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## THE COMPOSITION OF TWO DERIVATIVES HAS A FIXED POINT

### Abstract

We show that if  $f, g: [0, 1] \rightarrow [0, 1]$  are both Darboux Baire-1 functions, then their composition,  $f \circ g$ , possesses a fixed point.

In [3], Gibson and Natkaniec refer to a problem of K. C. Ciesielski who asked whether the composition of two derivative functions from the unit interval to the unit interval necessarily possesses a fixed point. A partial solution to this problem was given by P. Humke, R. Svetic and C. Weil in [4]. In this note we answer the question affirmatively. (An alternative proof has also been found by M. Elekes, T. Keleti and V. Prokaj, see [2]).

Recalling that a Baire-1 function is the pointwise limit of a sequence of continuous functions and that a Darboux function is one for which the image of any interval in its domain is connected, we can formulate our main result as follows.

**Theorem 1.** *If  $f, g: [0, 1] \rightarrow [0, 1]$  are both Darboux Baire-1 functions, then there is an  $x \in [0, 1]$  for which  $(f \circ g)(x) = x$ .*

Since derivative functions are examples of Darboux Baire-1 functions, this answers Ciesielski’s question. The rest of the paper consists of a proof of this theorem.

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## 1 Proof of Theorem

For  $\phi: I \rightarrow \mathbf{R}$ ,  $I$  an interval, we define

$$\text{graph}_X(\phi) = \{(x, \phi(x)) : x \in I\}$$

and

$$\text{graph}_Y(\phi) = \{(\phi(y), y) : y \in I\}$$

and given  $a, b \in \mathbf{R}$ , we let  $[a, b], (a, b)$  denote the closed, and open intervals connecting them, respectively. For a point  $p$  we let  $(p)_x$  and  $(p)_y$  denote the  $x$  and  $y$  coordinates of  $p$ , respectively.

Fix two Darboux Baire-1 functions  $f, g: [0, 1] \rightarrow [0, 1]$ . Set

$$F = \text{graph}_X(f) = \{(x, f(x)) \in [0, 1]^2 : x \in [0, 1]\}$$

and

$$G = \text{graph}_Y(g) = \{(g(y), y) \in [0, 1]^2 : y \in [0, 1]\},$$

then in order to prove the theorem it is sufficient to show that  $F \cap G \neq \emptyset$ . We may assume without loss of generality that

$$f(0) = 0, f(1) = 1$$

and

$$g(0) = 1, g(1) = 0$$

for, by considering the square  $[-1, 2] \times [-1, 2]$  and extending the sets  $F$  and  $G$  as indicated in Figure 1, and then rescaling, we can define two new Darboux Baire-1 functions  $\tilde{f}, \tilde{g}: [0, 1] \rightarrow [0, 1]$  with  $\tilde{f}(0) = 0, \tilde{f}(1) = 1, \tilde{g}(0) = 1$  and  $\tilde{g}(1) = 0$  whose composition possesses a fixed point if and only if the original functions did.

Throughout this note, by *rectangle* we understand a rectangle whose sides are parallel to the usual coordinate axes. Topological notions like open, closed, etc., will be considered relatively to  $[0, 1]^2$ .

**Definition 1.** We define a *crossing-configuration*,  $\mathcal{R} = (A, B)$  to be an ordered pair consisting of non-empty finite subsets  $A$  and  $B$  of  $F$  and  $G$ , respectively, such that whenever  $I$  and  $J$  are closed intervals with  $A \cup B \subset I \times J$  and  $\phi: I \rightarrow \mathbf{R}$  and  $\psi: J \rightarrow \mathbf{R}$  are continuous functions with:

$$A \subset \text{graph}_X(\phi) \quad \text{and} \tag{1}$$

$$B \subset \text{graph}_Y(\psi), \tag{2}$$

then

$$\text{graph}_X(\phi) \cap \text{graph}_Y(\psi) \neq \emptyset.$$

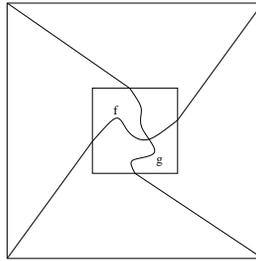


Figure 1: Ensuring that  $f(0) = 0, f(1) = 1$  and  $g(0) = 1, g(1) = 0$

**Remark 1.** If  $\mathcal{R} = (A, B)$  is a crossing configuration, and if  $\phi, \psi: I, J \rightarrow [0, 1]$  are continuous functions satisfying (1) and (2) respectively, then for any rectangle  $R \subset [0, 1]^2$  which contains  $A \cup B$ , we know that

$$\text{graph}_X(\phi) \cap \text{graph}_Y(\psi) \cap R \neq \emptyset.$$

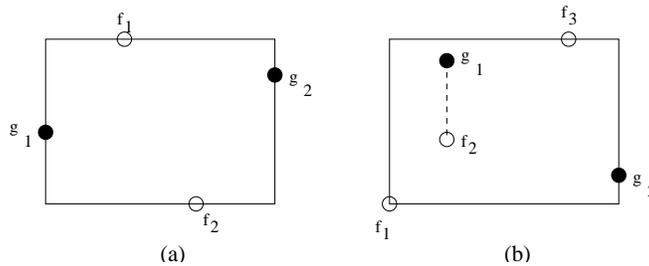


Figure 2: Two examples of crossing configurations: points denoted by  $\circ$  lie in  $F$  and points denoted by  $\bullet$  lie in  $G$ .

**Lemma 1.** The configurations illustrated in Figure 2 are crossing configurations.

PROOF. Figure 2(a): Here  $f_1, f_2$  are points from  $F$  lying on the top and bottom edges of a closed rectangle, and  $g_1, g_2$  are points of  $G$  lying on the left and right edges of the rectangle, respectively. Suppose that  $\phi$  and  $\psi$  are continuous functions with  $\{f_1, f_2\} \subset \text{graph}_X(\phi)$  and  $\{g_1, g_2\} \subset \text{graph}_Y(\psi)$ . Notice that the part of  $\text{graph}_X(\phi)$  lying within the vertical strip whose edges



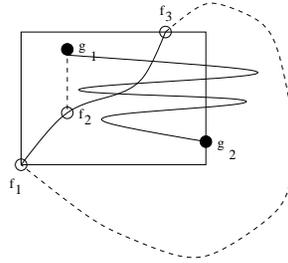


Figure 4:  $\text{graph}_X(\phi) \cap ((f_1)_x, (f_3)_x] \times \mathbf{R}$  may be extended to form a Jordan curve separating  $g_1$  and  $g_2$ .

**Proposition 2.** *For all crossing-configurations  $\mathcal{R} = (A, B)$  and for all open rectangles  $R \supset A \cup B$  and open sets  $U \supset F$  (or  $V \supset G$ ), we can find a crossing-configuration  $\mathcal{R}' = (A', B')$  and a closed rectangle  $R'$  with  $A' \cup B' \subset R' \subset U \cap R$  (or  $V \cap R$ ).*

Before proving this, we show how it immediately leads to a proof of Theorem 1: Since  $f$  and  $g$  are Darboux, Baire-1 functions, their graphs are  $G_\delta$  subsets of the plane (see [5, Ch.II.§31, VII, Thm 1]), so

$$F = \bigcap_{n=1}^{\infty} U_n, \quad \text{where } U_1 \supset U_2 \supset \dots \text{ are open sets}$$

and

$$G = \bigcap_{n=1}^{\infty} V_n, \quad \text{where } V_1 \supset V_2 \supset \dots \text{ are open sets.}$$

We recall, from Remark 2, that

$$\mathcal{R}_0 = (\{(0, 0), (1, 1)\}, \{(0, 1), (1, 0)\})$$

is a crossing-configuration. We now use the proposition to find a sequence of crossing-configurations  $\mathcal{R}_i = (A_i, B_i)$  and open rectangles  $R_i \supset A_i \cup B_i$  such that

$$\text{cl}(R_{2i+1}) \subset R_{2i} \cap U_{i+1}$$

and

$$\text{cl}(R_{2(i+1)}) \subset R_{2i+1} \cap V_{i+1}$$

for  $i = 0, 1, 2, \dots$ . Hence

$$\text{cl}(R_0) = R_0 \supset \text{cl}(R_1) \supset R_2 \supset \text{cl}(R_3) \supset \dots,$$

$$\emptyset \neq \bigcap_{n=1}^{\infty} \text{cl}(R_n) \subset \bigcap_n U_n = F$$

and

$$\emptyset \neq \bigcap_{n=1}^{\infty} \text{cl}(R_n) \subset \bigcap_n V_n = G,$$

which together imply that  $F \cap G \neq \emptyset$  as required.

PROOF. We prove the proposition 2 for the case when  $U \supset F$ , the case when  $V \supset G$  is similar.

Suppose that  $\mathcal{R} = (A, B)$  is a crossing configuration,  $R \supset A \cup B$  is an open rectangle, and let  $S = [x_1, x_2] \times [y_1, y_2] \subset R$  be a closed rectangle whose (relative) interior contains  $A \cup B$ .

Observe that if  $\phi: [x_1, x_2] \rightarrow \mathbf{R}$  were a continuous function for which  $A \subset \text{graph}_X(\phi)$  and  $\text{graph}_X(\phi) \cap G \cap S = \emptyset$ , then  $G \subset [0, 1]^2 \setminus (\text{graph}_X(\phi) \cap S)$  would be a relatively open set. Since  $g \in DB_1$ , it is almost continuous [1], so there exists a continuous function  $\psi: [0, 1] \rightarrow \mathbf{R}$  with  $\text{graph}_Y(\psi) \subset \mathbf{R} \times [0, 1] \setminus (\text{graph}_X(\phi) \cap S)$ . Moreover, we can require that  $B \subset \text{graph}_Y(\psi)$ . (See e.g. [6, Lemma 6.2].) Hence  $\text{graph}_X(\phi) \cap \text{graph}_Y(\psi) \cap S = \emptyset$ . But  $(A, B)$  is a crossing configuration — a contradiction.

We suppose that there are no crossing-configurations  $(A', B')$  and closed rectangles  $R'$  with

$$A' \cup B' \subset R' \subset U \cap R$$

(noting that this implies  $F \cap G \cap R = \emptyset$ ). We will prove that this implies that there is a continuous function  $\phi: [x_1, x_2] \rightarrow \mathbf{R}$  with  $A \subset \text{graph}_X(\phi)$  and  $\text{graph}_X(\phi) \cap G \cap S = \emptyset$  giving us our required contradiction.

We do this via the method of regular intervals: we say an interval  $I \subset [x_1, x_2]$  is *regular*, if for all  $s, t \in I$ ,  $s < t$ , we can find a continuous function  $\phi: [s, t] \rightarrow \mathbf{R}$  for which

$$\phi(s) = f(s), \quad \phi(t) = f(t)$$

and

$$\text{graph}_X(\phi) \cap G \cap S = \emptyset.$$

(Note that regular intervals need neither be open nor closed.) If we show that  $[x_1, x_2]$  is itself regular, then we are done.

It is easy to see that:

- (1) If  $I$  and  $J$  are regular intervals and  $I \cap J \neq \emptyset$ , then  $I \cup J$  is regular;
- (2) If  $I$  is an interval which is the (finite or infinite) union of *open* regular intervals, then  $I$  is regular.

It is slightly trickier to verify:

(3) If  $I$  is regular, then  $\text{cl}(I)$  is regular.

*Proof of (3):* Let  $r$  be the left endpoint of  $I$ . (The proof for the right endpoint is similar.) It is enough to show that we can find  $r' > r$  arbitrarily close to  $r$  for which there is a continuous function  $\phi: [r, r'] \rightarrow \mathbf{R}$  such that  $\phi(r) = f(r)$ ,  $\phi(r') = f(r')$  and  $\text{graph}_X(\phi) \cap G \cap S = \emptyset$ .

Choose  $r_i \in I$  such that  $r_1 > r_2 > \dots > r$ ,  $r_n \rightarrow r$  and for which  $f(r_n) \rightarrow f(r)$  (the Darboux property for  $f$  ensures we can find such a sequence). For each  $k \in \mathbf{N}$ , we can find  $n_k$  such that

$$f(r_{n_k}) \in (f(r) - 2^{-k}, f(r) + 2^{-k})$$

and both

$$g(f(r) - 2^{-k}) \text{ and } g(f(r) + 2^{-k}) \notin (r, r_{n_k}).$$

Fix a sequence  $n_1 < n_2 < n_3 < \dots$  with this property. Since  $I$  is regular we can find a sequence of continuous functions  $\phi_k: [r_{n_{k+1}}, r_{n_k}] \rightarrow \mathbf{R}$  for which

$$\phi_k(r_{n_k}) = f(r_{n_k}), \phi_k(r_{n_{k+1}}) = f(r_{n_{k+1}}) \text{ and } \text{graph}_X(\phi_k) \cap G \cap S = \emptyset.$$

Then the function  $\tilde{\phi}: [r, r_{n_1}] \rightarrow \mathbf{R}$  defined by

$$\tilde{\phi}(x) = \begin{cases} f(r) & \text{if } x = r \\ \max\{\min\{\phi_k(x), f(r) + 2^{-k}\}, f(r) - 2^{-k}\} & \text{if } x \in [r_{n_{k+1}}, r_{n_k}] \end{cases}$$

is a well-defined continuous function for which  $\tilde{\phi}(r_{n_k}) = f(r_{n_k})$  for all  $k$ ,  $\tilde{\phi}(r) = f(r)$  and  $\text{graph}_X(\tilde{\phi}) \cap G \cap S = \emptyset$ . □

Suppose that  $[x_1, x_2]$  is not a regular interval and let

$$P = [x_1, x_2] \setminus \bigcup \{I \subset [x_1, x_2] : I \text{ is relatively open and regular}\}.$$

Then  $P$  is closed, and observations (2) and (3) imply that  $P \cap (x_1, x_2)$  is non-empty and has no isolated points. Since  $f$  is a Baire-1 function, the continuity points of  $f|_P$  form a dense  $G_\delta$  set in  $P$  [5, Ch.II.§31, X, Thm 1]. Thus we can choose  $r \in P \cap (x_1, x_2)$  such that

- $f|_P$  is continuous at  $r$ ; and
- $r$  is not the endpoint of any interval in  $[x_1, x_2] \setminus P$  which is contiguous to  $P$ .



$$\text{graph}_X(f|_{P \cap [r, r^{***}]}) \subset R_2.$$

Finally by the Darboux property, we can find  $r^{***} \in (r, r^{**})$  for which  $f(r^{***}) = f(r)$ .

**Claim:**  $([r, r^{***}] \times [z'_1, z'_2]) \cap G = \emptyset$ .

*Proof of claim:* For suppose  $(g(v), v) \in [r, r^{***}] \times [z'_1, z'_2]$ , without loss of generality we can assume that  $v > f(r)$ . By the Darboux property applied to  $g$  we can find  $v' \in [f(r), v]$  with  $g(v') = r'_2$ . But then

$$(\{(r^{***}, f(r^{***})), (r'_2, f(r'_2))\}, \{(g(v), v), (g(v'), v')\})$$

is a crossing-configuration of the type (a) illustrated in Figure 2 contained in

$$[g(v), g(v')] \times [f(r), f(r'_2)] \subset R \cap U,$$

which is a contradiction. □

But now clearly the interval  $(r, r^{***})$  is regular.

**Case 3:**  $(r, f(r)) \in S$  and there is  $r^* > r$  such that  $f|_{(r, r^*)} \geq f(r)$ . (Or  $(r, f(r)) \in S$  and there is  $r^* > r$  such that  $f|_{(r, r^*)} \leq f(r)$ .)

Without loss of generality, we do the case when there is an  $r^* > r$  with  $f|_{(r, r^*)} \geq f(r)$  and there is no  $r' > r$  for which  $f|_{(r, r')}$  is constant. Choose  $z_2 > f(r)$  and  $r^* > r$  for which  $f|_{(r, r^*)} \geq f(r)$  and  $[r, r^*] \times [f(r), z_2] \subset R \cap U$  and set  $R_1 = [r, r^*] \times [f(r), z_2]$ .

Since  $f|_{(r, r^*)}$  is not constant (and so  $f(r) < 1$ ), then we can find  $r < r'_2 < r^*$  such that  $f(r) < f(r'_2) < z_2$ . Choose  $R_2 = [r, r'''] \times [f(r), z'_2]$  such that

$$f(r) < z'_2 < f(r'_2), \quad r < r''' < r'_2 \quad \text{and}$$

$$\text{graph}_X(f|_{P \cap [r, r''']}) \subset R_2.$$

We show that there are no points  $(u, f(u)), (g(v), v)$  in  $R_2$  for which  $u = g(v)$  and  $f(u) < v$ . For if there were, we could use the fact that  $g$  is Darboux to find  $w \in (f(r), v)$  for which  $g(w) = r'_2$  and then we would have a crossing-configuration, namely  $(\{(u, f(u)), (r, f(r)), (r'_2, f(r'_2))\}, \{(g(v), v), (g(w), w)\})$  contained in  $R_1$ , see Lemma 1 and Figure 6.

**Lemma 2.** *There is a rectangle  $R_3 = [r, r'''] \times [f(r), z''_2] \subset R_2$  such that  $\text{graph}_X(f|_{P \cap [r, r''']}) \subset R_3$  and there are no points  $(u, f(u)), (g(v), v) \in R_3$  for which  $g(v) \leq u, f(u) \leq v$ .*

**PROOF.** If  $R_3 = R_2$  does not satisfy the lemma, then there is  $(u_0, f(u_0))$  and  $(g(v_0), v_0) \in R_2$  for which

$$g(v_0) \leq u_0 \quad \text{and} \quad f(u_0) \leq v_0.$$

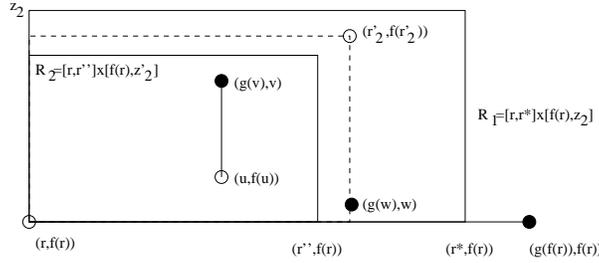


Figure 6: Diagram illustrating occurrence of a crossing-configuration in Case 3 if we can find  $u = g(v)$  and  $f(u) < v$  in  $R_2$ .

Choose  $f(r) < z_2'' < f(u_0)$  and use the fact that  $r$  is a continuity point of  $f|_P$  to find  $r < r''' < g(v_0)$  for which  $\text{graph}_X(f|_{P \cap [r, r''']}) \subset R_3 = [r, r'''] \times [f(r), z_2'']$ .

Suppose now that we can find  $(u, f(u))$  and  $(g(v), v)$  in  $R_3$  for which  $g(v) \leq u$  and  $f(u) \leq v$ . Then by the Darboux property for  $g$ , we can find a point  $(g(v_2), v_2)$  with  $v_2 \in (v, v_0)$  and  $g(v_2) = f(u)$  but then  $u = g(v_2)$  and  $f(u) < v_2$  contradicting the observation made just before this lemma; see Figure 7. Hence the lemma holds.  $\square$

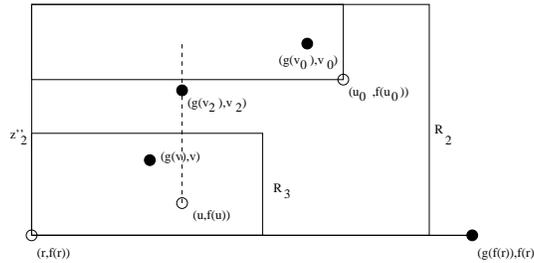


Figure 7: The figure for Lemma 2

We now prove that the interval  $(r, r''')$  we have constructed is regular; that is, for all  $s, t \in P \cap [r, r''']$  with  $s < t$ , we can find a continuous function  $\phi: [s, t] \rightarrow \mathbf{R}$  for which  $\text{graph}_X(\phi) \cap G \cap S = \emptyset$ , and  $\phi(s) = f(s)$  and  $\phi(t) = f(t)$ . We know that  $(s, f(s))$  and  $(t, f(t))$  are in  $R_3$ . We can assume that  $s$  is not the left endpoint of an interval contiguous to  $P$ . If  $f(s) \geq f(t)$ , then Lemma 2 allows us to choose  $\phi$  to be the affine function joining  $f(s)$  and  $f(t)$ . If  $f(s) < f(t)$  we distinguish two cases:

**(A):** There is  $u \in (s, t)$  such that  $f(u) \leq f(s)$ .

In this case, Lemma 2 implies that  $G$  does not meet the shaded region of Figure 8 and we can join  $(s, f(s))$ ,  $(u, f(t))$  and  $(t, f(t))$  by a piecewise linear function.

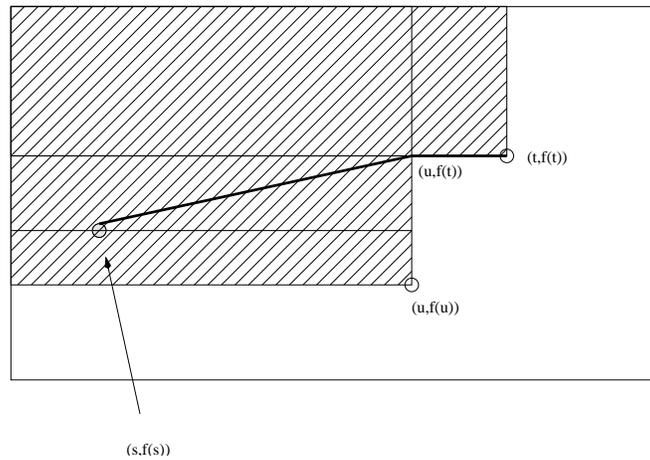


Figure 8: Constructing our continuous function  $\phi$  in Case (A).

**(B):** There is a sequence  $(u_n)$  with  $t > u_1 > u_2 > \dots \rightarrow s$  such that  $f(t) > f(u_1) > f(u_2) > \dots \rightarrow f(s)$ . In this case we can join the points  $(t, f(t))$ ,  $(u_1, f(t))$ ,  $(u_2, f(u_1))$ ,  $(u_3, f(u_2))$ ,  $\dots$  piecewise linearly, see Figure 9.  $\square$

## 2 Open problems

There are a couple of natural questions suggested by this result:

1. is the graph of the composition of two Darboux Baire-1 functions connected?
2. does a similar result hold for the composition of  $n$  Darboux Baire-1 functions when  $n \geq 3$ ?

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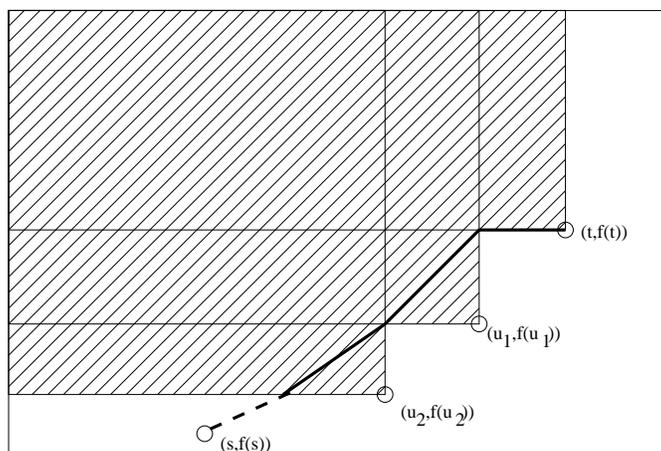


Figure 9: Constructing our continuous function  $\phi$  in Case (B).

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