RESEARCH

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F. S. Cater, Department of Mathematics and Statistics, Portland State University, Portland, Oregon 97207.

SOME LATTICES OF CONTINUOUS FUNCTIONS ON LOCALLY COMPACT SPACES

Abstract

Let U be a locally compact Hausdorff space that is not compact. Let $L(U)$ denote the family of continuous real valued functions on U such that for each $f \in L(U)$ there is a nonzero number p (depending on f) for which $f-p$ vanishes at infinity. Then $L(U)$ is obviously a lattice under the usual ordering of functions.

In this paper we prove that $L(U)$, as a lattice alone, characterizes the locally compact space U.

Let S be a locally compact Hausdorff space. Define $T(S)$ to be $L(S)$ if S is not compact, and $T(S)$ to be $C(S)$ if S is compact. We prove that any locally compact Hausdorff spaces S_1 and S_2 are homeomorphic if and only if their associated lattices $T(S_1)$ and $T(S_2)$ are isomorphic.

In [1] it was proved that for the compact Hausdorff spaces X, the lattice $C(X)$ of continuous real valued functions on X, as a lattice alone, characterizes the space X. The details are in [1], so we will not repeat them here.

So now let U be a locally compact Hausdorff space that is not compact. Let $L(U)$ denote the family of continuous real valued functions on U such that for each $f \in L(U)$, there is a nonzero number p (depending on f) for which $f-p$ vanishes at infinity. Then $L(U)$ is obviously a lattice under the usual ordering of functions.

In this paper we prove that $L(U)$, as a lattice alone, characterizes the locally compact space U.

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Thus $L(U)$ does for locally compact spaces U what $C(X)$ does for compact spaces X. On the other hand, $C(U)$ will not suffice for locally compact U. We begin with the following theorem.

Theorem 1. *Let X and Y be compact Hausdorff spaces. Fix $x_\infty \in X$ and $y_\infty \in Y$. Let*

$$L(X\,,\,x_\infty) = \Big\{g \in C(X)\ :\ g(x_\infty) \neq 0\Big\}\,,$$
$$L(Y\,,\,y_\infty) = \Big\{g \in C(Y)\ :\ g(y_\infty) \neq 0\Big\}\,.$$

Let $f \mapsto f^*$ be a lattice isomorphism of $L(X\,,\,x_\infty)$ onto $L(Y\,,\,y_\infty)$. Then there is a homeomorphism $y \mapsto y'$ of Y onto X that maps y_∞ to x_∞. Moreover,

$$f(x_\infty)f^*(y_\infty) > 0 \quad \textit{for all } f \in L(X\,,\,x_\infty)\,.$$

PROOF. Let $y \mapsto y'$ be the homeomorphism of Y onto X as in [1]. The arguments in [1] for $C(X)$ and $C(Y)$ go through verbatim for $L(X\,,\,x_\infty)$ and $L(Y\,,\,y_\infty)$. This homeomorphism also enjoys the property

for each $y \in Y$, the set
$$\Big\{f(y')\ :\ f \in L(X\,,\,x_\infty)\,,\ f^*(y) < 0\Big\} \tag{$*$}$$
is bounded above.

(To prove $(*)$, observe that the set $\{f^* \in L(Y\,,\,y_\infty)\ :\ f^*(y) < 0\}$ is a prime ideal in $L(Y\,,\,y_\infty)$ associated with the point y, and the corresponding prime ideal in $L(X\,,\,x_\infty)$ is associated with the point y'.)

It remains to prove that $y'_\infty = x_\infty$. So assume to the contrary, that $y'_\infty = x_0 \neq x_\infty$. Choose $g \in L(X\,,\,x_\infty)$ so that $g^*(y_\infty) < 0$. Choose $g_0 \in L(X\,,\,x_\infty)$ so that $g_0(x_\infty)$ and $g(x_\infty)$ have the same sign, but $g_0(x_0)$ is so large that $g_0^*(y_\infty) > 0$ by $(*)$. Then $g(x_\infty)$ and $g_0(x_\infty)$ have the same sign, but $g^*(y_\infty)$ and $g_0^*(y_\infty)$ have opposite sign. Put $F_1 = g \cup g_0$ and $f_1 = g \cap g_0$. Then $F_1(x_\infty)$ and $f_1(x_\infty)$ have the same sign, but $F_1^*(y_\infty)$ and $f_1^*(y_\infty)$ have opposite sign. Moreover $F_1 \geq f_1$.

Let $g_1 = (F_1 + f_1)/2$. Let F_2 and f_2 be two of the functions F_1, g_1, f_1 such that

$$F_2^*(y_\infty) > 0 > f_2^*(y_\infty)$$

and one of the functions F_2 or f_2 is g_1. Then

$$F_1 \geq F_2 \geq f_2 \geq f_1$$

and

$$F_1^*(y_\infty) \geq F_2^*(y_\infty) > 0 > f_2^*(y_\infty) \geq f_1^*(y_\infty)\,.$$

Furthermore

$$F_2 - f_2 = \frac{F_1 - f_1}{2}\,.$$

If $2F_2^*(y_\infty) + f_2^*(y_\infty) \neq 0$, choose $g_2 \in L(X\,,\,x_\infty)$ such that

$$g_2^* = \frac{2F_2^* + f_2^*}{3}\,;$$

otherwise choose $g_2 \in L(X\,,\,x_\infty)$ such that

$$g_2^* = \frac{F_2^* + 2f_2^*}{3}\,.$$

Let F_3 and f_3 be two of the functions F_2, g_2, f_2 such that

$$F_3^*(y_\infty) > 0 > f_3^*(y_\infty)$$

and one of the functions F_3 or f_3 is g_2. Then

$$F_2 \geq F_3 \geq f_3 \geq f_2$$

and

$$F_2^*(y_\infty) \geq F_3^*(y_\infty) > 0 > f_3^*(y_\infty) \geq f_2^*(y_\infty)\,.$$

Furthermore

$$F_3^*(y_\infty) - f_3^*(y_\infty) \leq \frac{2\big(F_2^*(y_\infty) - f_2^*(y_\infty)\big)}{3}\,.$$

We use the technique of the preceding two paragraphs and inductive construction to construct sequences of functions $(f_n) \subset L(X\,,\,x_\infty)$ and $(F_n) \subset L(X\,,\,x_\infty)$ such that

$$F_{n-1} \geq F_n \geq f_n \geq f_{n-1}\,, \tag{1}$$

$$F_{n-1}^*(y_\infty) \geq F_n^*(y_\infty) > 0 > f_n^*(y_\infty) \geq f_{n-1}^*(y_\infty)\,,\ \text{for } n > 1\text{, and} \tag{2}$$

$$F_n - f_n = \frac{F_{n-1} - f_{n-1}}{2}\,,\ \text{for } n \text{ even, and} \tag{3}$$

$$F_n^*(y_\infty) - f_n^*(y_\infty) \leq \frac{2\big(F_{n-1}^*(y_\infty) - f_{n-1}^*(y_\infty)\big)}{3}\,,\ \text{for } n \text{ odd}\,. \tag{4}$$

It follows from (1) and (3) that the sequences of functions (F_n) and (f_n) each converges uniformly to a continuous function H on X, and furthermore $F_n \geq H \geq f_n$ for each n. Plainly $H(x_\infty)$ has the same sign as $F_1(x_\infty)$ and $f_1(x_\infty)$, and it follows that $H \in L(X\,,\,x_\infty)$.

On the other hand, it follows from (2) and (4) that the sequences of numbers $\big(F_n^*(y_\infty)\big)$ and $\big(f_n^*(y_\infty)\big)$ each converges to 0. We deduce from (1) that

$$F_n^* \geq H^* \geq f_n^* \quad \text{and} \quad F_n^*(y_\infty) \geq H^*(y_\infty) \geq f_n^*(y_\infty)$$

for each index n. Necessarily, then, $H^*(y_\infty) = 0$ and consequently $H^* \notin L(Y\,,\,y_\infty)$, contrary to hypothesis. This proves that $y'_\infty = x_\infty$.

Let $s \in L(X\,,\,x_\infty)$ such that $s(x_\infty) > 0$. Choose $r \in L(X\,,\,x_\infty)$ such that $r(x_\infty) > 0$ and $r(x_\infty)$ is so large that $r^*(y_\infty) > 0$ by $(*)$. Then $s^*(y_\infty)$ is necessarily positive; for otherwise we could repeat our argument with r and s in place of g_0 and g. It follows that for $s \in L(X\,,\,x_\infty)$, the inequality $s(x_\infty) > 0$ implies $s^*(y_\infty) > 0$. For the converse implication, reverse the roles of the spaces X and Y. □

Before we turn to locally compact Hausdorff spaces that are not compact, we offer one corollary.

Corollary 1. *Let X and Y be compact Hausdorff spaces, let $x_0 \in X$ and $y_0 \in Y$. Then a necessary and sufficient condition that there exists a homeomorphism $y \mapsto y'$ of Y onto X that maps y_0 to x_0 is that there exists a lattice isomorphism $f \mapsto f^*$ of $L(X\,,\,x_0)$ onto $L(Y\,,\,y_0)$.*

PROOF. Sufficiency. Theorem 1.
Necessity. For each $f \in L(X\,,\,x_0)$, put $f^*(y) = f(y')$. We leave the rest. □

We now come to the result we stated in our introductory comments.

Corollary 2. *Let U and V be locally compact Hausdorff spaces, not compact. Then a necessary and sufficient condition that U and V be homeomorphic is that the lattices $L(U)$ and $L(V)$ be isomorphic.*

PROOF. Let $X = U \cup \{x_\infty\}$ and $Y = V \cup \{y_\infty\}$ be the one point compactifications of U and V respectively where x_∞ and y_∞ are the points at infinity.

Sufficiency. Theorem 1.

Necessity. Let $y \mapsto y'$ be the homeomorphism. For $f \in L(U)$ put $f^*(y) = f(y')$. We leave the rest. □

Next we see how $C(X)$ and $L(V)$ compare when X is compact Hausdorff and V is only locally compact Hausdorff.

Corollary 3. *Let X be compact Hausdorff and V be locally compact Hausdorff but not compact. Then $C(X)$ and $L(V)$ are not isomorphic lattices.*

PROOF. Let $Y = V \cup \{y_\infty\}$ be the one point compactification of V. Use the construction in the proof of Theorem 1 to show that $C(X)$ and $L(V)$ can not be isomorphic lattices. (Just delete any references to x_∞.) $\square$

Say that a compact space X is homogeneous if for any $a, b \in X$, there is a homeomorphism of X onto X that maps a to b. For example, a circle is homogeneous but the compact interval $[0, 1]$ is not.

Corollary 4. *Let X be a compact Hausdorff space. Then X is homogeneous if and only if $L(X\,,\,a)$ and $L(X\,,\,b)$ are isomorphic lattices for any $a \in X$, $b \in X$.*

PROOF. Theorem 1. $\square$

We conclude with an example.

Example 1. Let U be the linearly ordered space consisting of the real line followed by all the countable ordinal numbers in their usual order. Let V be the linearly ordered space U with one final point p adjoined. In V every neighborhood of p contains an uncountable totally disconnected neighborhood of p. But U contains no such point, so U and V are not homeomorphic spaces. However both U and V are locally compact Hausdorff spaces that are not compact. From Theorem 1 we deduce that $L(U)$ and $L(V)$ are not isomorphic. On the other hand, the lattices $C(U)$ and $C(V)$ are essentially identical, and likewise $C^*(U)$ and $C^*(V)$ are essentially identical lattices.

Finally, let S be a locally compact Hausdorff space. Define $T(S)$ to be $L(S)$ if S is not compact, and $T(S)$ to be $C(S)$ if S is compact. From reference [1] and Corollaries 2 and 3 we deduce that any locally compact Hausdorff spaces S_1 and S_2 are homeomorphic if and only if their associated lattices $T(S_1)$ and $T(S_2)$ are isomorphic.

References

[1] I. Kaplansky, *Lattices of continuous functions*, Bull. Amer. Math. Soc. **53** (1947), 617–623.