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HENSTOCK-STIELTJES INTEGRAL NOT INDUCED BY MEASURE

Abstract

The present paper concerns with the introduction of a new type of generalized Stieltjes integral with an integrator function which depends on multiple points in a division and cannot be induced by a measure. Some properties of this integral were studied.

1 Introduction

The Stieltjes integral integrates a function f with respect to another function g on $[a,b]$. In other words, the integral is approximated by Stieltjes sums $(D)\sum f(\xi)g(u,v)$ using Henstock notation where D is a division of $[a,b]$ and $g(u,v) = g(v) - g(u)$. In the literature there are many Stieltjes-type integrals where the integrator functions g in the Stieltjes sums are taken to depend on more than the endpoints of interval $[u,v]$, for example RG_k, RS_k^* as in [1,2]. Motivated by an attempt to present a uniform approach to these integrals, we here define an integral called GR_k integral, where GR refers to generalized Riemann. In the GR_k integral the integrator is a function from $[a,b]^{k+1}$ to R. Also a new concept of jump is introduced which plays an important role in formulating the properties of the integral. The GR_k integral includes the classical Stieltjes integral as a special case.

We shall generalize the δ-fine division of Henstock [3]. Using it we shall define our new integral. Let k be a fixed positive integer and δ a positive function defined on $[a,b]$. We shall call a division D of $[a,b]$ given by $a = x_0 < x_1 < \cdots < x_n = b$ with associated points $\{\xi_0, \xi_1, \cdots, \xi_{n-k}\}$ satisfying

$$\xi_i \in [x_i, x_{i+k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \text{ for } i = 0, 1, \cdots, n-k,$$

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a δ^k-*fine division* of $[a, b]$. For a given positive function δ, we denote a δ^k-fine division D by $\{([x_i, x_{i+k}], \xi_i)\}_{i=0,1,\cdots,n-k}$. When $k = 1$, it coincides with the usual definition of a δ-fine division.

Let g be a real-valued function defined on a closed interval $[a, b]^{k+1}$ in the (k+1)-dimensional space, and f a real-valued function defined on $[a, b]$. We say that f is GR_k *integrable* with respect to g to I on $[a, b]$ if for every $\epsilon > 0$ there is a function $\delta(\xi) > 0$ such that for any δ^k-fine division $D = \{([x_i, x_{i+k}], \xi_i)\}_{i=0,1,\cdots,n-k}$ we have

$$\Big|\sum_{i=0}^{n-k} f(\xi_i)g(x_i, \cdots, x_{i+k}) - I\Big| < \epsilon.$$

We shall denote the above Riemann sum by $s(f, g; D)$. If f is integrable with respect to g in the above sense, we write $(f, g) \in GR_k[a, b]$ and denote the integral by $\int_a^b f dg$.

Let $x \in [x_i, x_{i+k}]$ where $x_i < x_{i+1} < \cdots < x_{i+k}$. The *jump* of g at x, denoted by $J(g; x)$, is defined by

$$J(g; x) = \lim_{x_i \to x, x_{i+k} \to x} g(x_i, \cdots, x_{i+k}),$$

if the limit exists finitely. Next, let $[a_i, b_i]$, $i = 1, 2, \cdots, p$, be pairwise non-overlapping, and $\cup_{i=1}^p [a_i, b_i] \subset [a, b]$. Then $\{D_i\}_{i=1,2,\cdots,p}$ is said to be a δ^k-fine *partial division* of $[a, b]$ if each D_i is a δ^k-fine division of $[a_i, b_i]$. Its corresponding partial Riemann sum is given by $\sum_{i=1}^p s(f, g; D_i)$.

We remark that some authors [1] defined Riemann sums over $[a', b']$ with $a' < a < b < b'$, $a' \to a$, and $b' \to b$. Eventually it reduces to the case on $[a, b]$. Hence for simplicity we consider only $[a, b]$. Note that using compactness of $[a, b]$ it is easy to show that for a positive function δ there exists a δ^k-fine division of $[a, b]$. For $k = 1$ and $g(u, v) = \alpha(v) - \alpha(u)$, we obtain the Henstock-Stieltjes integral. Also for $g(x_i, x_{i+1}, \cdots, x_{i+k}) = (x_{i+k} - x_i)Q_k(\alpha; x_i, x_{i+1}, \cdots, x_{i+k})$ where $Q_k(\alpha; x_i, x_{i+1}, \cdots, x_{i+k})$ is the k-th divided difference of α, the GR_k integral includes the RS_k^* integral of Russell [5] and its corresponding Henstock type version. See Section 3 for other examples.

2 Some Properties

We shall prove some simple properties of the integral. The first theorem follows readily from the definition.

Theorem 2.1. *Let $(f_i, g) \in GR_k[a,b]$ and $(f, g_i) \in GR_k[a,b]$ for $i = 1, 2, \cdots, n$. Then for real numbers $\lambda_1, \lambda_2, \cdots, \lambda_n$ we have*

(i) $(\sum_{i=1}^n \lambda_i f_i, g) \in GR_k[a,b]$ *and* $\int_a^b \sum_{i=1}^n (\lambda_i f_i) dg = \sum_{i=1}^n \lambda_i (\int_a^b f_i dg)$.

(ii) $(f, \sum_{i=1}^n \lambda_i g_i) \in GR_R[a,b]$ *and* $\int_a^b f d(\sum_{i=1}^n \lambda_i g_i) = \sum_{i=1}^n \lambda_i \int_a^b f dg_i$.

(iii) If $f_1(x) \leq f_2(x)$ *for all* $x \in [a,b]$ *and* $g : [a,b]^{k+1} \to [0, \infty)$, *then* $\int_a^b f_1 dg \leq \int_a^b f_2 dg$.

Theorem 2.2. *Let $(f, g) \in GR_k[a,c]$ and $(f, g) \in GR_k[c,b]$. If $J(g;c)$ exists then $(f,g) \in GR_k[a,b]$ and*

$$\int_a^b f dg = \int_a^c f dg + \int_c^b f dg + (k-1) f(c) J(g;c).$$

PROOF. Since $(f,g) \in GR_k[a,c]$ and $(f,g) \in GR_k[c,b]$, given $\epsilon > 0$ there exist positive functions δ_1, δ_2 defined on $[a,c]$ and $[c,b]$ respectively such that for any δ_1^k-fine division D_1 of $[a,c]$ and any δ_2^k-fine division D_2 of $[c,b]$ we have

$$|s(f,g;D_1) - I_1| < \epsilon \text{ and } |s(f,g;D_2) - I_2| < \epsilon,$$

where I_1 and I_2 denote the integrals of (f,g) on $[a,c]$ and on $[c,b]$ respectively. Now we define $\delta(\xi) = \min\{\delta_1(\xi), c - \xi\}$ when $\xi \in [a,c)$, $\min\{\delta_1(c), \delta_2(c)\}$ when $\xi = c$, and $\min\{\delta_2(\xi), \xi - c\}$ when $\xi \in (c,b]$.

Let $D = \{([x_i, x_{i+k}], \xi_i)\}_{i=0,1,\cdots,m-k}$ be a δ^k-fine division of $[a,b]$. Note that $c = x_n$ for some $n < m$. Also by definition of δ, no other interval will cover c except the one using c as an associated point. More precisely, c is the associated point of the $(k+1)$ intervals $[x_{n-k}, x_n]$, $[x_{n-k+1}, x_{n+1}], \cdots,$ $[x_n, x_{n+k}]$. Since $J(g;c)$ exists, there exists $\eta > 0$ such that for $0 < x_{i+k} - x_i < \eta$ we have

$$|g(x_i, \cdots, x_{i+k}) - J(g;c)| < \epsilon.$$

Next we modify δ at c if necessary by $0 < \delta(c) < \eta/2$. Then for any δ^k-fine D we have

$$|s(f,g;D) - I_1 - I_2 - (k-1) f(c) J(g;c)| \leq |\sum_{i=0}^{n-k} f(\xi_i) g(x_i, \cdots, x_{i+k}) - I_1|$$
$$+ |\sum_{i=n}^{m-k} f(\xi_i) g(x_i, \cdots, x_{i+k}) - I_2|$$

$$
\begin{aligned}
&+\Big|\sum_{i=n-k+1}^{n-1} f(c)g(x_i,\cdots,x_{i+k})-(k-1)f(c)J(g;c)\Big| \\
&<2\epsilon+(k-1)|f(c)|\epsilon.
\end{aligned}
$$

Hence $(f,g)\in GR_k[a,b]$ and the equality holds. □

Theorem 2.3 (Cauchy Condition)**.** *The pair $(f,g)\in GR_k[a,b]$ if and only if for every $\epsilon>0$ there is a positive function δ on $[a,b]$ such that for all δ^k-fine divisions D_1 and D_2 of $[a,b]$ we have $|s(f,g;D_1)-s(f,g;D_2)|<\epsilon$.*

PROOF. We prove only the sufficiency. Suppose the condition holds. We can choose $\delta_n(\xi)>0$ and δ_n^k-fine divisions D_n such that for all $n,\ m$

$$|s(f,g;D_n)-s(f,g;D_m)|<\frac{1}{n}+\frac{1}{m}.$$

Hence $\{s(f,g;D_n)\}$ is a Cauchy sequence and the limit exists, say, I. Thus, for every $\epsilon>0$ choose N such that $|s(f,g;D_N)-I|<\epsilon$. We may assume $\delta_N\le\delta$. Then we find that for any δ_N^k-fine division D of $[a,b]$ we have

$$|s(f,g;D)-I|\le|s(f,g;D)-s(f,g;D_N)|+|s(f,g;D_N)-I|<2\epsilon.$$

That is, $(f,g)\in GR_k[a,b]$. □

Theorem 2.4. *If $(f,g)\in GR_k[a,b]$ and $a\le c<d\le b$, then $(f,g)\in GR_k[c,d]$ provided $J(g;c)$ and $J(g;d)$ exist.*

PROOF. It is sufficient to prove that $(f,g)\in GR_k[a,c]$ provided $J(g;c)$ exists. Since $(f,g)\in GR_k[a,b]$, for every $\epsilon>0$ there is $\delta(\xi)>0$ such that for any δ^k-fine divisions D and D' of $[a,b]$ we have $|s(f,g;D)-s(f,g;D')|<\epsilon$. Also, since $J(g;c)$ exists, there exists $\eta>0$ such that for $0<u_{i+k}-u_i<\eta$ and $c\in[u_i,u_{i+k}]$ we have $|g(u_i,\cdots,u_{i+k})-J(g;c)|<\epsilon$. Modify δ if necessary as in the proof of Theorem 2.2 so that c is always a division point of any δ^k-fine division D of $[a,b]$ and furthermore $\delta(\xi)\le\eta/2$ for all ξ.

Fix a δ^k-fine division D_0 of $[c,b]$, where $D_0=\{([x_i,x_{i+k}],\xi_i)\}_{i=0,1,\cdots,m-k}$ and $\xi_0=x_0=c$. Next, take any two δ^k-fine divisions D_1 and D_2 of $[a,c]$. Write $D_1=\{([y_i,y_{i+k}],\eta_i)\}_{i=0,1,\cdots,n-k}$ and $D_2=\{([z_i,z_{i+k}],\zeta_i)\}_{i=0,1,\cdots,p-k}$. Note that $\eta_{n-k}=y_n=\zeta_{p-k}=z_p=c$. Define

$$
\begin{aligned}
D&=D_1\cup\{([y_{i+n-k},x_i],c)\}_{i=1,2,\cdots,k-1}\cup D_0,\\
D'&=D_2\cup\{([z_{i+p-k},x_i],c)\}_{i=1,2,\cdots,k-1}\cup D_0.
\end{aligned}
$$

Then we obtain

$$|s(f,g;D_1)-s(f,g;D_2)|\le|s(f,g;D)-s(f,g;D')|+2(k-1)|f(c)|\epsilon$$

$$<(1+2(k-1)|f(c|)\epsilon.$$

By Theorem 2.3, $(f,g) \in GR_k[a,c]$. □

Theorem 2.5 (Saks-Henstock Lemma). *If $(f,g) \in GR_k[a,b]$ and $J(g;c)$ exists for all $c \in (a,b)$, then for every $\epsilon > 0$, there exists a positive function δ on $[a,b]$ such that for any δ^k-fine division D of $[a,b]$ and for any δ^k-fine partial division $\{D_i\}_{i=1,2,\cdots,p}$ of $[a,b]$ we have*

$$|s(f,g;D)-F(a,b)|<\epsilon \text{ and } \Big|\sum_{i=1}^{p}\{s(f,g;D_i)-F(a_i,b_i)\}\Big|<(k+1)\epsilon,$$

where each D_i is a δ^k-fine division of $[a_i,b_i]$ and $F(u,v)$ denotes the GR_k integral on $[u,v] \subset [a,b]$.

Proof. Since $(f,g) \in GR_k[a,b]$, for every $\epsilon > 0$ there exists $\delta_1(\xi) > 0$ such that for any δ_1^k-fine division D of $[a,b]$ we have $|s(f,g;D)-F(a,b)| < \epsilon$, where $F(a,b)$ denotes the integral of (f,g) on $[a,b]$. Let $\{D_i\}_{i=1,1,\cdots,p}$ be a δ_1^k-fine partial division of $[a,b]$ where each D_i is a δ_1^k-fine division of $[a_i,b_i]$. By Theorem 2.4, (f,g) is integrable on the complement of $\cup_{i=1}^{p}[a_i,b_i]$ called $[c_j,d_j]$ for $j = 1,2,\cdots,q$. So there exists $\delta_2(\xi) > 0$ defined on $\cup_{j=1}^{q}[c_j,d_j]$ such that for any δ_2^k-fine division P_j of $[c_j,d_j]$ for $j = 1,2,\cdots,q$ we have

$$\Big|\sum_{j=1}^{q}s(f,g;P_j)-\sum_{j=1}^{q}F(c_j,d_j)\Big|<\epsilon,$$

where again $F(c_j,d_j)$ denotes the integral of (f,g) on $[c_j,d_j]$. Let Λ denote the set of endpoints of $[a_i,b_i]$ for $i = 1,2,\cdots,p$ and $[c_j,d_j]$ for $j = 1,2,\cdots,q$. The set Λ is finite, say consisting of m elements. Since $J(g;c)$ exists for each $c \in (a,b)$, there exists $\eta_c > 0$ such that for $0 < x_k - x_0 < \eta_c$ with $c \in [x_0,x_k]$ and $x_0 < x_1 < \cdots < x_k$ we have

$$|g(x_0,\cdots,x_k)-J(g;c)|<\frac{\epsilon}{m|f(c)|}.$$

Now define $\delta = \min\{\delta_1,\delta_2\}$ and $\eta = \min\{\eta_c : c \in \Lambda\}$. Modify δ if necessary so that points in Λ are always division points of any δ^k-fine division of $[a,b]$, and $\delta(x) < \eta/2$ for all x. Let $\{D_i\}_{i=1,2,\cdots,p}$ be a δ^k-fine partial division of $[a,b]$. Then we may choose δ^k-fine division P_j of $[c_j,d_j]$ for $j = 1,2,\cdots,q$ such that for $x \in \Lambda$,

$$x \in [x_\ell, x_{\ell+k}] \subset (x-\delta(x), x+\delta(x)) \text{ for } \ell = 1,2,\cdots,k-1,$$

in which x_ℓ are division points in D_i or P_j for some $i,\ j$.

Clearly the δ^k-fine partial divisions $\{D_i\}_{i=1,2,\cdots,p}$, $\{P_j\}_{j=1,2,\cdots,q}$ and the partial division $\{([x_\ell, x_{\ell+k}, x)\}_{\ell=1,2,\cdots,k-1}$ corresponding to each $x \in \Lambda$ together constitute a δ^k-fine division D of $[a,b]$. Then

$$s(f,g;D) = \sum_{i=1}^{p} s(f,g;D_i) + \sum_{j=1}^{q} s(f,g;P_j) + \sum_{x\in\Lambda}\sum_{\ell=1}^{k-1} f(x)g(x_\ell \cdots, x_{\ell+k}),$$

and in view of Theorem 2.2

$$I = \sum_{i=1}^{p} F(a_i,b_i) + \sum_{j=1}^{q} F(c_j,d_j) + \sum_{x\in\Lambda}(k-1)f(x)J(g;x).$$

Hence $|s(f,g;D) - F(a,b)| < \epsilon$. This proves the first inequality. Also for any δ^k-fine partial division $\{D_i\}_{i=1,2,\cdots,p}$ of $[a,b]$, where each D_i is a δ^k-fine division of $[a_i,b_i]$, we have

$$\begin{aligned}\Big|\sum_{i=1}^{p} s(f,g;D_i) - \sum_{i=1}^{p} F(a_i,b_i)\Big| &\le |s(f,g;D) - F(a,b)| \\ &\quad + |\sum_{j=1}^{q} s(f,g;P_j) - \sum_{j=1}^{q} F(c_j,d_j)| \\ &\quad + |\sum_{x\in\Lambda}\sum_{\ell=1}^{k-1} f(x)g(x_0,\cdots,x_k) - \sum_{x\in\Lambda}(k-1)f(x)J(g;x)| \\ &\quad < 2\epsilon + (k-1)\epsilon. \end{aligned}$$ □

3 Examples

Let $X \subset [a,b]$. We define $V_g^k(X) = \inf_\delta \sup_D \sum_{\xi_i\in X} |g(x_i,\cdots,x_{i+k})|$, $D = \{[x_i,x_{i+k}],\xi_i\}_{i=0,\cdots .n-k}$ being any δ^k-fine division of $[a,b]$. A set $X \subset [a,b]$ is said to be of g^k-variation zero if $V_g^k(X) = 0$. Let g be a function from $[a,b]^{k+1}$ to R. Then g is said to be of $BV^k[a,b]$ if $V_g^k[a,b]$ is finite. Clearly, g is of $BV^k[a,b]$ if and only if $\sum_{i=0}^{n-k} |g(x_i,\cdots,x_{i+k})| < M$ for all δ^k-fine division $D = \{[x_i,x_{i+k}],\xi_i\}_{I=0,\cdots,n-k}$ of $[a,b]$. For example, let α be any increasing function on $[a,b]$ and g defined on $[a,b]^3$ as $g(u,v,w) = \alpha(u) - 2\alpha(v) + \alpha(w)$. Then g is of $BV^2[a,b]$.

Theorem 3.1. *Let $f(x) = 0$ for all x in $[a,b]$ except for a set X of g^k-variation zero, then $(f,g) \in GR_k[a,b]$ and $\int_a^b f dg = 0$.*

Proof. Let $X_i = \{x \in X : i-1 < |f(x)| \le i\},\ i = 1, 2, \cdots$. So $X_i \subset X$ and $\cup_{i=1}^{\infty} X_i = X$, and $V_g^k(X_i) = 0$ for $i = 1, 2, \cdots$. Hence given $\epsilon > 0$, there exists $\delta_i(x) > 0$ defined on $[a,b]$ such that $\sup_D \sum_{\xi_i \in X_i} |g(x_j, \cdots, x_{j+k})| < \dfrac{\epsilon}{i2^i}$ for $i = 1, 2, \cdots$, $D = \{([x_j, x_{j+k}], \xi_j)\}_{j=0,1,\cdots,n-k}$ being a δ^k-fine division of $[a,b]$.

We define $\delta(x) = \delta_i(x)$ for $x \in X_i$, and 1 otherwise. If

$$D = \{([y_i, y_{j+k}], \eta_j)\}_{j=0,1,\cdots,m-k}$$

is a δ^k-fine division of $[a,b]$, then

$$|s(f,g;D)| = |\sum_{\eta_j \in X} f(\eta_j) g(y_j, \cdots, y_{j+k})| < \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon.$$

So $(f,g) \in GR_k[a,b]$ and $\int_a^b f dg = 0$.

A property is said to hold g^k-almost everywhere (g^k a.e.) in $[a,b]$ if it holds everywhere in $[a,b]$ except in a set of g^k-variation zero. Thus if $f_1 = f_2,\ g^k$ a.e. in $[a,b]$ then $\int_a^b f_1 dg = \int_a^b f_2 dg$.

The following is easily verified.

Theorem 3.2. *Let f be continuous on $[a,b]$ and g of $BV^k[a,b]$. Then $(f,g) \in GR_k[a,b]$.*

4 Integration by parts

We now establish an integration by parts formula for $k = 2$ for particular functions.

Theorem 4.1. *Let g be continuous on $[a,b]$ and f_1 of $BV^2[a,b]$. Then $\int_a^b f d^2 g$ exists and $\int_a^b f d^2 g = \int_a^b g d^2 f + g(a)[f(a+) - f(a)] + g(b)[f(b-) - f(b)]$, provided $f(a+),\ f(b-)$ exist and where $\int_a^b f d^2 g$ and $\int_a^b g d^2 f$ mean $\int_a^b f dg_1$ and $\int_a^b g df_1$ respectively with*

$$f_1(u,v,w) = f(u) - 2f(v) + f(w),\ \ g_1(u,v,w) = g(u) - 2g(v) + g(w).$$

Proof. Since g is continuous on $[a,b]$ and f_1 is of $BV^2[a,b]$, by Theorem 3.2, $(g, f_1) \in GR_2[a,b]$. Also since g is uniformly continuous on $[a,b]$ and f_1 is of $BV^2[a,b]$, given $\epsilon > 0$ there is a constant $\eta > 0$ such that for any division $D_1 = \{([x_i, x_{i+2}], \xi_i)\}_{i=0,1,\cdots,n-2}$ with $\xi_i \in [x_i, x_{i+2}] \subset (\xi_i - \eta, \xi_i + \eta)$ we have

$$\sum_{i=0}^{n-2} g(\xi_i) f_1(x_i, x_{i+1}, x_{i+2}) - I| < \epsilon$$

where $I = \int_a^b g df_1$. Also since $f(a+),\ f(b-)$ exist we can find $\delta_1 > 0$ such that $x - a < \delta_1$ implies $|f(x) - f(a+)| < \epsilon,\ |g(x) - g(a)| < \epsilon$ and $b - y < \delta_1$ implies $|f(y) - f(b-)| < \epsilon,\ |g(y) - g(b)| < \epsilon$. Let $\delta(\xi)$ be any function satisfying $0 < \delta(\xi) < \frac{1}{2}\min\{\delta_1, \eta\}$. For any δ^2-fine division $D = \{([x_i, x_{i+2}], \xi_i)\}_{i=0,1,\cdots,n-2}$ of $[a, b]$ we have

$$\begin{aligned}
|s(f, g; D) - I - g(a)[f(a+) - f(a)] - g(b)[f(b-) - f(b)]| \\
\leq |\sum_{i=1}^{n-1}[f(\xi_{i-2}) - 2f(\xi_{i-1}) + f(\xi_i)] - I| \\
+ |g(a)||f(\xi_0) - f(a+)| + |f(a)||g(x_1) - g(a)| \\
+ |g(b)||f(\xi_{n-2}) - f(b-)| + |f(b)||g(x_{n-1}) - g(b)| \\
< \epsilon(1 + |g(a)| + |f(a)| + |g(b)| + |f(b)|),
\end{aligned}$$

where $\xi_{-1} = a,\ \xi_{n-1} = b$. □

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