RESEARCH

Real Analysis Exchange
Vol. 33(2), 2007/2008, pp. 279–284

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THE EQUIVALENCE RELATION OF BEING OF THE SAME KIND*

Abstract

Our purpose in this article is to prove that the equivalence relation of being of the same kind is not classifiable by countable structures.

1 Introduction.

A way to measure the complexity of an equivalence relation E defined on some Polish space X is to determine whether there exists a countable language L and a non-trivial Baire measurable function $f : X \to X_L$ with the property that

$$(\forall(x, y) \in X^2)(xEy \Rightarrow f(x) \cong f(y)). \qquad (\star)$$

Here X_L is the *Polish space of countably infinite structures for* L (see, for example, 16.5 on page 96 of [2]) and $\cong$ stands for the equivalence relation of isomorphism between structures for L, while $f : X \to X_L$ is said to be *trivial* if there exists a E-invariant comeager subset A of X for which all countable structures in $f[A]$ are isomorphic. When such a countable language L and such a non-trivial Baire measurable function $f : X \to X_L$ exist, we say that E is *classifiable by countable structures* and E is considered to be "less complicated" than the equivalence relation of isomorphism between countable structures. But if for any countable language L, every Baire measurable function $f : X \to X_L$ with the property $(\star)$ is trivial, then we say that E is not classifiable by

Key Words: same kind, turbulence
Mathematical Reviews subject classification: 03E15, 40A99
Received by the editors March 3, 2006
Communicated by: Steve Jackson
*Dedicated to the family of my brother Konstantinos Sofronidis

countable structures and E is considered to be "more complicated" than the equivalence relation of isomorphism between countable structures.

In what follows, let $\mathbf{P} = \{\mathbf{x} \in l^1 : (\forall n \in \mathbb{N})(\mathbf{x}(n) > 0)\}$. It is not difficult to see that $\mathbf{P}$ constitutes a G_δ subset of the separable Banach space l^1 and consequently it constitutes a Polish space, which we call the *Polish space of convergent series with positive terms.* (See, for example, 3.11 on page 17 of [2].) If $\mathbf{x} \in \mathbf{P}$ and for any $n \in \mathbb{N}$, we set $R_{\mathbf{x}}(n) = \sum_{m=n}^{\infty} \mathbf{x}(m)$, then we call $R_{\mathbf{x}}$ the *remainder sequence* of $\mathbf{x}$. A natural way to determine the relative rapidity of the convergence of two convergent series with positive terms is by examining the quotient of their remainder sequences. In particular, if $\mathbf{x} \in \mathbf{P}$ and $\mathbf{y} \in \mathbf{P}$, then the convergence of $\mathbf{x}$ is said to be of the *same kind* as that of $\mathbf{y}$, in symbols $\mathbf{x}E_{SK}\mathbf{y}$, if the following conditions hold: $\liminf_{n\to\infty} \frac{R_{\mathbf{y}}(n)}{R_{\mathbf{x}}(n)} > 0$ and $\limsup_{n\to\infty} \frac{R_{\mathbf{y}}(n)}{R_{\mathbf{x}}(n)} < \infty$. (See, for example, 162 on pages 279-280 of [4].) It is not difficult to prove that E_{SK} constitutes an equivalence relation and our purpose in this article is to prove the following result.

Theorem 1.1. *E_{SK} is not classifiable by countable structures.*

So, for convergent series with positive terms, the equivalence relation of being of the same kind is, in a sense, "more complicated" than the equivalence relation of isomorphism between countable structures.

2 The Theory of Turbulence.

A method to prove that an equivalence relation E defined on some Polish space X is not classifiable by countable structures is to show that there exists a Polish group G acting continuously on X with the following properties:

- $E_G^X \subseteq E$, where E_G^X is the corresponding *orbit equivalence relation*, namely $xE_G^X y \iff (\exists g \in G)(g \cdot x = y)$, whenever x, y are in X.
- The action of G on X is *generically turbulent.*

We explain what we mean below (see, for example, Chapter 3 on pages 37-58 of [1]):

Definition 2.1. *(Hjorth) Let G be any Polish group acting continuously on a Polish space X and let $x \in X$. For any open neighborhood U of x in X and for any symmetric open neighborhood V of 1^G in G, the (U,V)-local orbit $O(x,U,V)$ of x in X is defined as follows:*

> *$y \in O(x, U, V)$ if there exist g_0, ..., g_k in V ($k \in \mathbb{N}$) such that if $x_0 = x$ and $x_{i+1} = g_i \cdot x_i$ for every $i \in \{0, ..., k\}$, then all the x_i are in U and $x_{k+1} = y$.*

The action of G on X is said to be turbulent at the point x, in symbols $x \in T_G^X$, if for any such U and V, there exists an open neighborhood U' of x in X such that $U' \subseteq U$ and $O(x, U, V)$ is dense in U'.

Theorem 2.2. *(Hjorth) Let G be any Polish group acting continuously on a Polish space X in such a way that the orbits of the action are meager and at least one orbit is dense. Then the following are equivalent:*

- *The action of G on X is generically turbulent, in the sense that T_G^X is comeager in X.*
- *For any countable language L and for any Baire measurable function $f : X \to X_L$ with the property that $(\forall (x, y) \in X^2)(xE_G^X y \Rightarrow f(x) \cong f(y))$, there exists a E_G^X-invariant comeager subset A of X for which all countable structures in $f[A]$ are isomorphic.*

Indeed, if $f : X \to X_L$ has the property that $(\forall (x, y) \in X^2)(xEy \Rightarrow f(x) \cong f(y))$, then f has also the property that $(\forall (x, y) \in X^2)(xE_G^X y \Rightarrow f(x) \cong f(y))$ and, by virtue of Theorem 2.2, there exists a E_G^X-invariant comeager subset A of X for which all countable structures in $f[A]$ are isomorphic. So if we set $A^* = \{x \in X : (\exists a \in A)(xEa)\}$, then it is not difficult to verify that A^* constitutes a E-invariant comeager subset of X such that all countable structures in $f[A^*]$ are isomorphic.

3 The Proof of the Theorem.

By virtue of the discussion in Section 2, in order to prove Theorem 1.1, it is enough to show the following result.

Theorem 3.1. *If $\mathbf{G} = \left\{\mathbf{g} \in (0, \infty)^{\mathbb{N}} : \lim_{n\to\infty} \mathbf{g}(n) = 1\right\}$ and $(\mathbf{g} \cdot \mathbf{x})(n) = \mathbf{g}(n)\mathbf{x}(n)$, whenever $\mathbf{g} \in \mathbf{G}$, $\mathbf{x} \in \mathbf{P}$ and $n \in \mathbb{N}$, then the following are true:*

(i) *$\mathbf{G}$ constitutes a commutative Polish group under pointwise multiplication.*

(ii) *$\mathbf{G} \times \mathbf{P} \ni (\mathbf{g}, \mathbf{x}) \mapsto \mathbf{g} \cdot \mathbf{x} \in \mathbf{P}$ constitutes a continuous Polish group action.*

(iii) *The action of $\mathbf{G}$ on $\mathbf{P}$ is turbulent.*

(iv) *$E_{\mathbf{G}}^{\mathbf{P}} \subseteq E_{SK}$.*

The Proof of (i)

PROOF. It is well-known that $(0, \infty)$ constitutes a commutative Polish group under multiplication and if $d(x,y) = |x-y| + \left|\frac{1}{x} - \frac{1}{y}\right|$, whenever x and y are in $(0,\infty)$, then d constitutes a complete compatible metric on $(0,\infty)$. (See, for example, 9.A on page 58 of [2].) Given any $\mathbf{g} \in \mathbf{G}$ and any $\mathbf{h} \in \mathbf{G}$, we set $\rho(\mathbf{g},\mathbf{h}) = \sup_{n\in\mathbf{N}} d\left(\mathbf{g}(n),\mathbf{h}(n)\right)$ and it is not difficult to verify that ρ constitutes a metric on $\mathbf{G}$. So let $(\mathbf{g}_k)_{k\in\mathbb{N}}$ be any Cauchy sequence in $(\mathbf{G},\rho)$ and let $\epsilon > 0$. Then there exists $K \in \mathbb{N}$ such that for any integer $k \geq K$ and for any integer $l \geq K$, we have $|\mathbf{g}_k(n) - \mathbf{g}_l(n)| \leq d\left(\mathbf{g}_k(n),\mathbf{g}_l(n)\right) \leq \rho\left(\mathbf{g}_k,\mathbf{g}_l\right) < \frac{\epsilon}{2}$, whenever $n \in \mathbb{N}$. So for any $n \in \mathbb{N}$, $(\mathbf{g}_k(n))_{k\in\mathbf{N}}$ constitutes a Cauchy sequence in $((0,\infty),d)$ and consequently it has a limit, say $\mathbf{g}(n) = \lim_{k\to\infty} \mathbf{g}_k(n)$.

Moreover, since $\lim_{n\to\infty} \mathbf{g}_K(n) = 1$, there exists $N \in \mathbb{N}$ such that for any integer $n \geq N$, we have $|\mathbf{g}_K(n) - 1| < \frac{\epsilon}{2}$ and hence $|\mathbf{g}(n) - 1| = \lim_{l\to\infty} |\mathbf{g}_l(n) - 1| \leq \sup_{l\geq K} \left(|\mathbf{g}_l(n) - \mathbf{g}_K(n)| + |\mathbf{g}_K(n) - 1|\right) \leq \sup_{l\geq K} |\mathbf{g}_l(n) - \mathbf{g}_K(n)| + |\mathbf{g}_K(n) - 1| < \epsilon$, which implies that $\mathbf{g} \in \mathbf{G}$, while for any integer $k \geq K$ and for any $n \in \mathbb{N}$, we have $d\left(\mathbf{g}_k(n),\mathbf{g}(n)\right) = \lim_{l\to\infty} d\left(\mathbf{g}_k(n),\mathbf{g}_l(n)\right) \leq \frac{\epsilon}{2}$, hence $\rho\left(\mathbf{g}_k,\mathbf{g}\right) = \sup_{n\in\mathbb{N}} d\left(\mathbf{g}_k(n),\mathbf{g}(n)\right) \leq \frac{\epsilon}{2} < \epsilon$ and consequently $\mathbf{g}_k \to \mathbf{g}$ in $(\mathbf{G},\rho)$ as $k\to\infty$, which implies that ρ constitutes a complete metric on $\mathbf{G}$. If $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{h}$ are any elements of $\mathbf{G}$, then it is not difficult to prove that $\rho\left(\mathbf{f}^{-1},\mathbf{g}^{-1}\right) = \rho\left(\mathbf{f},\mathbf{g}\right)$ and $\rho\left(\mathbf{fh},\mathbf{gh}\right) \leq \max\left\{\sup_{n\in\mathbb{N}} \mathbf{h}(n), \sup_{n\in\mathbf{N}} \frac{1}{\mathbf{h}(n)}\right\} \rho\left(\mathbf{f},\mathbf{g}\right)$, which imply that inversion is continuous and multiplication is separately continuous and consequently $\mathbf{G}$ constitutes a topological group. (See, for example, 9.15 on page 62 of [2].)

What is left to show is that $(\mathbf{G},\rho)$ is separable. But it is not difficult to verify that $\mathcal{C} = \left\{\mathbf{g} \in (\mathbb{Q}\cap(0,\infty))^{\mathbb{N}} : \exists m \forall n \geq m(\mathbf{g}(n) = 1)\right\}$ constitutes a countable dense subset of $(\mathbf{G},\rho)$. Indeed, it is not difficult to see that $\mathcal{C}$ is equinumerous to the countable set $(\mathbb{Q}\cap(0,\infty))^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} (\mathbf{Q}\cap(0,\infty))^n$, while if $\mathbf{g} \in \mathbf{G}$ and $\epsilon > 0$, then since $\lim_{n\to\infty} \mathbf{g}(n) = 1$, and hence $\lim_{n\to\infty} \frac{1}{\mathbf{g}(n)} = 1$, there exists $N \in \mathbb{N}$ such that for any integer $n > N$, we have $|\mathbf{g}(n) - 1| < \frac{\epsilon}{2}$ and $\left|\frac{1}{\mathbf{g}(n)} - 1\right| < \frac{\epsilon}{2}$, which implies that $d\left(\mathbf{g}(n),1\right) < \epsilon$. Moreover, if $n \in \{0,...,N\}$, then since $\mathbb{Q}\cap(0,\infty)$ is dense in $(0,\infty)$, there exists an $r_n \in \mathbb{Q}\cap(0,\infty)$ such that $d\left(\mathbf{g}(n),r_n\right) < \epsilon$. So if $\mathbf{c} = (r_0,...,r_N,1,1,1,...)$, then $\mathbf{c} \in \mathcal{C}$ and $\rho\left(\mathbf{g},\mathbf{c}\right) \leq \epsilon$. $\square$

The Proof of (ii)

Proof. If $\mathbf{g} \in \mathbf{G}$ and $\mathbf{x} \in \mathbf{P}$, then $\|\mathbf{g} \cdot \mathbf{x}\|_1 \leq \left(\sup\limits_{n\in\mathbb{N}} \mathbf{g}(n)\right) \|\mathbf{x}\|_1$ and consequently $\mathbf{g} \cdot \mathbf{x} \in \mathbf{P}$. So the map $\mathbf{G} \times \mathbf{P} \ni (\mathbf{g}, \mathbf{x}) \mapsto \mathbf{g} \cdot \mathbf{x} \in \mathbf{P}$ is well-defined and it is not difficult to verify that it constitutes a group action. Moreover, if $\mathbf{g}$ and $\mathbf{h}$ are any elements of $\mathbf{G}$, while $\mathbf{x}$ and $\mathbf{y}$ are any elements of $\mathbf{P}$, then $\|\mathbf{g} \cdot \mathbf{x} - \mathbf{h} \cdot \mathbf{x}\|_1 \leq \rho(\mathbf{g}, \mathbf{h}) \|\mathbf{x}\|_1$ and $\|\mathbf{g} \cdot \mathbf{y} - \mathbf{g} \cdot \mathbf{x}\|_1 \leq \left(\sup\limits_{n\in\mathbb{N}} \mathbf{g}(n)\right) \|\mathbf{y} - \mathbf{x}\|_1$, which imply that the group action in question constitutes a continuous action. (See, for example, 9.14 on page 62 of [2].) □

The Proof of (iii)

Lemma 3.2. *For any* $\mathbf{x} \in \mathbf{P}$*,* $\mathbf{G} \cdot \mathbf{x}$ *is dense in* $\mathbf{P}$*.*

Proof. It is enough to notice that if $\mathbf{y} \in \mathbf{P}$ and $N \in \mathbb{N}$, while

$$\mathbf{g}_N(n) = \begin{cases} \frac{\mathbf{y}(n)}{\mathbf{x}(n)} & \text{if } n \in \{0, ..., N\} \\ \\ 1 & \text{if } n \in \mathbb{N} \setminus \{0, ..., N\} \end{cases}$$

then $\mathbf{g}_N \in \mathbf{G}$ and $\|\mathbf{g}_N \cdot \mathbf{x} - \mathbf{y}\|_1 = \sum\limits_{n>N} |\mathbf{x}(n) - \mathbf{y}(n)| \to 0$ as $N \to \infty$. □

Lemma 3.3. *For any* $\mathbf{x} \in \mathbf{P}$*,* $\mathbf{G} \cdot \mathbf{x}$ *is meager in* $\mathbf{P}$*.*

Proof. If $\mathbf{y} \in \mathbf{G} \cdot \mathbf{x}$, then it is not difficult to see that $\lim\limits_{n\to\infty} \frac{\mathbf{y}(n)}{\mathbf{x}(n)} = 1$ and consequently there exists $m \in \mathbb{N}$ such that for any integer $n \geq m$, we have $\frac{\mathbf{y}(n)}{\mathbf{x}(n)} \leq \frac{3}{2}$. So $\mathbf{G} \cdot \mathbf{x} \subseteq \mathcal{M}$, where $\mathcal{M} = \bigcup\limits_{m\in\mathbb{N}} \bigcap\limits_{n\geq m} \left\{\mathbf{y} \in \mathbf{P} : \frac{\mathbf{y}(n)}{\mathbf{x}(n)} \leq \frac{3}{2}\right\}$ is easily seen to be F_σ. So it is enough to show that $\mathbf{P} \setminus \mathcal{M}$ is dense in $\mathbf{P}$. Indeed, if $\mathbf{z} \in \mathbf{P}$ and $N \in \mathbb{N}$, while

$$\mathbf{z}_N(n) = \begin{cases} \mathbf{z}(n) & \text{if } n \in \{0, ..., N\} \\ \\ 2\mathbf{x}(n) & \text{if } n \in \mathbb{N} \setminus \{0, ..., N\}, \end{cases}$$

then it is enough to notice that $\mathbf{z}_N \in \mathbf{P} \setminus \mathcal{M}$ and $\|\mathbf{z}_N - \mathbf{z}\|_1 = \sum\limits_{n>N} |2\mathbf{x}(n) - \mathbf{z}(n)| \to 0$ as $N \to \infty$. □

If for an arbitrary $\mathbf{x} \in \mathbf{P}$ and for an arbitrary $\epsilon > 0$, we set $U(\mathbf{x}, \epsilon) = \{\mathbf{y} \in \mathbf{P} : \|\mathbf{y} - \mathbf{x}\|_1 < \epsilon\}$, then it is not difficult to see that the $U(\mathbf{x}, \epsilon)$ form a base of open neighborhoods of $\mathbf{x}$ in $\mathbf{P}$.

Lemma 3.4. *If $\mathbf{x} \in \mathbf{P}$ and $\epsilon > 0$, while $\mathbf{g} \in \mathbf{G}$ and $\mathbf{g} \cdot \mathbf{x} \in U(\mathbf{x}, \epsilon)$, then there exists a continuous path $[0, 1] \ni t \mapsto \mathbf{g}_t \in \mathbf{G}$ such that $\mathbf{g}_0 = 1^{\mathbf{G}}$, $\mathbf{g}_1 = \mathbf{g}$ and $\mathbf{g}_t \cdot \mathbf{x} \in U(\mathbf{x}, \epsilon)$ for every $t \in [0, 1]$.*

PROOF. Given any $t \in [0, 1]$, we set $\mathbf{g}_t = (1 - t)1^{\mathbf{G}} + t\mathbf{g}$ and it is not difficult to verify that $\mathbf{g}_t \in \mathbf{G}$, while obviously $\mathbf{g}_0 = 1^{\mathbf{G}}$ and $\mathbf{g}_1 = \mathbf{g}$. Moreover, if s, t are in $[0, 1]$, then it is not difficult to prove that $\rho(\mathbf{g}_s, \mathbf{g}_t) \leq |s - t| \sup_{n \in \mathbb{N}} \left(|\mathbf{g}(n) - 1| + \frac{|\mathbf{g}(n)-1|}{(\min\{1, \mathbf{g}(n)\})^2} \right)$ and consequently $[0, 1] \ni t \mapsto \mathbf{g}_t \in \mathbf{G}$ is continuous. What is left to show is that $\mathbf{g}_t \in U(\mathbf{x}, \epsilon)$ for every $t \in [0, 1]$. But this follows from the fact that for any $t \in [0, 1]$, we have $\|\mathbf{g}_t \cdot \mathbf{x} - \mathbf{x}\|_1 = t \|\mathbf{g} \cdot \mathbf{x} - \mathbf{x}\|_1$. $\square$

Now, Lemmas 3.3 – 3.4 and Lemma 5.7 on page 1472 of [3] imply that the action of $\mathbf{G}$ on $\mathbf{P}$ is turbulent.

The Proof of (iv)

PROOF. If $\mathbf{x} \in \mathbf{P}$, $\mathbf{y} \in \mathbf{P}$, $\mathbf{g} \in \mathbf{G}$ and $\mathbf{y} = \mathbf{g} \cdot \mathbf{x}$, then $\lim_{n \to \infty} \frac{\mathbf{y}(n)}{\mathbf{x}(n)} = 1$, hence there exists $m \in \mathbb{N}$ such that for any integer $n \geq m$, we have $\left| \frac{\mathbf{y}(n)}{\mathbf{x}(n)} - 1 \right| < \frac{1}{2}$, hence $\frac{1}{2}\mathbf{x}(n) < \mathbf{y}(n) < \frac{3}{2}\mathbf{x}(n)$ and consequently $\frac{1}{2}R_{\mathbf{x}}(n) \leq R_{\mathbf{y}}(n) \leq \frac{3}{2}R_{\mathbf{x}}(n)$, which implies that $\frac{1}{2} \leq \frac{R_{\mathbf{y}}(n)}{R_{\mathbf{x}}(n)} \leq \frac{3}{2}$, which implies in its turn that $\mathbf{y} E_{SK} \mathbf{x}$. $\square$

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