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ON A CONSTRUCTION OF J. TKADLEC CONCERNING σ-POROUS SETS

Abstract

In the paper [2] Josef Tkadlec gave an example of a finite singular Borel measure μ on the real line such that all σ-porous sets are of μ-measure zero. We give an alternative proof, i.e. probably a simpler construction, of his theorem [2, theorem] and we also give a similar example in $\mathbb{R}^n$.

A subset S of a metric space (X, d) is said to be porous at a point x,

$$\limsup_{\varepsilon \to 0} \frac{f(x, \varepsilon)}{\varepsilon} > 0$$

where $f(x, \varepsilon) = \sup\{r : \exists p,\, B(p, r) \subset B(x, \varepsilon) \setminus S\}$ and $B(p, r)$ denotes the open ball around p with radius r. The set S is porous if it is porous at each of its points.

Any porous set S in $\mathbb{R}^n$ is of Lebesgue measure zero, because its density is smaller then 1 at any point of S. Let μ be a Borel measure on $[0, 1]^n$ such that for any $0 < c < 1$ there exists a constant $d(c)$ depending only on c such that

$$\frac{\mu(B(p, cr))}{\mu(B(x, r))} > d(c) \tag{1}$$

provided that $B(p, cr) \subset B(x, r) \subset [0, 1]^n$, and r is small enough. Then the above argument concerning the μ-density of a porous set S shows that $\mu(S)$ has to be zero.

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We are going to construct Borel probability measures on $[0,1]^n$ satisfying inequality (1), and singular with respect to the Lebesgue measure. It is clear that we can use the "max" distance in $\mathbb{R}^n$; i.e.

$$d_n(x,y) = \max_{1\le i\le n} |x_i - y_i| .$$

It is enough to show the existence of a singular probability measure on the real line satisfying (2). Indeed if we have such a μ, a singular probability Borel measure on $[0,1]$ then $\mu_n = \mu \times \cdots \times \mu$ (n-fold product) is a singular probability measure on $[0,1]^n$ such that

$$\frac{\mu_n(\times_i(a_i,b_i))}{\mu_n(\times_i(c_i,d_i))} = \prod_{i=1}^n \frac{\mu((a_i,b_i))}{\mu((c_i,d_i))} > d(c)^n > 0$$

provided that $(a_i,b_i) \subset (c_i,d_i) \subset [0,1]$, $|b_i - a_i| > c\,|d_i - c_i|$ and $\max(|d_i - c_i|)$ is small enough.

We do something more that is needed. Almost with same effort we can construct a measure such that $d(c)$ can be chosen $d\cdot c$ with ($d = 1/216$).

So let $0 \le \alpha_n < 1/4$ be a decreasing sequence of non-negative numbers. We will denote by $|I|$ the length of the interval I. Put

$$\mu_{n,\alpha}(\{i\}) = \begin{cases} \frac{1-\alpha_n}{3} & i = 0,2 \\ \frac{1+2\alpha_n}{3} & i = 1; \end{cases}$$

i.e. we defined a probability measure on the power set of $A = \{0,1,2\}$. Let us form the infinite product measure $\mu_\alpha = \prod_n \mu_{n,\alpha}$ on the Borel sets of $A^{\mathbb{N}}$. Denote $\nu = \mu_0$ the measure we get in this way from the zero sequence.

Define a mapping $T : A^{\mathbb{N}} \to [0,1]$ by the formula

$$T(x) = \sum_{k=1}^{\infty} \frac{x(k)}{3^k},$$

i.e. the sequence x is the ternary expansion of $T(x)$. It is well known that there are countable sets $H \subset A^{\mathbb{N}}$ and $H' \subset [0,1]$ such that $T|_{A^{\mathbb{N}}\setminus H} : A^{\mathbb{N}} \setminus H \to [0,1] \setminus H'$ is bijective, measurable and the inverse is also measurable. So $\mu_\alpha \circ T^{-1}$ and $\nu \circ T^{-1}$ are singular if and only if μ_α and ν is such. Observe that countable sets are of μ_α–measure zero, whatever is the sequence α satisfying $0 \le \alpha_n < 1/4$.

It is also clear that $\nu \circ T^{-1}$ is just the Lebesgue measure on $[0,1]$. An old and famous theorem of Kakutani [1] can be applied to decide which sequences give singular measure $\mu_\alpha \circ T^{-1}$ with respect to the Lebesgue measure. He introduced the "inner product" of probability measures τ, ϑ defined on the same measurable space as follows:

$$\rho(\tau, \vartheta) = \int \left(\frac{d\tau}{d\pi}\right)^{1/2} \left(\frac{d\vartheta}{d\pi}\right)^{1/2} d\pi$$

where both τ, and ϑ are absolutely continuous with respect to the measure π. ρ does not depend on the choice of π. The two measures are the same if ρ is 1 and mutually singular if ρ is 0.

Theorem 1. *([1, Kakutani 1948]) Let μ_n, ν_n be a sequence of equivalent measures. $\prod_n \mu_n$ and $\prod_n \nu_n$ are either singular or equivalent according to*

$$\prod \rho(\mu_n, \nu_n) = 0 \text{ or } > 0.$$

Corollary 2. *$\mu_\alpha \circ T^{-1}$ is singular with respect to the Lebesgue measure if and only if*

$$\sum_{k=1}^{\infty} \alpha_k^2 = \infty$$

Proof. $\rho(\mu_{k,\alpha}, \mu_{k,0}) = 1/3(2\sqrt{1-\alpha_k} + \sqrt{1+2\alpha_k})$ which has logarithm of order $-\alpha_k^2$. □

In what follows let $0 < \alpha_n < 1/4$ be a fixed decreasing sequence and μ denotes $\mu_\alpha \circ T^{-1}$. Put

$$\mathcal{I}_n = \left\{ \left[\frac{k}{3^n}, \frac{k+1}{3^n}\right] \ : \ k = 0, 1, \ldots, 3^n - 1 \right\}.$$

We say that two intervals are joining if one of their endpoints coincide, more precisely I and J are joining if $I \cup J$ is interval and $I \cap J$ has at most one point.

The following two lemmas are trivial corollaries of the definition of μ

Lemma 3. *Let $I, J \in \mathcal{I}_n$ be two joining intervals, then $1/2 < \frac{\mu(I)}{\mu(J)} < 2$.*

Proof. By induction on n. For $n = 0$ there is nothing to prove. Let $I, J \in \mathcal{I}_{n+1}$ two joining intervals. Either there is $K \in \mathcal{I}_n$ such that $I, J \subset K$ or there are joining intervals $K_0, K_1 \in \mathcal{I}_n$ such that $I \subset K_0$ and $J \subset K_1$. In the first case

$$\frac{1}{2} < \frac{1-\alpha_{n+1}}{1+2\alpha_{n+1}} \le \frac{\mu(J)}{\mu(I)} \le \frac{1+2\alpha_{n+1}}{1-\alpha_{n+1}} < 2$$

In the second case

$$\frac{\mu(J)}{\mu(I)} = \frac{\frac{1-\alpha_{n+1}}{3}\mu(K_1)}{\frac{1-\alpha_{n+1}}{3}\mu(K_0)} = \frac{\mu(K_1)}{\mu(K_0)}$$

which is between 1/2 and 2 by induction hypothesis. □

Lemma 4. *Let $J \in \mathcal{I}_n$ and $I \in \mathcal{I}_{n+k}$ such that $I \subset J$, then*

$$\prod_{l=1}^{k}(1-\alpha_{n+l}) \leq \frac{3^k \mu(I)}{\mu(J)} \leq \prod_{l=1}^{k}(1+2\alpha_{n+l}).$$

Lemma 5. *Let $c > 0$ and I, J two subintervals of $[0,1]$ such that $I \subset J$ and $|I| > c\,|J|$, then*

$$\frac{\mu(I)}{\mu(J)} \geq c\frac{\prod_{l=1}^{m}(1-\alpha_{n+l})}{108},$$

where $n = [-\log_3(|J|)]$ and $m = [-\log_3(c/12)] + 1$ ($[a]$ denotes the integer part of a and $\log_3$ stands for the logarithm of base 3.)

PROOF. Let n be an integer such that $1/3 < 3^n\,|J| \leq 1$, i.e $n = [-\log_3(|J|)]$. Then there are joining intervals $J_1, J_2 \in \mathcal{I}_n$ such that $J \subset J_1 \cup J_2$, e.g. let $J_1 = [a,b] \in \mathcal{I}_n$ such that $(a,b]$ contains the left endpoint of J and let $J_2 = [b, b+3^{-n}]$. Therefore

$$\mu(J) \leq \mu(J_1) + \mu(J_2) \leq 3\min(\mu(J_1), \mu(J_2)) \tag{2}$$

Either for $k=1$ or $k=2$ we have that

$$|I \cap J_k| \geq \frac{1}{2}\,|I| \geq \frac{c}{2}\,|J| \geq \frac{c}{2}\frac{|J_k|}{3}$$

Let m be an integer such that $3^{-m} < \frac{c}{12} \leq 3\cdot 3^{-m}$, i.e. $m = [-\log_3(c/12)] + 1$, there is an interval I_0 in $\mathcal{I}_{n+m}$ such that $I_0 \subset I \cap J_k$ because in $\mathcal{I}_{m+n}$ there are intervals only of length at most the half of the length of $I \cap J_k$ and $\cup\mathcal{I}_{n+m} = [0,1]$. So by lemma 4 we get

$$\mu(I) \geq \mu(I_0) \geq \mu(J_k)3^{-m}\prod_{l=1}^{m}(1-\alpha_{n+l}) \geq \mu(J_k)\frac{c}{36}\prod_{l=1}^{m}(1-\alpha_{n+l}) \tag{3}$$

Comparing the inequalities (3) and (2) we get the statement. □

Corollary 6. *Let $0 < \alpha_n < 1/4$ be an arbitrary sequence such that*

1. $\alpha_n \to 0$

2. $\sum_n \alpha_n^2 = \infty$.

The associated Borel measure $\mu_\alpha \circ T^{-1}$ is singular and takes on a value of zero on any σ-porous Borel set.

Proof. By property 1 for any $c > 0$ there is an n_0 such that for $n \geq n_0$, $\prod_{l=1}^{m}(1-\alpha_{n+l}) > 1/2$ where $m = [-\log_3(c/12)] + 1$. This means that if J is small enough and $I \subset J$, $|I| > c\,|J|$ then

$$\frac{\mu(I)}{\mu(J)} > \frac{c}{216}$$

So the sufficient condition (1) is satisfied, μ assigns measure zero to porous sets. By the theorem of Kakutani this measure is singular with respect to the Lebesgue measure. □

Corollary 7. *There is a continuum family of pairwise mutually singular Borel measures on $[0,1]$ such that each measure is singular with respect to the Lebesgue measure and takes on a value of zero on σ-porous sets.*

Proof. Let μ_a be the measure corresponding to $\alpha_n = a \cdot n^{-1/2}$ where $a \in (0, 1/4)$. Using the previous corollary we have to prove only that if $a, b \in (0, 1/4)$ and $a \neq b$ then μ_a and μ_b are mutually singular. We can apply Kakutani's theorem again since

$$\rho\left(\mu_{n,a}, \mu_{n,b}\right) = \frac{2}{3}\sqrt{(1-\frac{a}{\sqrt{n}})(1-\frac{b}{\sqrt{n}})} + \frac{1}{3}\sqrt{(1+2\frac{a}{\sqrt{n}})(1+2\frac{b}{\sqrt{n}})}$$

which has logarithm of order $-n^{-1}$. □

The above construction gives something more that we actually need. Indeed for the constructed measure the constant $d(c)$ can depend on c linearly. If we do not care about this we can choose $0 < \alpha_n < 1/4$ to be a constant sequence and then instead of Kakutani's theorem we can apply the strong law of large numbers to see that μ_α is singular with respect to the Lebesgue measure.

References

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[2] Josef Tkadlec, *Construction of a finite Borel measure with σ-porous sets as null sets*, Real Anal. Exchange, 12(1)(1986/87), 349–353.