial sums $S_{2^{\mathbf{k}}}(\mathbf{t})$ of these series are identically equal. Hence the $\mathcal{B}$-interval functions defined for these series by (11) are identically equal too. Moreover the following three objects are isomorphic (in the obvious sense): the set of all series of the form (4), the set of all series of the form (3), and the set of all additive $\mathcal{B}$-interval functions. These observations allow to obtain the following statements.

Lemma 1. *Let the number $\rho = 2^{-M}$ $(M = 0, 1, \ldots)$ be chosen. Assume that the series of the form* (3) *ρ-regularly converges at some point $\mathbf{t} \in G^d$. Then the corresponding d-multiple Haar series also converges ρ-regularly at the point $\mathbf{t}$.*

Proof of lemma. Let the number ρ satisfies the condition of the theorem. For ρ-regular convergence of the d-multiple Haar series it is sufficient that the partial sums $S_{2^{\mathbf{k}}}(\mathbf{t})$ are ρ-regularly convergent. But the partial sums $S_{2^{\mathbf{k}}}$ of d-multiple Haar series are identically equal to the partial sums $S_{2^{\mathbf{k}}}$ of the corresponding d-multiple Walsh series. The last sums are ρ-regularly convergent at the point $\mathbf{t}$ because the Walsh series ρ-regularly converges at this point. Lemma is proved. □

Theorem 8. *For any $\rho \in (0,1]$ there exists a double Haar series such that it is ρ-regularly convergent to a finite sum everywhere on G^d, while $\mathcal{B}$-interval function ψ defined for this series by* (11) *is not continuous in the sense of Saks.*

Proof. The statement of the theorem follows from Lemma 1 and Corollary 5. $\square$

Theorem 9. *For any $\rho \in (0,1]$ there exists a double Haar series ρ-regularly convergent to a finite sum everywhere on G^d, but not satisfying the condition* (19) *at the point* $(0,0)$.

Proof. Fix $\rho \in (0,1]$. Assume that a sequence $\{C_n\}$ of real numbers satisfies the condition (29). Construct a double Walsh series by (21) and by (22). Let the corresponding double Haar series be

$$\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1,n_2}\chi_{n_1,n_2}(t^1,t^2). \tag{31}$$

Consider the coinciding partial sums $S_{2^{k_1},2^{k_2}}(t^1,t^2)$ of these series. We have for $K \geq M+1$ (see Theorem 6)

$$a_{2^{K-M},2^K}\chi_{2^{K-M},2^K}(0,0) = S_{2^{K-M+1},2^{K+1}}(0,0) - S_{2^{K-M},2^{K+1}}(0,0)$$

$$-S_{2^{K-M+1},2^K}(0,0) + S_{2^{K-M},2^K}(0,0) = \sum_{n_1=2^{K-M}}^{2^{K-M+1}-1}\sum_{n_2=2^K}^{2^{K+1}-1} b_{n_1,n_2}\omega_{n_1,n_2}(0,0)$$

$$= \sum_{n_1=2^{K-M}}^{2^{K-M+1}-1}\sum_{n_2=2^K}^{2^{K+1}-1} b_{n_1,n_2} \overset{\text{by (21)}}{=} -C_K 2^{K-M} \overset{\text{by (29)}}{\neq} \bar{\bar{o}}(2^{2K}) \text{ as } K\to\infty.$$

This concludes the proof. $\square$

Remark 1. In our work we do not use the order of d-multiple Walsh functions within the dyadic 'packages'. Hence, if we consider the rearrangements $\omega_{\mathbf{n}}(\mathbf{t}) \leftrightarrow \omega_{\mathbf{n}^*}(\mathbf{t})$ of d-multiple Walsh system **saving the rank of all functions**, the results of this paper concerning the series of the form (3) are true for the series

$$\sum_{\mathbf{n}=\mathbf{0}}^{\infty} b_{\mathbf{n}}\omega_{\mathbf{n}^*}(\mathbf{t}). \tag{32}$$

In particular the theorems 4, 3, 4, 5, 6, and 7 are true for the series of the form (32).

In conclusion we ask two questions which seem to be of interest.

Question 1. Is any countable set $L \subset G^d$ a set of uniqueness of d-multiple Walsh series for the cubical (i.e., 1-regular) convergence?

Question 2. Suppose that a d-multiple Walsh series cubically converges to a finite sum at every point $\mathbf{t} \in G^d$ except possibly a countable set L. Does this imply that the condition (13) is satisfied?

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