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Lee Tuo-Yeong, Mathematics and Mathematics Education Academic Group, National Institute of Education, Nanyang Technological University, 1 Nanyang Walk, Singapore 637616, Republic of Singapore. e-mail: `tylee@nie.edu.sg`

A SOLUTION TO PFEFFER'S PROBLEM

Abstract

We give an example of a non-integrable function f on $[0,1]\times[0,1]$ such that

$$\int_\alpha^\beta (\int_\gamma^\delta f(x,y)\,dy)\,dx = \int_\gamma^\delta (\int_\alpha^\beta f(x,y)\,dx)\,dy$$

for each subinterval $[\alpha,\beta]\times[\gamma,\delta]$ of $[0,1]\times[0,1]$.

In [9, problem 6.2], Pfeffer posed the following problem:

Problem. *Let f be a function on an interval $A=[a,b]\times[c,d]$, and*

$$\int_\alpha^\beta (\int_\gamma^\delta f(x,y)\,dy)\,dx = \int_\gamma^\delta (\int_\alpha^\beta f(x,y)\,dx)\,dy$$

for each subinterval $[\alpha,\beta]\times[\gamma,\delta]$ of A. Is f integrable in A? If so is

$$\int_A f = \int_a^b (\int_c^d f(x,y)\,dy)\,dx?$$

In the above problem, "integrable" is understood to be in the sense of [8, Definition 3.1]. For other equivalent definitions of this integral, see [4]. By taking A to be the unit square $[0,1]\times[0,1]$, we give a negative answer to the above problem.

Theorem. *There exists a non-integrable function f on $[0,1]\times[0,1]$ such that*

$$\int_\alpha^\beta (\int_\gamma^\delta f(x,y)\,dy)\,dx = \int_\gamma^\delta (\int_\alpha^\beta f(x,y)\,dx)\,dy$$

for each subinterval $[\alpha,\beta]\times[\gamma,\delta]$ of $[0,1]\times[0,1]$.

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PROOF. We shall construct a sequence of subintervals of $[0,1]\times[0,1]$ similar to that of [7, Lemma 4.1]. Then we construct the required function f. For each positive integer n, we let $a_n = \sum_{k=1}^{n}(\frac{1}{2})^k$, $c_n = \frac{1}{2}(a_n + a_{n+1})$ and $I_n = [a_n, a_{n+1}]\times[a_n, a_{n+1}]$. Define $f_n : I_n \to \mathbb{R}$ by

$$f_n(x,y) = \begin{cases} \dfrac{4}{(a_{n+1}-a_n)^2} & \text{if } (x,y) \in ((a_n,c_n)\times(a_n,c_n)) \\ & \cup((c_n,a_{n+1})\times(c_n,a_{n+1})) \\ \dfrac{-4}{(a_{n+1}-a_n)^2} & \text{if } (x,y) \in ((a_n,c_n)\times(a_n,c_{n+1})) \\ & \cup((c_n,a_{n+1})\times(c_n,a_n)) \\ 0 & \text{otherwise.} \end{cases}$$

Now define $f : [0,1]\times[0,1] \longrightarrow \mathbb{R}$ by

$$f(x,y) = \begin{cases} f_n(x,y) & \text{if } (x,y) \in I_n \text{ for some positive integer } n \\ 0 & \text{otherwise.} \end{cases}$$

By our definition of f, we have

$$\Big| \int_{[a_n,c_n]\times[a_n,c_n]} f \Big| = 1 \tag{1}$$

for all positive integers n. Since the two-dimensional Lebesgue measure of $[a_n,c_n]\times[a_n,c_n]$ tends to zero as $n\to\infty$, it follows from [8, Proposition 4.10] and (1) that f is not integrable on $[0,1]\times[0,1]$. However, we shall prove that

$$\int_\alpha^\beta \Big(\int_\gamma^\delta f(x,y)\,dy\Big)\,dx = \int_\gamma^\delta \Big(\int_\alpha^\beta f(x,y)\ dx\Big)\,dy \tag{2}$$

for each subinterval $[\alpha,\beta]\times[\gamma,\delta]$ of $[0,1]\times[0,1]$.

If $[\alpha,\beta]\times[\gamma,\delta] \subset [0,1]\times[0,1]$ with $\big([\alpha,\beta]\times[\gamma,\delta]\big)\cap\{(1,1)\} = \emptyset$, then f is Lebesgue integrable on $[\alpha,\beta]\times[\gamma,\delta]$, so (**??**) is true for all such subintervals of $[0,1]\times[0,1]$. It remains to verify that

$$\int_\alpha^1 \Big(\int_\gamma^1 f(x,y)\,dy\Big)\,dx = \int_\gamma^1 \Big(\int_\alpha^1 f(x,y)\ dx\Big)\,dy \tag{3}$$

Let q be the minimum positive integer such that $\big((\alpha,1]\times(\gamma,1]\big)\cap I_q \neq \emptyset$. If $a_i \le x \le a_{i+1}$ for some $i > q$, then we have

$$\int_\gamma^1 f(x,y)\ dy = \int_{a_{i+1}}^1 f(x,y)\ dy + \int_{a_i}^{a_{i+1}} f(x,y)\ dy + \int_\gamma^{a_i} f(x,y)\,dy = 0.$$

If $x \in [a_q, a_{q+1}]$, then we have

$$\int_\gamma^1 f(x,y)\ dy = \int_{a_{q+1}}^1 f(x,y)\ dy + \int_\gamma^{a_{q+1}} f(x,y)\ dy = \int_\gamma^{a_{q+1}} f(x,y)\ dy.$$

Thus we have

$$\int_\gamma^1 f(x,y)\ dy = \begin{cases} \int_\gamma^{a_{q+1}} f(x,y)\ dy & \text{if } x \in [a_q, a_{q+1}] \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, we have

$$\int_\alpha^1 f(x,y)\, dx = \begin{cases} \int_\alpha^{a_{q+1}} f(x,y)\, dx & \text{if } y \in [a_q, a_{q+1}] \\ 0 & \text{otherwise.} \end{cases}$$

From the above computations, we have

$$\int_\alpha^1 (\int_\gamma^1 f(x,y)\, dy)\, dx = \int_\alpha^{a_{q+1}} (\int_\gamma^{a_{q+1}} f(x,y)\ dy)\, dx$$

and

$$\int_\gamma^1 (\int_\alpha^1 f(x,y)\, dx)\, dy = \int_\gamma^{a_{q+1}} (\int_\alpha^{a_{q+1}} f(x,y)\, dx)\, dy.$$

So (3) is true. Thus f is the function with the desired properties. $\square$

Remark The above function f is neither M_1 (see [2]) nor strongly ρ-integrable (see [5]) on $[0,1] \times [0,1]$. In particular, this function cannot be Henstock-Kurzweil integrable on $[0,1] \times [0,1]$ (see [6]). If a function is integrable in the sense of [8, Definition 3.1], then it is shown in [4] that it is also ρ-integrable (see [3]) there with the same integral value. Must this function be strongly ρ-integrable on this interval?

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1. Assuming that the double integral and the iterated integrals exist, do they have the same value?

2. Assuming that the double integral exists and the iterated integrals are equal, is their common value equal to the double integral?

References

[1] Z. Buczolich. Thierry De Pauw and W.F. Pfeffer, Charges, BV functions, and Multipliers for Generalized Riemann integrals, *Indiana Univ. Math. J.,* Vol **48**(1999), 1471−1511.

[2] J. Jarńik, J. Kurzweil, S. Schwabik, On Mawhin's approach to multiple nonabsolutely convergent integral, Časopis pěst Mat. **108** (1983), 356−380.

[3] J. Jarnik and J. Kurzweil, Perron-type integration on n-dimensional intervals and its properties, *Czechoslovak Math. J.* **45**(**120**) (1995), no. 1, 79−106.

[4] J. Kurzweil and J. Jarńik, Equivalent definitions of regular generalized Perron integral, Czech. Math. J. **42** (117) (1992), 365−378.

[5] J. Kurzweil and J. Jarńik, Perron-type integration on n-dimensional intervals as an extension of integration of stepfunctions by strong equiconvergence, Czech. Math. J. **46** (121) (1996), 1−20.

[6] Lee Peng Yee and Rudolf Vyborny, The integral, An Easy Approach after Kurzweil and Henstock, Australian Mathematical Society Lecture Series 14, Cambridge University Press 2000.

[7] Lee Tuo Yeong, Chew Tuan Seng and Lee Peng Yee, Characterisation of multipliers for the double Henstock integrals, Bull. Austral. Math Soc **54** (1996), 441− 449.

[8] W.F. Pfeffer, The divergence Theorem, Trans. Amer. Math. Soc. **295** (1986), 665−685.

[9] W.F. Pfeffer, A Riemann type integration and the fundamental Theorem of Calculus, Rend. Circ. Mat. Palermo **36** (1987), 482−506.