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A NEW ELEMENTARY PROOF OF A THEOREM OF DE LA VALLÉE POUSSIN

Abstract

We give a new elementary proof of the Classical Theorem: Let f be of bounded variation on $[a, b]$ and let V be its total variation function. Then there is a set N such that $m\big(V(N)\big) = m\big(f(N)\big) = m(N) = 0$, and for each x not in N, f and V have derivatives, finite or infinite, and $V'(x) = |f'(x)|$.

Let f be a real valued function of bounded variation on $[a, b]$, and for $x \in [a, b]$, let $V(x)$ denote the total variation of f on the interval $[a, x]$. In this note we give a new elementary proof of a classical result of de la Valée Poussin (see [3, Theorem 9.6 (ii), chapter IV]).

Classical Theorem. *There exists a set $N \subset [a, b]$ such that*

$$m\big(V(N)\big) = m\big(f(N)\big) = m(N) = 0$$

(where m denotes Lebesgue outer measure) and such that for $x \in [a, b] \setminus N$, $V'(x)$ and $f'(x)$ exist (finite or infinite) and $|f'(x)| = V(x)$.

Recently Vasile Ene in [1] revived interest in the Theorem by giving analogues for it and other work involving functions of generalized bounded variation. Thus he found new proofs of the Classical Theorem by these sophisticated means. We will not require integrals nor arc length. We use derived numbers [2, p. 207] and the Vitali Covering Theorem [2, p. 81] and little else. The proof is almost immediate from our Lemma 1.

Lemma 1. *Let h and k be positive numbers with $h < k$ and let $E \subset [a, b]$ be a set such that at each $x \in E$, V has a derived number greater than k and f*

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has a derived number whose absolute value is less than h. Let $S \subset [a,b]$ be a set such that at each $x \in S$, f has a positive and a negative derived number. Then

$$m\big(V(E \cup S)\big) = m\big(f(E \cup S)\big) = m(E \cup S) = 0\,.$$

PROOF. Take any $\epsilon > 0$. Let $a = u_o < u_1 < \cdots < u_n = b$ be a partition of $[a,b]$ such that for any partition $a = z_o < z_1 < \cdots < z_t = b$ that contains all the u_i we have

$$V(b) - V(a) < \sum_{i=1}^{t} \big|f(z_i) - f(z_{i-1})\big| + \epsilon\,,$$

and hence

$$V(b) - V(a) = \sum_{i=1}^{t} \big(V(z_i) - V(z_{i-1})\big) < \sum_{i=1}^{t} \big|f(z_i) - f(z_{i-1})\big| + \epsilon\,. \quad (1)$$

Let P denote the finite set $\{u_0, u_1, \ldots, u_n\}$.

Without loss of generality we assume that V and f are continuous at each point of $E \cup S$. Let U be an open set with $V(E) \subset U$ and $m(U) < m\big(V(E)\big) + \epsilon$. We use the Vitali Covering Theorem (on the y-axis) to cover $V(E)$ almost everywhere with mutually disjoint intervals $\big[V(a_i), V(b_i)\big]$ where

$$\big[V(a_i), V(b_i)\big] \subset U \text{ and } V(b_i) - V(a_i) > k(b_i - a_i) \text{ for each } i.$$

Then the intervals $[a_i, b_i]$ are also mutually disjoint, and

$$m\big(V(E)\big) + \epsilon > m(U) \geq \sum_i \big(V(b_i) - V(a_i)\big) > k \sum_i (b_i - a_i)\,,$$

so

$$\sum_i (b_i - a_i) < \Big(m\big((V(E)\big) + \epsilon\Big) k^{-1}\,. \quad (2)$$

Again we use the Vitali Covering Theorem (on the y-axis) to cover $V(E)$ almost everywhere with mutually disjoint intervals $\big[V(c_j), V(d_j)\big]$ where $[c_j, d_j] \subset \cup_i [a_i, b_i]$, $P \cap [c_j, d_j]$ is void, and

$$\big|f(d_j) - f(c_j)\big| < h(d_j - c_j) \text{ for each } j\,.$$

Then

$$m\big(V(E)\big) \leq \sum_j \big(V(d_j) - V(c_j)\big)\,. \quad (3)$$

But we deduce from (1) that

$$\sum_j \big(V(d_j)-V(c_j)\big) < \sum_j \big|f(d_j)-f(c_j)\big| + \epsilon \tag{4}$$

and from $\cup_j [c_j, d_j] \subset \cup_i [a_i, b_i]$ we deduce that

$$\sum_j (d_j - c_j) \le \sum_i (b_i - a_i)\,. \tag{5}$$

We combine (3), (4) and (5) to obtain

$$\begin{aligned} m\big(V(E)\big) \le & \sum_j \big(V(d_j)-V(c_j)\big) \le \sum_j \big|f(d_j)-f(c_j)\big| + \epsilon \\ \le & h\sum_j (d_j - c_j) + \epsilon \le h\sum_i (b_i - a_i) + \epsilon \end{aligned}$$

and

$$\Big(m\big(V(E)\big) - \epsilon\Big)h^{-1} \le \sum_i (b_i - a_i)\,. \tag{6}$$

We combine (2) and (6) to obtain $\Big(m\big(V(E)\big)+\epsilon\Big)k^{-1} \ge \Big(m\big(V(E)\big)-\epsilon\Big)h^{-1}$. But ϵ was arbitrary, so $m\big(V(E)\big)k^{-1} \ge \Big(m\big(V(E)\big)\Big)h^{-1}$. Now $0 < h < k$, and it follows that $m\big(V(E)\big) = 0$.

We use the definition of S and the Covering Theorem again to cover $V(S)$ almost everywhere with mutually disjoint intervals $\big[V(r_i), V(s_i)\big]$ so that for each i, $f(s_i) > f(r_i)$, $P \cap [r_i, s_i]$ is void and

$$m\big(V(S)\big) \le \sum_i \big(V(s_i) - V(r_i)\big) \le \sum_i \big(f(s_i) - f(r_i)\big) + \epsilon\,. \tag{7}$$

Again cover the set $V(S)$ almost everywhere with mutually disjoint intervals $\big[V(p_j), V(q_j)\big]$ with $[p_j, q_j] \subset \cup_i [r_i, s_i]$, $P \cap [p_j, q_j]$ is void and $f(q_j) < f(p_j)$ for each j, so that

$$m\big(V(S)\big) \le \sum_j \big(V(q_j) - V(p_j)\big) \le \sum_j \big(f(p_j) - f(q_j)\big) + \epsilon. \tag{8}$$

We compare upper and lower variations of f and deduce that

$$\sum_j \big(f(p_j) - f(q_j)\big) + \sum_i \big(f(s_i) - f(r_i)\big) \le \sum_i \big(V(s_i) - V(r_i)\big).$$

By (7) and (8) we have

$$\sum_j \big(V(q_j) - V(p_j)\big) - \epsilon + \sum_i \big(V(s_i) - V(r_i)\big) - \epsilon \leq \sum_i \big(V(s_i) - V(r_i)\big)$$

and $\sum_j \big(V(q_j) - V(p_j)\big) \leq 2\epsilon$. We invoke (8) and obtain $m\big(V(S)\big) \leq 2\epsilon$. But ϵ was arbitrary, so $m\big(V(S)\big) = 0$. It follows that $m\big(V(E \cup S)\big) = 0$. Cover $V(E \cup S)$ with intervals I_j such that $\sum_j m(I_j) < \epsilon$. Now each set $f\big(V^{-1}(I_j)\big)$ is contained in an interval whose length does not exceed that of I_j. It follows that $m\big(f(E \cup S)\big) < \epsilon$. Hence $m\big(f(E \cup S)\big) = 0$.

Finally at each point of $E \cup S$, V has a positive derived number. That $m(E \cup S) = 0$ can be proved by a standard Vitali covering argument, this time on the x-axis. The argument can be found, for example, in [2, pp. 210, 211], so we leave it. □

Proof of Theorem. We let h and k run through all the positive rational numbers ($h < k$) and deduce from Lemma 1 that there is a set $N \subset [a, b]$ such that

$$m\big(V(N)\big) = m\big(f(N)\big) = m(N) = 0$$

and for any $x \in [a, b] \setminus N$, for any derived number $DV(x)$ of V at x and for any derived number $Df(x)$ of f at x, we have $DV(x) \leq \big|Df(x)\big|$; moreover f does not have two derived numbers of opposite sign at x.

Let $Df(x)$ be a derived number of f at x. Then there cannot be a derived number $D_1f(x)$ such that $\big|Df(x)\big| > \big|D_1f(x)\big|$; otherwise V would have a derived number as large as $\big|Df(x)\big|$ and larger than $\big|D_1f(x)\big|$, contrary to the choice of x. Thus $\big|Df(x)\big|$ and $-\big|Df(x)\big|$ are the only possible derived numbers of f at x, so there can be only one. It follows that f has a derivative at x, finite or infinite. If $DV(x)$ is a derived number of V at x, then necessarily $DV(x) \geq \big|f'(x)\big|$. We deduce from the choice of x that $DV(x) = \big|f'(x)\big|$, V has only one derived number at x, V has a derivative at x, and moreover $V'(x) = \big|f'(x)\big|$. □

References

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