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Bożena Szkopińska, Institute of Mathematics,Technical University of Łódź, Poland. e-mail: `bozenasz@ck-sg.p.lodz.pl`

FUNCTIONS $\mathcal{I}$-APPROXIMATELY CONTINUOUS IN $\mathcal{I}_1$ A. E. DIRECTION AT EVERY POINT

Abstract

In this paper we shall show that every function $f\colon \Re^2 \to \Re$ having the Baire property and $\mathcal{I}$-approximately continuous in $\mathcal{I}_1$-almost everywhere direction at every point is of the first class of Baire.

Let $\mathbb{R}^k$ denote the k-dimensional Euclidean space, $k = 1, 2$, $\mathcal{N}$ the set of positive integers, $\mathcal{W}$ the set of rational numbers and $\mathbb{R}_+$ the set of positive real numbers. The ball centred at a point p and with radius $r > 0$ will be denoted by $K(p, r)$. We introduce the following notations:

$\mathcal{S}_k$ – the σ-field of subsets of $\mathbb{R}^k$ with the Baire property,

$\mathcal{I}_k$ – the σ-ideal of first category subsets of $\mathbb{R}^k$.

A set $A \subset \mathbb{R}^k$ is $\mathcal{S}_k$-measurable if and only if $A \in \mathcal{S}_k$. For $A \in \mathcal{S}_1$, denote by $\phi(A)$ the set of all $\mathcal{I}_1$-density points of A. It is known [3] that the mapping $\phi\colon \mathcal{S}_1 \to 2^{\mathbb{R}^1}$ is a lower density operator. If a plane set A is contained in a line, then we use linear $\mathcal{I}_1$-density points of the set $A \in \mathcal{S}_1$.

Let L_θ $(L_\theta(x, y), respectively)$ denote a line passing through the point $(0, 0)$ (respectively, the point (x, y)) and forming an angle θ with the x-axis for $\theta \in [0, \pi)$. Denote by $A \triangle B$ the symmetric difference of A and B. If $A, B \in \mathcal{S}_k$, then $A \sim B$ means that $A \triangle B \in \mathcal{I}_k$, $k = 1, 2$.

Set $M \in \mathcal{S}_2$ and $\theta \in [0, \pi)$. Put

$$S_\theta(M) = \{(x, y) \in \mathbb{R}^2 \colon \exists_{r \in \mathbb{R}_+} M \cap L_\theta(x, y) \cap K((x, y), r) \in \mathcal{S}_1\},$$

$$\Phi_\theta(M) = \{(x, y) \in S_\theta(M) \colon (x, y) \in \phi(M \cap L_\theta(x, y))\}$$

and

$$\Phi(M) = \{(x, y) \in \mathbb{R}^2 \colon (x, y) \in \Phi_\theta(M) \text{ in } \mathcal{I}_1\text{-almost every direction } \theta \in [0, \pi)\}.$$

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Definition 1. *A point $(x,y)\in\mathbb{R}^2$ is an $\mathcal{I}_1$-density point of M in the direction θ if and only if $(x,y)\in\Phi_\theta(M)$.*

By [2] we have the following

Proposition 2. *For any $\theta\in[0,\pi)$ and $M\subset\mathbb{R}^2$, $(x,y)\in\Phi_\theta(M)$ if and only if $M\cap L_\theta(x,y)\in\mathcal{S}_1$ and, for each $n\in\mathcal{N}$, there exist $k,p\in\mathcal{N}$ such that, for any $h\in\left(0,\frac{1}{p}\right)$ and $i\in\{1,\dots,n\}$, there exist $j_1,j_2\in\{1,\dots,k\}$ such that*

$$\left\{(x+t\cos\theta,y+t\sin\theta)\colon t\in\left(\frac{(i-1)k+j_1-1}{kn}h,\frac{(i-1)k+j_1}{kn}h\right)\right\}\setminus(M\cap L_\theta(x,y))\in\mathcal{I}_1$$

and

$$\left\{(x+t\cos\theta,y+t\sin\theta)\colon t\in\left(-\frac{(i-1)k+j_2}{kn}h,-\frac{(i-1)k+j_2-1}{kn}h\right)\right\}\setminus(M\cap L_\theta(x,y))\in\mathcal{I}_1.$$

By [1] we have the following three theorems:

Theorem 3. *For all $A,B\in\mathcal{S}_2$,*

I. $\Phi(A)\sim A$,

II. $A\sim B\Rightarrow\Phi(A)=\Phi(B)$,

III. $\Phi(\emptyset)=\emptyset$, $\Phi(\mathbb{R}^2)=\mathbb{R}^2$,

IV. $\Phi(A\cap B)=\Phi(A)\cap\Phi(B)$.

Theorem 4. *The family $\tau=\{A\in\mathcal{S}_2\colon A\subset\Phi(A)\}$ is a topology in $\mathbb{R}^2$.*

Theorem 5. *For any $P\in\mathcal{I}_2$ and $(x,y)\in\mathbb{R}^2$ there exists a $\Theta\subset[0,\pi)$ such that $[0,\pi)\setminus\Theta\in\mathcal{I}_1$ and, for each $\theta\in\Theta$, $P\cap L_\theta(x,y)\in\mathcal{I}_1$.*

Definition 6. *A function $f\colon\mathbb{R}^2\to\mathbb{R}$ is $\mathcal{I}$-approximately continuous in $\mathcal{I}_1$-almost every direction at every point if and only if the functions f is a continuous with respect to the topology τ.*

Theorem 7. *Every function $f\colon\mathbb{R}^2\to\mathbb{R}$ having the Baire property and which is $\mathcal{I}$-approximately continuous in $\mathcal{I}_1$-almost every direction at every point is of the first class of Baire.*

Proof. Suppose the assertion of the theorem is false. Then we may assume that there exists a perfect set $T\subset\mathbb{R}^2$ and two real numbers a,b such that the set $E_a=\{(x,y)\colon f(x,y)<a\}$ is a second category subset of T and the set

$E^b = \{(x,y)\colon f(x,y) > b\}$ is a dense subset of T. By our assumption, there exists a residual set $P \subset \mathbb{R}^2$ such that the function $f_{|P}$ is continuous on P.

We introduce the following notation: for any $p, n \in \mathcal{N}$, and for each $(x,y) \in \mathbb{R}^2$, let $\Gamma(p,n,(x,y))$ be the set of all $\theta \in [0,\pi)$ such that for each $h \in \left(0, \frac{1}{p}\right)$ there exists an $i \in \{1,\ldots,n\}$ such that

$$\left\{(x + t\cos\theta, y + t\sin\theta)\colon t \in \left(\frac{i-1}{n}h, \frac{i}{n}h\right)\right\} \setminus (E_a \cap P \cap L_\theta(x,y)) \in \mathcal{I}_1$$

and, for any $\alpha, \beta \in [0,\pi)$ let

$$H(\alpha,\beta,p,n) = \{(x,y)\colon \Gamma(p,n,(x,y)) \text{ is a dense subset of } (\alpha,\beta) \subset [0,\pi)\}.$$

We shall show that

$$E_a \subset \bigcup_{n\in\mathcal{N}} \bigcup_{p\in\mathcal{N}} \bigcup_{\alpha\in\mathcal{W}} \bigcup_{\beta\in\mathcal{W}} H(\alpha,\beta,p,n).$$

Let $(x,y) \in E_a$. Then there exists a set $\Theta_1 \subset [0,\pi)$ such that $[0,\pi) \setminus \Theta_1 \in \mathcal{I}_1$ and for each $\theta \in \Theta_1$, $E_a \cap L_\theta(x,y) \in S_1$ and (x,y) is an $\mathcal{I}$-density point of the set E_a in the direction θ. Additionally, by $\mathbb{R}^2 \setminus P \in \mathcal{I}_2$, there exists a set $\Theta_2 \subset [0,\pi)$ such that $[0,\pi) \setminus \Theta_2 \in \mathcal{I}_1$ and for each $\theta \in \Theta_2$, $L_\theta(x,y) \setminus P \in \mathcal{I}_1$.

We put $\Theta = \Theta_1 \cap \Theta_2$. Then, for each $\theta \in \Theta$, there exist $n, p \in \mathcal{N}$ such that, for any $h \in \left(0, \frac{1}{p}\right)$ and $i \in \{1,\ldots,n\}$,

$$\left\{(x + t\cos\theta, y + t\sin\theta)\colon t \in \left(\frac{i-1}{n}h, \frac{i}{n}h\right)\right\} \setminus (E_a \cap P \cap L_\theta(x,y)) \in \mathcal{I}_1.$$

Thus

$$\Theta \subset \bigcup_{n\in\mathcal{N}} \bigcup_{p\in\mathcal{N}} \Gamma(p,n,(x,y)).$$

Therefore there exist two rational numbers $\alpha, \beta \in [0,\pi) \cap \mathcal{W}$ and two numbers $n, p \in \mathcal{N}$ such that $\Gamma(p,n,(x,y))$ is dense in (α,β).

Since $E_a \cap T$ is a second catgory subset of T,

$$E_a \cap T \subset \bigcup_{n\in\mathcal{N}} \bigcup_{p\in\mathcal{N}} \bigcup_{\alpha\in\mathcal{W}} \bigcup_{\beta\in\mathcal{W}} H(\alpha,\beta,p,n) \cap T$$

and $E^b \cap T$ is a dense subset of T we infer the existence of a point $(x_0,y_0) \in T \cap E^b$, a sequence $\{(x_k,y_k)\}_{k\in\mathcal{N}} \subset T \cap E_a$, two rational numbers α, β and two positive integers n,p such that $(x_k,y_k)\to(x_0,y_0)$ and, for each $k\in\mathcal{N}$, $\Gamma(p,n,(x_k,y_k))$ is a dense subset of (α,β).

As above we can show there exists a set $\Theta^*\subset[0,\pi)$ such that $[0,\pi)\setminus\Theta^*\in\mathcal{I}_1$ and for each $\theta\in\Theta^*$, (x_0,y_0) is an $\mathcal{I}_1$-density point of $E^b\cap P$ in the direction θ. Let $\theta\in(\alpha,\beta)\cap\Theta^*$. Then there exists $\delta\in\mathbb{R}$ such that, for any $h\in(0,\delta)$ and $i\in\{1,\dots,n\}$,

$$E^b\cap P\cap L_\theta(x_0,y_0)\cap\left\{(x_0+t\cos\theta,y_0+t\sin\theta)\colon t\in\left(\frac{i-1}{n}h,\frac{i}{n}h\right)\right\}\notin\mathcal{I}_1.$$

Let $h<\min\{\frac{1}{p},\delta\}$. Since $f_{|P}$ continuous on P, for each $i\in\{1,\dots,n\}$, there exist

$$(u_i,v_i)\in\left\{(x_0+t\cos\theta,y_0+t\sin\theta)\colon t\in\left(\frac{i-1}{n}h,\frac{i}{n}h\right)\right\}$$

and $r_i>0$ such that $P\cap K((u_i,v_i),r_i)\subset P\cap E^b$. Additionally, since $(x_k,y_k)\to(x_0,y_0)$, there exists $k\in\mathcal{N}$, such that, for each $i\in\{1,\dots,n\}$,

$$\left\{(x_k+t\cos\theta,y_k+t\sin\theta)\colon t\in\left(\frac{i-1}{n}h,\frac{i}{n}h\right)\right\}\cap K((u_i,v_i),r_i)\neq\emptyset.$$

We know that $\Gamma(p,n,(x_k,y_k))$ is a dense subset of (α,β). Thus there exists a $\theta_k\in\Gamma(p,n,(x_k,y_k))$ such that, for each $i\in\{1,\dots,n\}$,

$$\left\{(x_k+t\cos\theta_k,y_k+t\sin\theta_k)\colon t\in\left(\frac{i-1}{n}h,\frac{i}{n}h\right)\right\}\cap K((u_i,v_i),r_i)\neq\emptyset$$

and, for θ_k, there exists an $i_0\in\{1,\dots,n\}$ such that

$$\left\{(x_k+t\cos\theta_k,y_k+t\sin\theta_k)\colon t\in\left(\frac{i_0-1}{n}h,\frac{i_0}{n}h\right)\right\}\setminus(E_a\cap P\cap L_{\theta_k}(x_k,y_k))\in\mathcal{I}_1.$$

Let $t_0\in[\frac{i_0-1}{n}h,\frac{i_0}{n}h]$ be such that

$$(x_k+t_0\cos\theta_k,y_k+t_0\sin\theta_k)\in P\cap E_a\cap K((u_{i_0},v_{i_0}),r_{i_0}).$$

Then there exists an $r>0$ such that

$$K((x_k+t_0\cos\theta_k,y_k+t_0\sin\theta_k),r)\cap P\subset K((u_{i_0},v_{i_0}),r_i)\cap P\cap E_a\subset E^b\cap E_a,$$

which is a contradiction. Thus the proof of the theorem is completed.

References

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