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$\mathcal{I}$-CONVERGENCE*

Abstract

In this paper we introduce and study the concept of $\mathcal{I}$-convergence of sequences in metric spaces, where $\mathcal{I}$ is an ideal of subsets of the set $\mathbb{N}$ of positive integers. We extend this concept to $\mathcal{I}$-convergence of sequence of real functions defined on a metric space and prove some basic properties of these concepts.

1 Introduction

This paper was inspired by [14], where the concept of $\mathcal{I}$-convergence of sequences of real numbers is introduced. We will often quote some results from [14] that can be easily transferred to sequences of points in a metric space. In [14] it is shown that our $\mathcal{I}$-convergence is, in a sense, equivalent to μ-statistical convergence of J. Connor ([8]).

The concept of statistical convergence is introduced in [10] and [29] and developed in [6], [7], [8], [9], [10], [12], [26] and [28]. Some applications of statistical convergence in number theory and mathematical analysis can be found in [4], [5] and [19]. The concept of $\mathcal{I}$-convergence is a generalization of statistical convergence and it is based on the notion of the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of positive integers.

This paper consists of five sections with the new results in sections 3–5. In the Section 3 the concept of the $\mathcal{I}$-convergence of sequences in a metric space is introduced and its fundamental properties are studied. In Section 4 we introduce and study the concept of an $\mathcal{I}$-cluster point and an $\mathcal{I}$-limit

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point of a sequence in a metric space. In Section 5 we extend the concept of $\mathcal{I}$-convergence to sequences of real functions defined on a metric space and we will discuss some questions concerning the limit functions of $\mathcal{I}$-convergent sequences, specially in the case when all functions in the sequence are continuous.

2 Definitions and Notation

Throughout the paper $\mathbb{N}$ will denote the set of positive integers χ_A – the characteristic function of $A \subset \mathbb{N}$, $\mathbb{R}$ – the set of all real numbers, (X, ρ) – a metric space, $B(\xi, \epsilon)$ – an open ball in X with center $\xi \in X$ and radius $\epsilon > 0$. The topological terminology in taken from [20]. We recall the concept of asymptotic and logarithmic density of a set $A \subset \mathbb{N}$ (see [22], pp. 71, 95-96 and [27]).

Let $A \subset \mathbb{N}$. Put $d_n(A) = \frac{1}{n}\sum_{k=1}^{n} \chi_A(k)$ and $\delta_n(A) = \frac{1}{s_n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k}$ for $n \in \mathbb{N}$, where $s_n = \sum_{k=1}^{n} \frac{1}{k}$. The numbers $\underline{d}(A) = \liminf_{n\to\infty} d_n(A)$ and $\overline{d}(A) = \limsup_{n\to\infty} d_n(A)$ are called the lower and upper asymptotic density of A, respectively. Similarly, the numbers $\underline{\delta}(A) = \liminf_{n\to\infty} \delta_n(A)$ and $\overline{\delta}(A) = \limsup_{n\to\infty} \delta_n(A)$ are called the lower and upper logarithmic density of A, respectively. If $\underline{d}(A) = \overline{d}(A)$ ($\underline{\delta}(A) = \overline{\delta}(A)$), then $d(A) = \underline{d}(A)$ is called the asymptotic density of A ($\delta(A) = \underline{\delta}(A)$ is called the logarithmic density of A, respectively). It is well known that for each $A \subset \mathbb{N}$ $\underline{d}(A) \leq \underline{\delta}(A) \leq \overline{\delta}(A) \leq \overline{d}(A)$ (see [22], 70-75, 95-96). Hence if $d(A)$ exists, then also $\delta(A)$ exists and $d(A) = \delta(A)$. The converse is not true. Obviously all numbers $\underline{d}(A)$, $\overline{d}(A)$, $\underline{\delta}(A)$, $\overline{\delta}(A)$ (and so $d(A)$, $\delta(A)$) belong to $[0, 1]$. Since

$$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma + O(\frac{1}{n}) \tag{1}$$

where γ is an Euler constant, if we put $\delta_n^*(A) = \frac{1}{\ln n}\sum_{k=1}^{n} \frac{\chi_A(k)}{k}$ for $n \in \mathbb{N}$, then $\underline{\delta}(A) = \liminf_{n\to\infty} \delta_n^*(A)$, $\overline{\delta}(A) = \limsup_{n\to\infty} \delta_n^*(A)$ and if $\delta(A)$ exists, then $\delta(A) = \lim_{n\to\infty} \delta_n^*(A)$.

Now recall the concept of statistical convergence of real sequences (see [9], [28]).

Definition A. A sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be statistically convergent to $\xi \in \mathbb{R}$ provided that for each $\epsilon > 0$ we have $d(A(\epsilon)) = 0$, where $A(\epsilon) = \{n \in \mathbb{N} : |x_n - \xi| \geq \epsilon\}$.

Recall that if X is a non-empty set then a family of sets $\mathcal{I} \subset 2^X$ is an ideal if and only if for each $A, B \in \mathcal{I}$ we have $A \cup B \in \mathcal{I}$ and for each $A \in \mathcal{I}$

and each $B \subset A$ we have $B \in \mathcal{I}$. A non-empty family of sets $\mathcal{F} \subset 2^X$ is a filter on X if and only if $\phi \notin \mathcal{F}$, for each $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$ and for each $A \in \mathcal{F}$ and each $B \supset A$ we have $B \in \mathcal{F}$. An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. $\mathcal{I} \subset 2^X$ is a non-trivial ideal if and only if $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \setminus A : A \in \mathcal{I}\}$ is a filter on X. A non-trivial ideal $\mathcal{I} \subset 2^X$ is called admissible if and only if $\mathcal{I} \supset \{\{x\} : x \in X\}$.

3 $\mathcal{I}$-Convergence of Sequences of Elements of a Metric Space

3.1 $\mathcal{I}$-Convergence, Examples and Properties

In what follows (X, ρ) is a fixed metric space and $\mathcal{I}$ denotes a non-trivial ideal of subsets of $\mathbb{N}$.

Definition 3.1. A sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X is said to be $\mathcal{I}$-convergent to $\xi \in X$ ($\xi = \mathcal{I}\text{-}\lim_{n\to\infty} x_n$) if and only if for each $\epsilon > 0$ the set $A(\epsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \epsilon\}$ belongs to $\mathcal{I}$. The element ξ is called the $\mathcal{I}$-limit of the sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$.

Example 3.1. (a) Take for $\mathcal{I}$ the class $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$. Then $\mathcal{I}_f$ is a non-trivial admissible ideal and $\mathcal{I}_f$-convergence coincides with the usual convergence with respect to the metric ρ in X.

(b) Denote by $\mathcal{I}_d$ ($\mathcal{I}_\delta$) the class of all $A \subset \mathbb{N}$ with $d(A) = 0$ ($\delta(A) = 0$, respectively). Then $\mathcal{I}_d$ and $\mathcal{I}_\delta$ are non-trivial admissible ideals, $\mathcal{I}_d$-convergence coincides with the statistical convergence. $\mathcal{I}_\delta$-convergence is said to be logarithmic statistical convergence.

(c) The uniform density of a set $A \subset \mathbb{N}$ is defined as follows.For integers $t \geq 0$ and $s \geq 1$ let $A(t+1, t+s) = \text{card}\{n \in A : t+1 \leq n \leq t+s\}$. Put $\beta_s = \liminf_{t\to\infty} A(t+1, t+s)$, $\beta^s = \limsup_{t\to\infty} A(t+1, t+s)$. It can be shown (see [3]) that the following limits exist: $\underline{u}(A) = \lim_{s\to\infty} \frac{\beta_s}{s}$, $\overline{u}(A) = \lim_{s\to\infty} \frac{\beta^s}{s}$. If $\underline{u}(A) = \overline{u}(A)$, then $u(A) = \underline{u}(A)$ is called the uniform density of the set A.

Put $\mathcal{I}_u = \{A \subset \mathbb{N} : u(A) = 0\}$. Then $\mathcal{I}_u$ is a non-trivial ideal and $\mathcal{I}_u$-convergence is said to be the uniform statistical convergence.

(d) A wide class of $\mathcal{I}$-convergences can be obtained as follows. Let $T = \{t_{n,k}\}_{n,k\in\mathbb{N}}$ be a regular non negative matrix (see [23], p. 8). For $A \subset \mathbb{N}$ we put $d_T^{(n)}(A) = \sum_{k=1}^{\infty} t_{n,k} \cdot \chi_A(k)$ for $n \in \mathbb{N}$. If $\lim_{n\to\infty} d_T^{(n)}(A) = d_T(A)$ exists, then $d_T(A)$ is called a T-density of A (see [18]). From the regularity of T it follows that $\lim_{n\to\infty} \sum_{k=1}^{\infty} t_{n,k} = 1$ and from this we see that $d_T(A) \in [0, 1]$ (if it exists). Put $\mathcal{I}_{d_T} = \{A \subset \mathbb{N} : d_T(A) = 0\}$. Then $\mathcal{I}_{d_T}$ is a non-trivial ideal and $\mathcal{I}_{d_T}$ contains both $\mathcal{I}_d$- and $\mathcal{I}_\delta$-convergence as special cases. Indeed,

$\mathcal{I}_d$-convergence can be obtained by choosing $t_{n,k} = \frac{1}{n}$ for $k \leq n$, $t_{n,k} = 0$ for $k > n$ and $\mathcal{I}_\delta$-convergence by choosing $t_{n,k} = \frac{\frac{1}{k}}{s_n}$ for $k \leq n$, $t_{n,k} = 0$ for $k > 0$, where $s_n = \sum_{j=1}^{n} \frac{1}{j}$ for $n \in \mathbb{N}$. Choosing $t_{n,k} = \frac{\phi(k)}{n}$ for $k \leq n$, $k|n$ and $t_{n,k} = 0$ for $k \leq n$ $k \nmid n$ and $t_{n,k} = 0$ for $k > n$ we get ϕ-convergence of Schoenberg (see [29]), where ϕ is the Euler function.

Another special case of $\mathcal{I}_{d_T}$-convergence is the following. Take an arbitrary divergent series $\sum_{n=1}^{\infty} c_n$, where $c_n > 0$ for $n \in \mathbb{N}$ and put $t_{n,k} = \frac{c_k}{s_n}$ for $k \leq n$, where $s_n = \sum_{j=1}^{n} c_j$, and $t_{n,k} = 0$ for $k > n$ (see [1]).

(e) Let ν be an arbitrary finitely additive normed measure defined on a field $\mathcal{U} \subset 2^{\mathbb{N}}$. Suppose that $\mathcal{U}$ contains all singletons $\{n\}$, $n \in \mathbb{N}$. Then $\mathcal{I}_\nu = \{A \subset \mathbb{N} : \nu(A) = 0\}$ is a non-trivial ideal in $\mathbb{N}$ which generates the $\mathcal{I}_\nu$-convergence.

(f) Suppose that $\mu_m : 2^{\mathbb{N}} \to [0,1]$ is a finitely additive normed measure for $m \in \mathbb{N}$. If for $A \subset \mathbb{N}$ there exists $\mu(A) = \lim_{m\to\infty} \mu_m(A)$, then the set A is said to be measurable and $\mu(A)$ is called the measure of A. Obviously μ is a finitely additive measure on some field $\mathcal{S} \subset 2^{\mathbb{N}}$ (see [16]). The family $\mathcal{I}_\mu = \{A \subset \mathbb{N} : \mu(A) = 0\}$ is a non-trivial ideal which generates the $\mathcal{I}_\mu$-convergence.

For μ_m we can take d_m or δ_m (compare the definition).

(g) Let $\mathbb{N} = \bigcup_{j=1}^{\infty} \Delta_j$ be a decomposition of $\mathbb{N}$ such that each Δ_j is infinite and obviously $\Delta_i \cap \Delta_j = \emptyset$ for $i \neq j$. Denote by $\mathcal{E}$ the class of all $A \subset \mathbb{N}$ that intersect only a finite number of $\Delta_j' s$. Then $\mathcal{E}$ is a non-trivial ideal.

(h) Put $\mathcal{I}_c = \{A \subset \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ (see [23]). Then $\mathcal{I}_c$ is a non-trivial ideal. Since $\sum_{a\in A} a^{-1} < \infty$ implies $d(A) = 0$ (see [24]), we see that $\mathcal{I}_c$-convergence implies statistical convergence.

Remark 3.1. Note that if $\mathcal{I}$ is an admissible ideal, then the usual convergence in X implies $\mathcal{I}$-convergence in X.

We shall now investigate which axioms of convergence are satisfied by $\mathcal{I}$-convergence. The following properties are the most familiar axioms of convergence (see [17]):

(S) Every constant sequence $\{\xi, \xi, \ldots, \xi, \ldots\}$ converges to ξ.

(H) The limit of any convergent sequence is uniquely determined.

(F) If a sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ has the limit ξ, then each of its subsequences has the same limit.

(U) If each subsequence of the sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ has a subsequence which converges to ξ, then $\mathbf{x}$ converges to ξ.

Proposition 3.1. *Suppose that X has at least two points. Let $\mathcal{I} \subset 2^X$ be an admissible ideal.*

(i) The $\mathcal{I}$-convergence satisfies $(S),(H)$ and (U).

(ii) If $\mathcal{I}$ contains an infinite set, then $\mathcal{I}$-convergence does not satisfy (F).

PROOF. (i) (S) is obviously fulfilled. To prove (H) it is sufficient to observe that for any $A_1, A_2 \in \mathcal{I}$ we have $(\mathbb{N} \setminus A_1) \cap (\mathbb{N} \setminus A_2) \neq \emptyset$ since the last two sets belong to the filter associated with $\mathcal{I}$. If there are two limits $\xi, \eta \in X$, $\xi \neq \eta$, choose ϵ such that

$$0 < \epsilon < \frac{1}{2}\rho(\xi, \eta) \tag{2}$$

and put $A_1 = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \epsilon\}$, $A_2 = \{n \in \mathbb{N} : \rho(x_n, \eta) \geq \epsilon\}$.

Suppose now that (U) does not hold. Then there exists $\epsilon_0 > 0$ such that

$$A(\epsilon_0) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \epsilon_0\} \notin \mathcal{I}.$$

But then $A(\epsilon_0)$ is an infinite set since $\mathcal{I}$ is admissible. Let $A(\epsilon_0) = \{n_1 < n_2 < \cdots < n_k < \dots\}$. Put $y_k = x_{n_k}$ for $k \in \mathbb{N}$. Then $\mathbf{y} = \{y_k\}_{k \in \mathbb{N}}$ is a subsequence of $\mathbf{x}$ without a subsequence $\mathcal{I}$-convergent to ξ.

(ii) Suppose that $A \in \mathcal{I}$ is an infinite set, $A = \{n_1 < n_2 < \cdots < n_k < \dots\}$. $B = \mathbb{N} \setminus A = \{m_1 < m_2 < \cdots < m_k < \dots\}$. The set B is also infinite since $\mathcal{I}$ is non-trivial ideal. Define $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}}$ by choosing $\xi_1, \xi_2 \in X$, $\xi_1 \neq \xi_2$ and put $x_{n_k} = \xi_1$, $x_{m_k} = \xi_2$ for $k \in \mathbb{N}$. Obviously $\mathcal{I}\text{-}\lim_{k\to\infty} x_k = \xi_2$, but the subsequence $y_k = x_{n_k}$, $k \in \mathbb{N}$, $\mathcal{I}$-converges to ξ_1. □

Remark 3.2. If $\mathcal{I}$ is an admissible ideal which does not contains any infinite set, then $\mathcal{I}$-convergence coincides with the usual convergence and obviously fulfills (F).

3.2 $\mathcal{I}$-Convergence and $\mathcal{I}^*$-Convergence

The following result is well known in the theory of statistical convergence. A sequence $\{x_n\}_{n \in \mathbb{N}}$ of real numbers is statistically convergent to ξ if and only if there exists a set $M = \{m_1 < m_2 < \dots m_k < \dots\} \subset \mathbb{N}$ such that $d(M) = 1$ and $\lim_{k\to\infty} x_{m_k} = \xi$ (see [10], [11], [18], [26]).

This result suggests the introduction of the following concept of convergence (which we shall call $\mathcal{I}^*$-convergence) closely related to $\mathcal{I}$-convergence.

Definition 3.2. A sequence $\mathbf{x} = \{x_n\}_{n \in \mathbb{N}}$ of elements of X is said to be $\mathcal{I}^*$-convergent to $\xi \in X$ if and only if there exists a set $M \in \mathcal{F}(\mathcal{I})$ (i.e. $\mathbb{N} \setminus M \in \mathcal{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \dots\}$ such that $\lim_{k\to\infty} \rho(x_{m_k}, \xi) = 0$.

Proposition 3.2. *Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{I}^*$-$\lim_{n\to\infty} x_n = \xi$, then $\mathcal{I}$-$\lim_{n\to\infty} x_n = \xi$.*

PROOF. By assumption there exists a set $H \in \mathcal{I}$ such that for $M = \mathbb{N} \setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$ we have

$$\lim_{k\to\infty} x_{m_k} = \xi. \tag{3}$$

Let $\epsilon > 0$. By virtue of (3) there exists $k_0 \in \mathbb{N}$ such that $\rho(x_{m_k}, \xi) < \epsilon$ for each $k > k_0$. Then obviously

$$A(\epsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \epsilon\} \subset H \cup \{m_1 < m_2 < \dots < m_{k_0}\}. \tag{4}$$

The set on the right-hand side of (4) belongs to $\mathcal{I}$ (since $\mathcal{I}$ is admissible). So $A(\epsilon) \in \mathcal{I}$.

The converse implication between $\mathcal{I}$- and $\mathcal{I}^*$-convergence depends essentially on the structure of the metric space (X, ρ).

□

Theorem 3.1. *Let (X, ρ) be a metric space.*

(i) *If X has no accumulation point, then $\mathcal{I}$- and $\mathcal{I}^*$-convergence coincide for each admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$.*

(ii) *If X has an accumulation point ξ, then there exists an admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ of elements of X such that $\mathcal{I}$-$\lim_{n\to\infty} y_n = \xi$ but $\mathcal{I}^*$-$\lim y_n$ does not exist.*

PROOF. (i) Let $\xi \in X$ and $\mathcal{I}$-$\lim_{n\to\infty} x_n = \xi$. By virtue of Proposition 1.2. it suffices to prove that $\mathcal{I}^*$-$\lim_{n\to\infty} x_n = \xi$. Since X has no accumulation points, there exists $\delta > 0$ such that $B(\xi, \delta) = \{x \in X : \rho(x, \xi) < \delta\} = \{\xi\}$. From the assumption we have $\{n \in \mathbb{N} : \rho(x_n, \xi) \geq \delta\} \in \mathcal{I}$. Hence

$$\{n \in \mathbb{N} : \rho(x_n, \xi) < \delta\} = \{n \in \mathbb{N} : x_n = \xi\} \in \mathcal{F}(\mathcal{I})$$

and obviously $\mathcal{I}^*$-$\lim_{n\to\infty} x_n = \xi$.

(ii) Since ξ is an accumulation point of X, there exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X such that $\xi = \lim_{n\to\infty} x_n$ and the sequence $\{\rho(x_n, \xi)\}_{n\in\mathbb{N}}$ is decreasing to 0. Put $\epsilon_n = \rho(x_n, \xi)$ for $n \in \mathbb{N}$. For $\mathcal{I}$ we take the ideal $\mathcal{E}$ from Example 1.1. (g).

Define the sequence $\{y_n\}_{n\in\mathbb{N}}$ by $y_n = x_j$ if $n \in \Delta_j$. Let $\eta > 0$. Choose $\nu \in \mathbb{N}$ such that $\epsilon_\nu < \eta$. Then $A(\eta) = \{n : \rho(y_n, \xi) \geq \eta\} \subset \Delta_1 \cup \dots \cup \Delta_\nu$. Hence $A(\eta) \in \mathcal{E}$ and $\mathcal{E}$-$\lim_{n\to\infty} y_n = \xi$.

Suppose that $\mathcal{E}^*$-$\lim_{n\to\infty} y_n = \xi$. Then there exists a set $H \in \mathcal{E}$ such that for $M = \mathbb{N}\setminus H = \{m_1 < m_2 < \dots < m_k < \dots\}$ we have $\lim_{k\to\infty} \rho(y_{m_k}, \xi) = 0$. By definition of $\mathcal{E}$ there exists $l \in \mathbb{N}$ such that $H \subset \Delta_1 \cup \dots \cup \Delta_l$. But then $\Delta_{l+1} \subset \mathbb{N} \setminus H = M$, so for infinitely many k's (each Δ_i is an infinite set) we have $\rho(y_{m_k}, \xi) = \epsilon_{l+1} > 0$, which contradicts $y_{m_k} \to \xi$. Also the assumption $\mathcal{E}^*$-$\lim_{n\to\infty} y_n = y$ for $y \neq \xi$ leads to the contradiction. □

Now we shall formulate a necessary and sufficient condition (for the ideal $\mathcal{I}$) under which $\mathcal{I}$- and $\mathcal{I}^*$-convergence are equivalent. This condition (AP) is similar to the condition (APO) used in [7] and [10].

Definition 3.3. An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \dots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \dots\}$ such that $A_j \triangle B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$.

Remark 3.3. Observe that also $B_j \in \mathcal{I}$ for $j \in \mathbb{N}$.

Theorem 3.2. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal.*

(i) *If the ideal $\mathcal{I}$ has property (AP) and (X, ρ) is an arbitrary metric space, then for arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X $\mathcal{I}$-$\lim_{n\to\infty} x_n = \xi$ implies $\mathcal{I}^*$-$\lim_{n\to\infty} x_n = \xi$.*

(ii) *If (X, ρ) has at least one accumulation point and for arbitrary sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X and for each $\xi \in X$ $\mathcal{I}$-$\lim_{n\to\infty} x_n = \xi$ implies $\mathcal{I}^*$-$\lim_{n\to\infty} x_n = \xi$, then $\mathcal{I}$ has property (AP).*

Proof. (i) Suppose that $\mathcal{I}$ satisfies condition (AP). Let $\mathcal{I}$-$\lim_{n\to\infty} x_n = \xi$. Then $A(\epsilon) = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq \epsilon\} \in \mathcal{I}$ for $\epsilon > 0$. Put $A_1 = \{n \in \mathbb{N} : \rho(x_n, \xi) \geq 1\}$ and $A_n = \{n \in \mathbb{N} : \frac{1}{n} \leq \rho(x_n, \xi) < \frac{1}{n-1}\}$ for $n \geq 2$, $n \in \mathbb{N}$. Obviously $A_i \cap A_j = \emptyset$ for $i \neq j$. By condition (AP) there exists a sequence of sets $\{B_n\}_{n\in\mathbb{N}}$ such that $A_j \triangle B_j$ are finite sets for $j \in \mathbb{N}$ and $B = \bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. It is sufficient to prove that for $M = \mathbb{N} \setminus B$ we have

$$\lim_{\substack{n\to\infty \\ n\in M}} x_n = \xi. \tag{5}$$

Let $\eta > 0$. Choose $k \in \mathbb{N}$ such that $\frac{1}{k+1} < \eta$. Then $\{n \in \mathbb{N} : \rho(x_n, \xi) \geq \eta\} \subset \bigcup_{j=1}^{k+1} A_j$. Since $A_j \triangle B_j$, $j = 1, 2, \dots, k+1$ are finite sets, there exists $n_0 \in \mathbb{N}$ such that

$$\Big(\bigcup_{j=1}^{k+1} B_j\Big) \cap \{n \in \mathbb{N} : n > n_0\} = \Big(\bigcup_{j=1}^{k+1} A_j\Big) \cap \{n \in \mathbb{N} : n > n_0\}. \tag{6}$$

If $n > n_0$ and $n \notin B$, then $n \notin \bigcup_{j=1}^{k+1} B_j$ and, by (6), $n \notin \bigcup_{j=1}^{k+1} A_j$. But then $\rho(x_n, \xi) < \frac{1}{n+1} < \eta$; so (5) holds.

(ii) Suppose that $\xi \in X$ is an accumulation point of X. There exists a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X such that $\xi = \lim_{n\to\infty} x_n$ and the sequence $\{\rho(x_n, \xi)\}_{n\in\mathbb{N}}$ is decreasing to 0. For $n \in \mathbb{N}$ let $\epsilon_n = \rho(x_n, \xi)$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a disjoint family of non-empty sets from $\mathcal{I}$. Define a sequence $\{y_n\}_{n\in\mathbb{N}}$ by $y_n = x_j$ if $n \in A_j$. Let $\eta > 0$. Choose $m \in \mathbb{N}$ such that $\epsilon_m < \eta$. Then $A(\eta) = \{n \in \mathbb{N} : \rho(y_n, \xi) \geq \eta\} \subset A_1 \cup \cdots \cup A_m$. Hence $A(\eta) \in \mathcal{I}$ and $\mathcal{I}$-$\lim_{n\to\infty} y_n = \xi$. By virtue of our assumption we have also $\mathcal{I}^*$-$\lim_{n\to\infty} y_n = \xi$. Hence there exists a set $B \in \mathcal{I}$ such that if $M = \mathbb{N} \setminus B = \{m_1 < m_2 < \dots\}$, then

$$\lim_{k\to\infty} y_{m_k} = \xi. \tag{7}$$

Put $B_j = A_j \cap B$ for $j \in \mathbb{N}$. Then $B_j \in \mathcal{I}$ for each n. Further $\bigcup_{j=1}^{\infty} B_j = B \cap \bigcup_{j=1}^{\infty} A_j \subset B$. Hence $\bigcup_{j=1}^{\infty} B_j \in \mathcal{I}$. Fix $j \in \mathbb{N}$. From (7) it is clear that A_j has only a finite numbers of elements common with the set M. Thus there exists $k_0 \in \mathbb{N}$ such that $A_j \subset (A_j \cap B) \cup \{m_1, m_2, \dots, m_{k_0}\}$. Hence $A_j \triangle B_j = A_j \setminus B_j \subset \{m_1, m_2, \dots, m_{k_0}\}$; so $A_j \triangle B_j$ is a finite set. From the arbitrariness of $j \in \mathbb{N}$ it follows that $\mathcal{I}$ has property (AP). □

In [18] it is proved that $\mathcal{I}_{d_T}$- and $\mathcal{I}^*_{d_T}$-convergence are equivalent (in $\mathbb{R}$) provided that $T = \{t_{n,k}\}_{n,k\in\mathbb{N}}$ is a non-negative triangular matrix with $\sum_{k=1}^{n} t_{nk} = 1$ for $n \in \mathbb{N}$. From this we get that $\mathcal{I}_d$-, $\mathcal{I}_\delta$-convergence (Example 1.1. (b)) and $\mathcal{I}_\phi$-convergence (Example 1.1. (d)) coincide, respectively, with $\mathcal{I}^*_d$-, $\mathcal{I}^*_\delta$- and $\mathcal{I}^*_\phi$-convergence.

3.3 Functions Preserving $\mathcal{I}$-Convergence

Let (X, ρ) be a metric space and $\mathcal{I} \subset 2^{\mathbb{N}}$-an admissible ideal. As in [2] we say that a function $g : X \to X$ preserves $\mathcal{I}$-convergence in X if $\mathcal{I}$-$\lim_{n\to\infty} x_n = \xi$ implies $\mathcal{I}$-$\lim_{n\to\infty} g(x_n) = g(\xi)$ for each sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X and each $\xi \in X$. As is not difficult to predict, we have the following.

Proposition 3.3. *A function $g : X \to X$ preserves $\mathcal{I}$-convergence in X (for an arbitrary admissible ideal $\mathcal{I}$) if and only if g is continuous on X.*

Proof. 1. Let $\mathcal{I}$-$\lim_{n\to\infty} x_n = \xi$. If g is continuous, then for each $\eta > 0$ there exists $\delta > 0$ such that if $x \in B(\xi, \delta)$, then $g(x) \in B(g(\xi), \eta)$. But then we have

$$C(\delta) = \{n \in \mathbb{N} : \rho(x_n, \xi) < \delta\} \subset \{n \in \mathbb{N} : \rho(g(x_n), g(\xi)) < \eta\} = D(\eta)$$

and $D(\eta) \in \mathcal{F}(\mathcal{I})$, since $C(\delta) \in \mathcal{F}(\mathcal{I})$. Hence $\mathcal{I}$-$\lim_{n\to\infty} g(x_n) = g(\xi)$.

2. If g is not continuous at some $\xi \in X$, then there exists a number $\eta > 0$ and a sequence $\{x_n\}_{n\in\mathbb{N}}$ of elements of X such that $\lim_{n\to\infty} x_n = \xi$ and $\rho(g(x_n), g(\xi)) \geq \eta$ for $n \in \mathbb{N}$. Hence g does not preserve $\mathcal{I}$-convergence for any ideal $\mathcal{I}$. □

3.4 Relationship between $\mathcal{I}_d$- and I_δ-Convergence and Cesaro Summability

Recall that a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $(C,1)$-summable (or $(C,1)$-convergent) to $\xi \in \mathbb{R}$ if and only if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} x_i = \xi$ (see [23], p. 3) (abreviated $(C,1)$-$\lim_{n\to\infty} x_n = \xi$) and is said to be strongly $(C,1)$-summable to ξ if and only if $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |x_i - \xi| = 0$ (see [23], p. 5, [7]).

If $\{x_n\}_{n\in\mathbb{N}} \in \ell_\infty$, then $\mathcal{I}_d$-$\lim_{n\to\infty} x_n = \xi$ implies $(C,1)$-$\lim_{n\to\infty} x_n = \xi$ (see [7], [10], [26]). The converse is obviously not true (e.g. $\{0,1,0,1,\dots\}$). However, in ℓ^∞ the $\mathcal{I}_d$-convergence to some number is equivalent to strong Cesaro summability to the same number. For $\mathcal{I}_\delta$-convergence the situation is different.

Proposition 3.4. *In ℓ^∞ there is a sequence $\{x_n\}_{n\in\mathbb{N}}$ such that $\mathcal{I}_\delta$-$\lim_{n\to\infty} x_n = 0$ and $(C,1)$-$\lim_{n\to\infty} x_n$ does not exist.*

PROOF. Put $A = \bigcup_{k=2}^{\infty} A_k$, where $A_k = \{k^{k^2}+1, k^{k^2}+2, \dots, k^{k^2+1}\}$ for $k \in \mathbb{N}$, $k \geq 2$. If $A(n) = d_n(A)$ for $n \in \mathbb{N}$ (compare section 2), then

$$\overline{d}(A) \geq \limsup_{k\to\infty} \frac{A(k^{k^2+1})}{k^{k^2+1}} \geq \limsup_{k\to\infty} \frac{k^{k^2+1} - k^{k^2}}{k^{k^2+1}} = 1.$$

Hence

$$\overline{d}(A) = 1. \tag{8}$$

Simultaneously by (1) we have $\sum_{j\in A_n} \frac{1}{j} = \ln k + O(\frac{1}{k^{k^2}})$ for $k \in \mathbb{N},\ k \geq 2$. From this by a simple calculation we get

$$\overline{\delta}(A) \leq \lim_{n\to\infty} \frac{\sum_{k=1}^{n} \ln k + O(1)}{\sum_{j=1}^{n^{n^2+1}} \frac{1}{j}} \leq \lim_{n\to\infty} \frac{n\ln n + O(1)}{(n^2+1)\ln n + O(1)} = 0.$$

So we have $\delta(A) = 0$ and consequently

$$\underline{d}(A) = 0. \tag{9}$$

So $d(A)$ does not exist.

Define $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ by

$$x_n = \begin{cases} 0 & \text{if } n \in \mathbb{N} \setminus A \\ 1 & \text{if } n \in A. \end{cases}$$

Since $\delta(A) = 0$ we have $\mathcal{I}_\delta$-$\lim_{n\to\infty} x_n = 0$. But $(C,1)$-$\lim_{n\to\infty} x_n$ does not exist because $\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{A(n)}{n}$ for $n \in \mathbb{N}$ (compare (8), (9)). □

4 $\mathcal{I}$-Limit Points and $\mathcal{I}$-Cluster Points

Recall that a number $\xi \in \mathbb{R}$ is said to be a statistical limit point of a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers provided that there exists a set $M = \{m_1 < m_2 < \dots\} \subset \mathbb{N}$ such that $\overline{d}(M) > 0$ and $\lim_{k\to\infty} x_{m_k} = \xi$. A number $\xi \in \mathbb{R}$ is said to be a statistical cluster point of $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ provided that for each $\epsilon > 0$ we have $\overline{d}\{n \in \mathbb{N} : |x_n - \xi| < \epsilon\} > 0$ (see [9], [12], [13]).

We can extend these concepts to $\mathcal{I}$-convergence in the following way.

Definition 4.1. Let (X, ρ) be a metric space, $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ a sequence of elements of X.

a) An element $\xi \in X$ is said to be an $\mathcal{I}$-limit point of $\mathbf{x}$ provided that there is a set $M = \{m_1 < m_2 < \dots\} \subset \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim_{k\to\infty} x_{m_k} = \xi$.

b) An element $\xi \in X$ is said to be an $\mathcal{I}$-cluster point of $\mathbf{x}$ if and only if for each $\epsilon > 0$ we have $\{n \in \mathbb{N} : \rho(x_n, \xi) < \epsilon\} \notin \mathcal{I}$.

Denote by $\mathcal{I}(\Lambda_{\mathbf{x}})$ and $\mathcal{I}(\Gamma_{\mathbf{x}})$ the set of all $\mathcal{I}$-limit and $\mathcal{I}$-cluster points of $\mathbf{x}$, respectively.

Proposition 4.1. *Let $\mathcal{I}$ be an admissible ideal. Then for each sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of elements of X we have $\mathcal{I}(\Lambda_{\mathbf{x}}) \subset \mathcal{I}(\Gamma_{\mathbf{x}})$.*

PROOF. Let $\xi \in \mathcal{I}(\Lambda_{\mathbf{x}})$. Then there exists a set $M = \{m_1 < m_2 < \dots\} \notin \mathcal{I}$ such that

$$\lim_{k\to\infty} \rho(x_{m_k}, \xi) = 0. \tag{10}$$

Take $\delta > 0$. According to (10) there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have $\rho(x_{m_k}, \xi) < \delta$. Hence $\{n \in \mathbb{N} : \rho(x_n, \xi) < \delta\} \supset M \setminus \{m_1, \dots, m_{k_0}\}$ and so $\{n \in \mathbb{N} : \rho(x_n, \xi) < \delta\} \notin \mathcal{I}$, which means that $\xi \in \mathcal{I}(\Gamma_{\mathbf{x}})$. □

Theorem 4.1. *Let $\mathcal{I}$ be an admissible ideal.*

(i) *The set $\mathcal{I}(\Gamma_{\mathbf{x}})$ is closed in X for each sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of elements of X.*

(ii) *Suppose that (X,ρ) is a separable metric space. Suppose that there exists a disjoint sequence of sets $\{M_n\}_{n\in\mathbb{N}}$ such that $M_n \subset \mathbb{N}$ and $M_n \notin \mathcal{I}$ for $n \in \mathbb{N}$. Then for each closed set $F \subset X$ there exists a sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of elements of X such that $F = \mathcal{I}(\Gamma_{\mathbf{x}})$.*

PROOF. (i) Let $y \in \overline{\mathcal{I}(\Gamma_{\mathbf{x}})}$. Take $\epsilon > 0$. There exists $\xi_0 \in \mathcal{I}(\Gamma_{\mathbf{x}}) \cap B(y,\epsilon)$. Choose $\delta > 0$ such that $B(\xi_0,\delta) \subset B(y,\epsilon)$. We obviously have

$$\{n \in \mathbb{N} : \rho(y,x_n) < \epsilon\} \supset \{n \in \mathbb{N} : \rho(\xi_0,x_n) < \delta\}.$$

Hence $\{n \in \mathbb{N} : \rho(y,x_n) < \epsilon\} \notin \mathcal{I}$ and $y \in \mathcal{I}(\Gamma_{\mathbf{x}})$.

(ii) Let $A = \{a_1,a_2,\dots\} \subset F$ be a countable set dense in F. For $n \in M_i$ we put $x_n = a_i$. Obviously we have $\mathcal{I}(\Gamma_{\mathbf{x}}) \subset F$. To prove the converse inclusion take $z \in F$ and $\epsilon > 0$. There exists $i_0 \in \mathbb{N}$ such that $\rho(a_{i_0},z) < \epsilon$. Since $x_n = a_{i_0}$ for each $n \in M_{i_0}$, we obtain $\{n \in \mathbb{N} : \rho(x_n,z) < \epsilon\} \supset M_{i_0}$; so $\{n \in \mathbb{N} : \rho(x_n,z) < \epsilon\} \notin \mathcal{I}$ and $z \in \mathcal{I}(\Gamma_{\mathbf{x}})$. □

In [13] the following result has been established for sequences of real numbers.

Theorem A.

(i) *For each sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of real numbers the set $\mathcal{I}_d(\Lambda_{\mathbf{x}})$ is of type F_σ.*

(ii) *If $F \subset \mathbb{R}$ is a set of type F_σ, then there exists a sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of real numbers such that $F = \mathcal{I}_d(\Lambda_{\mathbf{x}})$.*

A detailed analysis of the proof of Theorem 4 in [13] shows that in this theorem $\mathcal{I}_d$ can be replaced by $\mathcal{I}_\delta$. It would be desirable to extend Theorem 4 for more general $\mathcal{I}$-convergence.

It is not difficult to observe that $\mathcal{I}$-convergence cannot in general be metrizable.

Proposition 4.2. *Suppose that X has at least two elements and $\mathcal{I} \subset 2^{\mathbb{N}}$ is an admissible ideal containing an infinite set $M \subset \mathbb{N}$. Then $\mathcal{I}$-convergence cannot be metrizable.*

PROOF. Suppose that there exists a metric σ on X such that $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi$ if and only if $\lim_{n\to\infty} \sigma(x_n,\xi) = 0$. Take $\xi_1,\xi_2 \in X$, $\xi_1 \neq \xi_2$ and put $x_n = \xi_1$ if $n \in M$, $x_n = \xi_2$ if $n \notin M$. Obviously we have $\mathcal{I}\text{-}\lim_{n\to\infty} x_n = \xi_2$ and $\lim_{n\to\infty} \sigma(x_n,\xi_2) = 0$, which implies $\sigma(\xi_1,\xi_2) = 0$ since $\mathbb{N} \setminus M$ is an infinite set. This contradicts $\xi_1 \neq \xi_2$. □

In connection with the above mentioned results about $\mathcal{I}_d(\Lambda_{\mathbf{x}})$ (compare [14]) we can conjecture that (at least under some reasonable conditions) the set $\mathcal{I}(\Lambda_{\mathbf{x}})$ is of type F_σ. To show that the separability of (X, ρ) is essential, we prove the following.

Theorem 4.2. *Suppose that (X, ρ) is not separable and $\mathcal{I} \subset 2^{\mathbb{N}}$ is an admissible ideal.*

(i) *There exists a closed set $F \subset X$ such that for each sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of elements of X we have $\mathcal{I}(\Gamma_{\mathbf{x}}) \neq F$.*

(ii) *There exists an open set $G \subset X$ (so also G is of type F_σ) such that for each sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of elements of X we have $\mathcal{I}(\Lambda_{\mathbf{x}}) \neq G$.*

In the proof we shall need the following lemma (see [21]).

Lemma 4.1. *Suppose that (X, ρ) is not separable. Then there exists $\epsilon_1 > 0$ and an uncountable set $D = \{d_0, d_1, \dots, d_\alpha, \dots\} \subset X$, $\alpha < \Omega$ such that for $\alpha, \beta < \Omega$, $\alpha \neq \beta$ we have $\rho(d_\alpha, d_\beta) \geq \epsilon_1$.*

Proof of Theorem 4.2 (i) Let D be the (closed) set from Lemma 4.1. We shall prove that $D \neq \mathcal{I}(\Gamma_{\mathbf{x}})$ for each sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of elements of X. Consider the uncountable family $\{B(d_\alpha, \frac{\epsilon_1}{2}) : \alpha < \Omega\}$ of disjoint balls. There exists $\alpha_0 < \Omega$ such that $\{n \in \mathbb{N} : x_n \in B(d_{\alpha_0}, \frac{\epsilon_1}{2})\} = \emptyset$. Hence $d_{\alpha_0} \notin \mathcal{I}(\Gamma_{\mathbf{x}})$.

(ii) Put $G = \bigcup_{\alpha<\Omega} B(d_\alpha, \frac{\epsilon_1}{2})$. The proof is similar to that of part (i). □

5 $\mathcal{I}$-Convergence of Sequences of Functions

In a natural manner we can extend the notion of $\mathcal{I}$-convergence of sequences in X to $\mathcal{I}$-convergence of sequences of functions.

Definition 5.1. Let X be a non-empty set and let (Y, τ) be a metric space. Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ transforming X into Y is said to $\mathcal{I}$-converge to a function $f : X \to Y$ provided that for each $x \in X$ we have $\mathcal{I}\text{-}\lim_{n\to\infty} f_n(x) = f(x)$.

The function f is called the $\mathcal{I}$-limit function of the sequence $\{f_n\}_{n\in\mathbb{N}}$ and we write $f = \mathcal{I}\text{-}\lim_{n\to\infty} f_n$.

Remark 5.1. (a) From Proposition 3.1 it follows that the $\mathcal{I}$-limit function is uniquely determined.

(b) From Proposition 4.2 it follows that the $\mathcal{I}$-convergence of sequences of functions is not metrizable.

If we additionally assume that (X,ρ) is a metric space and (Y,τ) is $\mathbb{R}$ equipped with the natural metric, the question arises whether the $\mathcal{I}$-limit function of a sequence of continuous functions belongs to B_1 (Baire class one) and a similar question arises for higher Baire classes. Taking into account that $\mathcal{I}_d$-convergence of bounded real sequences implies $(C,1)$-summability one can answer positively for questions concerning all Baire classes. However, for ideals different from $\mathcal{I}_d$ the situation is a little bit more complicated. Under some conditions (concerning the ideal $\mathcal{I}$) we are able to give the answer to the question concerning continuous functions.

We shall suppose that μ is a finitely additive, normed measure defined on some class of subsets of $\mathbb{N}$ (as in Example 3.1 (f)), $\mu(A) = \lim_{m\to\infty} \mu_m(A)$, where $\mu_m : 2^{\mathbb{N}} \to [0,1]$ is a finitely additive measure on $2^{\mathbb{N}}$. Further, we shall assume that each μ_m satisfies the following condition.

$$\begin{aligned}&\text{For each } m \in \mathbb{N} \text{ and for each } A \subset \mathbb{N} \quad \mu_m(A)\\ &= \lim_{k\to\infty} \mu_m(A \cap \{n \in \mathbb{N} : n \le k\}).\end{aligned} \tag{11}$$

Theorem 5.1. *Suppose that (X,ρ) is a complete metric space and $f_n : X \to \mathbb{R}$ are continuous functions for $n \in \mathbb{N}$. If $\mathcal{I}_\mu$-$\lim f_n = f$, then $f \in B_1$.*

Proof. Suppose that $f \notin B_1$. Then from [25] (see also [15], p. 39), we can conclude that there exists a perfect set $F \subset X$ and two numbers a, b, $a < b$ such that each of the sets $T_1 = \{x \in F : f(x) < a\}$, $T_2 = \{x \in F : f(x) > b\}$ is dense in F. Choose $x_1 \in T_1 \cap F$. Then $\mathcal{I}_\mu$-$\lim_{n\to\infty} f_n(x_1) = f(x_1) < a$. Put $\epsilon_1 = \frac{1}{2}(a - f(x_1))$ and $A(\epsilon_1) = \{n \in \mathbb{N} : f_n(x_1) < f(x_1) + \epsilon_1\} = \{n \in \mathbb{N} : f_n(x_1) < \frac{1}{2}(a + f(x_1)\}$. Then $\mu(A(\epsilon_1)) = 1$; so $\lim_{m\to\infty} \mu_m(A(\epsilon_1)) = 1$. Therefore there exists $m_1 \in \mathbb{N}$ such that $\mu_{m_1}(A(\epsilon_1)) > \frac{1}{2}$. By (11) we conclude that there exists $k_1 \in \mathbb{N}$ such that $\mu_{m_1}(A(\epsilon_1) \cap \{n \in \mathbb{N} : n \le k_1\}) > \frac{1}{2}$.

Since all functions f_j for $j \le k_1$, $j \in A(\epsilon_1)$ are continuous, there exists $\delta_1 > 0$ such that for each $x \in \overline{B(x_1,\delta_1)}$ and each $n \le k_1$, $n \in A(\epsilon_1)$ we have $f_n(x) \le a$ and so

$$\mu_{m_1}(\{n \in \mathbb{N} : n \le k_1, n \in A(\epsilon_1) \text{ and } f_n(x) \le a \text{ for each } x \in \overline{B(x_1,\delta_1)}\}) > \frac{1}{2}.$$

Choose arbitrary $x_2 \in T_2 \cap F \cap B(x_1,\delta_1)$. Then $\mathcal{I}_\mu$-$\lim_{n\to\infty} f_n(x_2) = f(x_2) > b$. Take $\epsilon_2 = \frac{1}{2}(f(x_2) - b)$ and put

$$A(\epsilon_2) = \{n \in \mathbb{N} : f_n(x_2) > f(x_2) - \epsilon_2\} = \{n \in \mathbb{N} : f_n(x_2) > \frac{1}{2}(f(x_2) + b\}.$$

We have $\mu(A(\epsilon_2)) = 1$, so as before we can find $m_2 \in \mathbb{N}$ and $k_2 \in \mathbb{N}$ (moreover, $k_2 > k_1$) such that $\mu_{m_2}(A(\epsilon_2) \cap \{n \in \mathbb{N} : n \le k_2\}) > \frac{1}{2}$. Again from the continuity of all functions f_j, $j \le k_2$, $j \in A(\epsilon_2)$ it follows that there exists $\delta_2 > 0$ such that $\delta_2 < \frac{1}{2}\delta_1$, $B(x_2,\delta_2) \subset B(x_1,\delta_1)$ and for each $x \in \overline{B(x_2,\delta_2)}$ and each $n \le k_1$, $n \in A(\epsilon_2)$ we have $f_n(x) \ge b$ and so

$$\mu_{m_2}(\{n \in \mathbb{N} : n \le k_2, n \in A(\epsilon_2) \text{ and } f_n(x) \ge b \text{ for each } x \in \overline{B(x_2,\delta_2)}\}) > \frac{1}{2}.$$

In this way (by induction) we construct a sequence $\{x_n\}_{n\in\mathbb{N}}$ of points of F, a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ decreasing to 0, a sequence $\{\epsilon_n\}_{n\in\mathbb{N}}$ of positive numbers, a sequence $A(\epsilon_n)$ of subsets of $\mathbb{N}$ and a descending sequence $\{\overline{B(x_n,\delta_n)}\}_{n\in\mathbb{N}}$ of closed sets with diameters tending to 0. Simultaneously we obtain two increasing sequences of natural numbers $\{m_i\}_{i\in\mathbb{N}}$ and $\{k_i\}_{i\in\mathbb{N}}$ such that

$$\begin{aligned}\mu_{m_{2i-1}}(\{n \in \mathbb{N} :& n \le k_{2i-1}, n \in A(\epsilon_{2i-1}) \text{ and } f_n(x) \le a \\ &\text{for each } x \in \overline{B(x_{2i-1},\delta_{2i-1})}\}) > \frac{1}{2}\end{aligned} \tag{12}$$

$$\begin{aligned}\mu_{m_{2i}}(\{n \in \mathbb{N} :& n \le k_{2i}, n \in A(\epsilon_{2i}) \text{and } f_n(x) \ge b \\ &\text{for each } x \in \overline{B(x_{2i},\delta_{2i})}\}) > \frac{1}{2}\end{aligned} \tag{13}$$

Let $x_0 \in \bigcap_{k=1}^{\infty} \overline{B(x_k,\delta_k)}$. By the monotonicity of μ_m, $m \in \mathbb{N}$ from (12) and (13) we obtain

$$\mu_{m_{2i-1}}(\{n \in \mathbb{N} : f_n(x_0) \le a\}) > \frac{1}{2} \tag{14}$$

$$\mu_{m_{2i}}(\{n \in \mathbb{N} : f_n(x_0) \ge b\}) > \frac{1}{2} \tag{15}$$

for $i \in \mathbb{N}$. Suppose that $\mathcal{I}_\mu$-$\lim_{n\to\infty} f_n(x_0) = f(x_0)$. If $f(x_0) \le a$, we obtain a contradiction to (15), if $f(x_0) \ge b$; a contradiction to (14), if $a < f(x_0) < b$; a contradiction to both (14) and (15). □

Remark 5.2. According to Theorem 5.1 $\mathcal{I}$-limit function of a sequence of continuous real functions belongs to Baire class one if $\mathcal{I} = \mathcal{I}_d, \mathcal{I}_\delta, \mathcal{I}_{d_T}, \mathcal{I}_\phi$. Using similar technique one can prove that the same holds also for the ideal from Example 3.1 (g).

Now we shall show that there are also admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ and sequences $\{f_n\}_{n\in\mathbb{N}}$ of continuous real functions defined on $[0,1]$ such that $f = \mathcal{I}$-$\lim_{n\to\infty} f_n$ does not belong to B_1. Using Zorn's lemma one can show that in the family of all admissible ideals $\mathcal{I} \subset 2^{\mathbb{N}}$ there exists a maximal ideal (with respect to inclusion). We shall need the following properties of maximal ideals.

Lemma 5.1. *If $\mathcal{I} \subset 2^{\mathbb{N}}$ is a maximal admissible ideal, then for each $A \subset \mathbb{N}$ we have either $A \in \mathcal{I}$, or $\mathbb{N} \setminus A \in \mathcal{I}$.*

Lemma 5.2. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a maximal admissible ideal. Then each bounded sequence $\mathbf{x} = \{x_n\}_{n\in\mathbb{N}}$ of real numbers is $\mathcal{I}$-convergent; i.e., there exists $\xi \in \mathbb{R}$ such that $\xi = \mathcal{I}\text{-}\lim_{n\to\infty} x_n$.*

PROOF. See Theorem 4.1. in [14]. □

Theorem 5.2. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be a maximal admissible ideal. There exists an $\mathcal{I}$-convergent sequence $\{f_n\}_{n\in\mathbb{N}}$ of continuous functions transforming $[0,1]$ onto $[0,1]$ such that $f = \mathcal{I}\text{-}\lim_{n\to\infty} f_n$ does not belong to B_1.*

PROOF. Let $\{r_i\}_{i\in\mathbb{N}}$ be a sequence containing each rational number from $[0,1]$ exactly once. For $n \in \mathbb{N}$, let $f_n : [0,1] \to [0,1]$ be a continuous function such that $f_n(r_i) = 1$ for $i \in \{1,\dots,n\}$ and $\lambda(\{x \in [0,1] : f_n(x) > 0\}) < 2^{-n}$, where λ is a Lebesgue measure. Then we have $\limsup_{n\to\infty} f_n(x) = 0$ a.e. on $[0,1]$ and $\lim_{n\to\infty} f_n(r_i) = 1$ for $i \in \mathbb{N}$. Since obviously $\mathcal{I}\text{-}\lim_{n\to\infty} f_n(x) = f(x)$ exists everywhere by Lemma 5.2 and $\mathcal{I}\text{-}\lim_{n\to\infty} f_n(x) \le \limsup_{n\to\infty} f_n(x)$, we have $\mathcal{I}\text{-}\lim_{n\to\infty} f_n(x) = 0$ a.e.; so f is 1 on a set dense in $[0,1]$ and is 0 on another set dense in $[0,1]$ and consequently $f \notin B_1$.

Observe that in the above proof f is Lebesgue measurable. We can construct another sequence $\{f_k\}_{k\in\mathbb{N}}$ of continuous real functions defined on $[0,1]$ such that $\mathcal{I}\text{-}\lim_{k\to\infty} f_k$ is non-measurable for each admissible maximal ideal $\mathcal{I} \subset 2^{\mathbb{N}}$.

A function f is a function of accumulation of $\{f_k\}_{k\in\mathbb{N}}$ if and only if for each $\epsilon > 0$ and for each finite set $\{x_1, x_2, \dots, x_m\}$ (included in the domain) there exists an infinite subset $K = \{k_1, k_2, \dots, k_p, \dots\} \subset \mathbb{N}$ such that for each $k \in K$ and for each $i \in \{1,\dots,m\}$ we have $|f_k(x_i) - f(x_i)| < \epsilon$.

Sierpiński in [30] found a sequence $\{f_k\}_{k\in\mathbb{N}}$ of continuous functions transforming $[0,1]$ onto $[0,1]$ such that each function of accumulation of this sequence is non-measurable.

Observe that if $f = \mathcal{I}\text{-}\lim_n f_n$, then f is a function of accumulation of $\{f_n\}$. Indeed, take $\epsilon > 0$ and a finite set $\{x_1, x_2, \dots, x_m\} \subset [0,1]$. Then for each $i \in \{1,\dots,m\}$ $E_i(\epsilon) = \{n : |f_n(x_i) - f(x_i)| \ge \epsilon\} \in \mathcal{I}$; so $\bigcup_{i=1}^m E_i(\epsilon) \in \mathcal{I}$. But then (for our ideals) $\mathbb{N} \setminus \bigcup_{i=1}^m E_i(\epsilon)$ is an infinite set (It does not belong to $\mathcal{I}$.) and for each $i \in \{1,\dots,m\}$ and each $k \in \mathbb{N} \setminus \bigcup_{i=1}^m E_i(\epsilon) = \bigcap_{i=1}^m (\mathbb{N} \setminus E_i(\epsilon))$ we have $|f_k(x_i) - f(x_i)| < \epsilon$. Now if $\mathcal{I}$ is a maximal ideal and $\{f_n\}_{n\in\mathbb{N}}$ is a sequence constructed by Sierpiński, then there exists a function $f = \mathcal{I}\text{-}\lim_n f_n$. Hence f is also a function of accumulation and therefore f is non-measurable. □

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