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Real Analysis Exchange
Vol. (), , pp. 141–154

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ON SOME PROPERTIES OF THE CLASS $\mathcal{A}^\star$

Abstract

In this paper we investigate problems connected with some generalization of spaces $\mathcal{A}^\star$ and $\mathcal{Q}_\mathcal{D}$ considered in [3].

In paper [2] the authors investigate problems connected with the linear space of Darboux derivatives. In their work they introduced some subsets of the set of all Darboux derivatives which possess interesting properties from a topological point of view. For example these subsets played an important role in consideration connected with stationary sets and some generalization of the notion of a retract with respect to the topological space formed by Darboux derivatives ([3]). In [4] some generalizations of classes of functions considered in [3] was introduced. Now we apply the following definitions (the notion of the class $\mathcal{Q}_\mathcal{D}$ is more general than in [4]).

By $\mathbb{R}$ ($\mathbb{Q}$, $\mathbb{N}$, $\mathbb{I}$) we denote the set of real numbers (rational numbers, natural numbers, segment [0,1]). The cardinality of $\mathbb{R}$ is denoted by $\mathfrak{c}$.

For any $x, y \in [0, 1]$ ($x \neq y$), denote by $I_{(x,y)}$ the closed interval $[x, y]$ if $x < y$, and the closed interval $[y, x]$ otherwise. For a function $f : \mathbb{I} \to \mathbb{I}$ and for $x, y \in \mathbb{I}$ let

$$A^f_{xy} = I_{(f(x),f(y))} \setminus f(I_{(x,y)}) \text{ and } A^f = \bigcup_{x,y\in\mathbb{I}} A^f_{xy}.$$

Let $\mathcal{J}$ be a family of all σ-ideals J of subsets of $\mathbb{I}$ such that:

- each $A \in J$ is boundary in $\mathbb{I}$;

Key Words: Darboux function, quasicontinuous function, porosity
Mathematical Reviews subject classification: 26A21, 26A24
Received by the editors December 18, 2000

- $\{x\} \in J$ for each $x \in \mathbb{I}$.

For fixed σ-ideal $J_0 \in \mathcal{J}$ let $\mathcal{A}^\star_{\mathcal{J}_\prime}$ denote the family of all nowhere constant functions $f : \mathbb{I} \to \mathbb{I}$ such that $A^f \in J_0$. Then let

$$\mathcal{A}^\star = \bigcup_{\mathcal{J}_\prime \in \mathcal{J}} \mathcal{A}^\star_{\mathcal{J}_\prime}.$$

For a fixed σ-ideal $J_0 \in \mathcal{J}$ let $\mathcal{Q}^{J_0}_{\mathcal{D}}$ denote the family of all nowhere constant functions $f : \mathbb{I} \to \mathbb{I}$ such that $A^f_{xy} \in J_0$ for every $x, y \in \mathbb{I}$. Then let

$$\mathcal{Q}_{\mathcal{D}} = \bigcup_{\mathcal{J} \in \mathcal{J}} \mathcal{Q}^{\mathcal{J}}_{\mathcal{D}}.$$

A function $f : X \to Y$ (where X, Y are topological spaces) is called a Darboux function if $f(C)$ is a connected set for each connected set $C \subset X$. The set of all nowhere constant Darboux functions $f : \mathbb{I} \to \mathbb{I}$ will be denoted by $\hat{\mathcal{D}}$.

It is obvious that $\hat{\mathcal{D}} \subset \mathcal{A}^\star \subset \mathcal{Q}_{\mathcal{D}}$ and $\hat{\mathcal{D}} \neq \mathcal{A}^\star$, but it was an open question if the classes $\mathcal{Q}_{\mathcal{D}}$ and $\mathcal{A}^\star$ are equal. It turns out that an answer is positive (Theorem 1).

In the second part of our work we consider the class of real functions defined on the space $\mathcal{A}^\star$. Our results are connected with a class of functions $\mathcal{D}_{\mathcal{P}}$; i.e., a class of functions $f : \mathcal{A}^\star \to \mathbb{R}$ such that $f(L)$ is a connected set for each arc[1] $L \subset \mathcal{A}^\star$. We prove that there exists a subset Φ of the space $\mathcal{A}^\star$ such that if f is a function of the class $\mathcal{D}_{\mathcal{P}}$ and $f \restriction_\Phi$ or $f \restriction_{\mathcal{A}^\star \setminus \oplus}$ is quasicontinuous (cliquish), f is also (but the corresponding assertion for continuity is false). We also show that the set $\mathcal{D}_{\mathcal{P}}$ is porous at each point of some subset of the space of all functions $f : \mathcal{A}^\star \to \mathbb{R}$. To formulate these all facts more precisely let us apply the following notion and definitions.

We say that $x \in \mathrm{cl}_a(A)$ if $x \in A$ or there exists an arc L such that $L \setminus \{x\} \subset A$.

A function $f : X \to \mathbb{R}$ (where X is a topological space) is cliquish at a point $x \in X$ if for each $\varepsilon > 0$ and each neighborhood U of x there exists a nonempty open set $G \subset U$ such that $|f(y) - f(z)| < \varepsilon$ for each $y, z \in G$. A function $f : X \to \mathbb{R}$ is said to be cliquish if it is cliquish at each point $x \in X$ ([1]).

A function $f : X \to Y$ (where X, Y are topological spaces) is quasicontinuous at a point $x \in X$ if for each neighborhood U of x and for each neighborhood

[1] A subset $L \subset X$ (where X is a topological space) is called an arc if there exists a homeomorphism $h : \mathbb{I} \to L$. The elements $h(0)$ and $h(1)$ will be called the endpoints of L. The arc with endpoints x and y is denoted by $L(x, y)$.

V of $f(x)$ there exists a nonempty open set $G \subset U$ such that $f(G) \subset V$. A function $f : X \to Y$ is said to be quasicontinuous if it is quasicontinuous at each point $x \in X$.

If (X, d) is a metric space, $x_0 \in X$, $r > 0$, $A \subset X$ let
$B(x_0, r) = \{y \in X : d(x_0, y) < r\}$,
$\overline{B}(x_0, r) = \{y \in X : d(x_0, y) \leq r\}$,
$S(x_0, r) = \{y \in X : d(x_0, y) = r\}$,
$d_A(x) = \inf\{d(x, a) : a \in A\}$.

Let $M \subset X$, $x \in X$, $R > 0$. Then $\gamma(x, R, M)$ denotes the supremum of the set of all $r > 0$ for which there exists $z \in X$ such that $B(z, r) \subset B(x, R) \setminus M$. The set M is porous at x if $\limsup_{R \to 0^+} \frac{\gamma(x,R,M)}{R} > 0$.

By ρ we denote the metric of uniform convergence (i.e., $\rho(f, g) = \sup\{|f(x) - g(x)| : x \in \mathbb{I}\}$).

We begin our consideration with two lemmas.

Lemma 1. *Let $x, y, z \in \mathbb{I}$ be such that $x < y < z$. Then $A^f_{xz} \subset A^f_{xy} \cup A^f_{yz}$.*

Let

$$\times_< \mathbb{I} = \{(x, y) \in \mathbb{I} \times \mathbb{I} : x < y$$
$$\mathbb{I}_q = \{(x, y) \in \mathbb{I} \times_< \mathbb{I} : x < q < y\} \text{ for } q \in (0, 1) \cap \mathbb{Q}.$$

Then $\mathbb{I} \times_< \mathbb{I} = \bigcup_{q \in \mathbb{I} \cap \mathbb{Q}} \mathbb{I}_q$.

Lemma 2. *If $f(0) = 0, 0 < x < y$ and $f(y) \leq f(x)$, then $A^f_{0x} \supset A^f_{0y}$.*

Theorem 1. $\mathcal{A}^\star = \mathcal{Q}_\mathcal{D}$.

PROOF. Obviously $\mathcal{A}^\star \subset \mathcal{Q}_\mathcal{D}$. We shall prove an inverse inclusion. Let $f \in \mathcal{Q}_\mathcal{D}$. Then there exists a σ-ideal $J_0 \in \mathcal{J}$ such that $A^f_{xy} \in J_0$ for every $x, y \in \mathbb{I}$. We have to prove that $A^f \in J_0$. Since

$$A^f = \bigcup_{x,y \in \mathbb{I}} A^f_{xy} = \bigcup_{q \in \mathbb{Q} \cap (0,1)} \bigcup_{(x,y) \in \mathbb{I}_q} A^f_{xy},$$

it suffices to show that for every $q \in \mathbb{Q} \cap (0, 1)$ we have $\bigcup_{(x,y) \in \mathbb{I}_q} A^f_{xy} \in J_0$. So let $q \in \mathbb{Q} \cap (0, 1)$ be fixed. Note that (by Lemma 1)

$$\bigcup_{(x,y) \in \mathbb{I}_q} A^f_{xy} \subset \bigcup_{(x,y) \in \mathbb{I}_q} (A^f_{xq} \cup A^f_{qy}) = \bigcup_{0 \leq x < q} A^f_{xq} \cup \bigcup_{1 \geq y > q} A^f_{qy}.$$

So it is sufficient to prove that $\bigcup_{0 < x \leq 1} A^f_{0x} \in J_0$. To simplify notation assume that $f(0) = 0$. Let $\{B_n\}_{n \in \mathbb{N}}$ be a base in $\mathbb{I}$. Fix $n \in \mathbb{N}$. Then we choose

$\{x_{n,k}\}_{k\in\mathbb{N}} \in B_n$ such that $f(x_{n,k}) \overset{k\to\infty}{\longrightarrow} \sup_{x\in B_n} f(x)$. Let

$$\overline{C} = \{x \in \mathbb{I} : f(x) > 0 \wedge \exists_{y\in[0,x)} f(y) \geq f(x)\},$$
$$\overline{D} = \{x \in \mathbb{I} : f(x) > 0\} \setminus \overline{C},$$
$$\overline{E} = \{x \in \mathbb{I} : f(x) = 0\}.$$

Then

$$\bigcup_{x\in\mathbb{I}} A^f_{0x} = \bigcup_{x\in\overline{C}} A^f_{0x} \cup \bigcup_{x\in\overline{D}} A^f_{0x} \cup \bigcup_{x\in\overline{E}} A^f_{0x}.$$

From the assumption that $f(0) = 0$ we deduce that $\bigcup_{x\in\overline{E}} A^f_{0x} = \emptyset$. So it is sufficient to show that $\bigcup_{x\in\overline{C}} A^f_{0x} \in J_0$ and $\bigcup_{x\in\overline{D}} A^f_{0x} \in J_0$.

If $x \in \overline{C}$, there exists $y \in [0,x)$ such that $f(y) \geq f(x) > 0$ and there exist $n,\, k \in \mathbb{N}$ such that $0 < x_{n,k} < x$ and $f(x_{n,k}) \geq f(x)$. So, by Lemma 2, $A^f_{0x} \subset A^f_{0x_{n,k}}$. Hence $\bigcup_{x\in\overline{C}} A^f_{0x} \subset \bigcup_{n,k\in\mathbb{N}} A^f_{0x_{n,k}} \in J_0$. Now let $\{t_i\}_{i\in\mathbb{N}}$ be a dense set in $\overline{D}$ containing all points which are left isolated in $\overline{D}$. Note that

$$\bigcup_{x\in\overline{D}} A^f_{0x} \subset \bigcup_{i=1}^{\infty} A^f_{0t_i}. \tag{1}$$

Indeed, let $z \in \bigcup_{x\in\overline{D}} A^f_{0x}$. Then there exists $x \in \overline{D}$ such that $z \in A^f_{0x}$. If there exists $i_0 \in \mathbb{N}$ such that $x = t_{i_0}$, $z \in \bigcup_{i\in\mathbb{N}} A^f_{0t_i}$. So let us assume that $x \notin \{t_i : i \in \mathbb{N}\}$. Hence x is a left accumulation point of $\overline{D}$ and hence also of $\{t_i : i \in \mathbb{N}\}$. It is also easy to show that $f \restriction_{\overline{D}}$ is strictly increasing and continuous from the left. Therefore it is not difficult to conclude that there exists $i_0 \in \mathbb{N}$ such that $f(t_{i_0}) > z$. Then $z \in [0, f(t_{i_0})] \setminus f([0, t_{i_0}]) = A^f_{0t_{i_0}}$, which finishes the proof of (1). From (1) we conclude that $\bigcup_{x\in\overline{D}} A^f_{0x} \in J_0$, which finishes the proof of the theorem. □

The following lemmas can be found in [4] (see also [3]).

Lemma 3. $\mathcal{A}^\star \subset \mathrm{cl}_a(\mathcal{A}^\star \setminus \hat{\mathcal{D}})$.

Lemma 4. $\mathcal{A}^\star \subset \mathrm{cl}_a(\hat{\mathcal{D}})$.

Theorem 2. *Let $f : \mathcal{A}^\star \to \mathbb{R}$ and let $f \in \mathcal{D}_{\mathcal{P}}$. Then the following conditions are equivalent:*

(i) the function f is quasi-continuous (cliquish),
(ii) the function $f \restriction_{\hat{\mathcal{D}}}$ is quasi-continuous (cliquish),
(iii) the function $f \restriction_{\mathcal{A}^\star\setminus\hat{\mathcal{D}}}$ is quasi- continuous (cliquish).

Proof. (for quasi-continuity). Implications (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) follows from the fact that sets $\hat{\mathcal{D}}$ and $\mathcal{A}^\star \setminus \hat{\mathcal{D}}$ are dense in $\mathcal{A}^\star$. Now we shall prove the implication (ii) $\Longrightarrow$ (i). (The proof of the implication (iii) $\Longrightarrow$ (i) is analogous.)

Let $\xi \in \mathcal{A}^\star$, let $\varepsilon > 0$ and let $\delta > 0$. We will show that there exists $\eta \in \hat{\mathcal{D}} \cap B(\xi, \delta)$ such that $f(\eta) \in (f(\xi) - \frac{\varepsilon}{2}, f(\xi) + \frac{\varepsilon}{2})$. Indeed, if $\xi \in \hat{\mathcal{D}}$, it suffices to put $\eta = \xi$. So assume that $\xi \notin \hat{\mathcal{D}}$. By Lemma 4 there exists an arc $L = L(\xi, a)$ such that $L \setminus \{\xi\} \subset \hat{\mathcal{D}}$. We can assume that $L \subset B(\xi, \delta)$. Since (by assumption) $f(L)$ is a connected set, there exists $\eta \in L$ such that $f(\eta) \subset (f(\xi) - \frac{\varepsilon}{2}, f(\xi) + \frac{\varepsilon}{2})$. Then $f \restriction_{\hat{\mathcal{D}}}$ is quasi-continuous at η. Hence there exists a nonempty open set $V \subset B(\xi, \delta)$ such that

$$f(V \cap \hat{\mathcal{D}}) \subset (f(\xi) - \frac{\varepsilon}{2}, f(\xi) + \frac{\varepsilon}{2}). \tag{2}$$

Consider $\phi \in V \setminus \hat{\mathcal{D}}$. Then ϕ is an endpoint of an arc $L^\star = L(\phi, b)$ such that $L^\star \setminus \{\phi\} \subset V \cap \hat{\mathcal{D}}$ (Lemma 4). By assumption $f(L^\star)$ is a connected set; so (from (2)) $f(\phi) \in [f(\xi) - \frac{\varepsilon}{2}, f(\xi) + \frac{\varepsilon}{2}]$. Hence $f(V \setminus \hat{\mathcal{D}}) \subset [f(\xi) - \frac{\varepsilon}{2}, f(\xi) + \frac{\varepsilon}{2}]$. From this and (2), it follows that $f(V) \subset (f(\xi) - \varepsilon, f(\xi) + \varepsilon)$, which finishes the proof of the quasi-continuity of f at ξ.

Proof. (for cliquish). Implications (i) $\Longrightarrow$ (ii) and (i) $\Longrightarrow$ (iii) follow from the fact that sets $\hat{\mathcal{D}}$ and $\mathcal{A}^\star \setminus \hat{\mathcal{D}}$ are dense in $\mathcal{A}^\star$. Now we shall prove the implication (ii) $\Longrightarrow$ (i). (The proof of the implication (iii) $\Longrightarrow$ (i) is analogous.) Let $\xi \in \mathcal{A}^\star$, let $\varepsilon > 0$ and let $\delta > 0$. Then from Lemma 4 we can infer that there exists $\eta \in \hat{\mathcal{D}} \cap B(\xi, \delta)$. Then $f \restriction_{\hat{\mathcal{D}}}$ is cliquish at η. Hence there exists a nonempty open set $V \subset B(\xi, \delta)$ such that

$$|f(\phi) - f(\psi)| < \frac{\varepsilon}{3} \text{ for each } \phi, \psi \in V \cap \hat{\mathcal{D}}. \tag{3}$$

Now let $\eta, \tau \in V$. There are three possible cases.

1^0 $\eta, \tau \in V \cap \hat{\mathcal{D}}$.

Then $|f(\eta) - f(\tau)| < \frac{\varepsilon}{3} < \varepsilon$ from (3).

2^0 $\eta \in V \cap \hat{\mathcal{D}}$, $\tau \in V \setminus \hat{\mathcal{D}}$.

Then by Lemma 4 there exists an arc $L^* = L(\tau, \tau_1)$ such that $L^* \setminus \{\tau\} \subset V \cap \hat{\mathcal{D}}$. Hence from (3) we infer that

$$f(a) \in (f(\eta) - \frac{\varepsilon}{3}, f(\eta) + \frac{\varepsilon}{3}) \text{ for each } a \in L^* \setminus \{\tau\}.$$

According to our assumption $f(L^*)$ is connected; so $f(\tau) \in [f(\eta) - \frac{\varepsilon}{3}, f(\eta) + \frac{\varepsilon}{3}]$. Therefore $|f(\tau) - f(\eta)| \leq \frac{\varepsilon}{3} < \varepsilon$.

3^0 $\eta, \tau \in V \setminus \hat{\mathcal{D}}$.

Then by Lemma 4there exist arcs $L_1 = L(\eta, \eta_1)$ and $L_2 = L(\tau, \tau_1)$ such that $L_1 \setminus \{\eta\} \subset V \cap \hat{\mathcal{D}}$ and $L_2 \setminus \{\tau\} \subset V \cap \hat{\mathcal{D}}$. Hence from (3) we infer that $f(a) \in (f(\eta_1) - \frac{\varepsilon}{3}, f(\eta_1) + \frac{\varepsilon}{3})$ for each $a \in L_1 \setminus \{\eta\}$ and $f(b) \in (f(\eta_1) - \frac{\varepsilon}{3}, f(\eta_1) + \frac{\varepsilon}{3})$ for each $b \in L_2 \setminus \{\tau\}$. According to our assumption $f(L_1)$ and $f(L_2)$ are connected sets, so

$$f(\eta) \in \Big[f(\eta_1) - \frac{\varepsilon}{3}, f(\eta_1) + \frac{\varepsilon}{3}\Big] \text{ and } f(\tau) \in \Big[f(\eta_1) - \frac{\varepsilon}{3}, f(\eta_1) + \frac{\varepsilon}{3}\Big].$$

Hence $|f(\eta) - f(\tau)| \le |f(\eta) - f(\eta_1)| + |f(\eta_1) - f(\tau)| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} < \varepsilon$. So we have shown that $|f(\eta) - f(\tau)| < \varepsilon$ for each $\eta, \tau \in V$, which finishes the proof of the fact that f is cliquish at ξ. □

Remark 1. The last theorem fails for continuity. For example let $\xi_0 \in \hat{\mathcal{D}}$ be fixed and define $f : \mathcal{A}^\star \to \mathbb{R}$ by

$$f(\xi) = \begin{cases} \sin \frac{1}{\rho(\xi, \xi_0)} & \text{for } \xi \in \mathcal{A}^\star \setminus \{\xi_0\} \\ 0 & \text{for } \xi = \xi_0. \end{cases}$$

Then $f \in \mathcal{D}_{\mathcal{P}}$, $f \restriction_{\mathcal{A}^\star \setminus \hat{\mathcal{D}}}$ is a continuous function, but f is not continuous at ξ.

Remark 2. The analogous theorem fails for Darboux functions $f : \mathbb{R} \to \mathbb{R}$. We will show that there exists no set $A \subset \mathbb{R}$ such that for each Darboux function $f : \mathbb{R} \to \mathbb{R}$:

(i) if $f \restriction_A$ is quasi-continuous, then f is quasi-continuous;

(ii) if $f \restriction_{\mathbb{R} \setminus A}$ is quasi-continuous, then f is quasi-continuous.

Proof. Suppose that there exists a set $A \subset \mathbb{R}$ such that for each Darboux function $f : \mathbb{R} \to \mathbb{R}$ conditions (i) and (ii) hold. First note that $\mathbb{R} \setminus A$ is dense in $\mathbb{R}$. Indeed, suppose that there exists a nonempty open interval $P \subset A$. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(x) = 0$ for $x \notin P$ and $f(P') = \mathbb{R}$ for each nonempty open interval $P' \subset P$. Then f is the Darboux function, $f \restriction_{\mathbb{R} \setminus A} = 0$ is quasi-continuous, but f is not quasi-continuous, which contradicts the condition (ii).

In the analogous way we can prove that A is dense in $\mathbb{R}$. Moreover note that there exists a nonempty interval (a, b) such that A is c-dense in (a, b) or $\mathbb{R} \setminus A$ is c-dense in (a, b). Suppose, for instance, that the set A is c-dense in some interval (a_0, b_0); i.e., the set $A \cap (a_0, b_0)$ is c-dense in itself. Hence

$$A \cap (a_0, b_0) = \bigcup_{\alpha < \mathfrak{c}} A_\alpha,$$

where A_α, $\alpha < \mathfrak{c}$, are dense in $A \cap (a_0, b_0)$ and these sets are pairwise disjoint. Let $\xi : \{A_\alpha : \alpha < \mathfrak{c}\} \to \mathbb{R}$ be a bijection. We define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \notin A \cap (a_0, b_0) \\ \xi(A_\alpha) & \text{if } x \in A_\alpha, \alpha < \mathfrak{c}. \end{cases}$$

First note that f is a Darboux function. Indeed, let $P \subset \mathbb{R}$ be a non-degenerate interval. If $P \subset \mathbb{R} \setminus (A \cap (a_0, b_0))$, then $f(P) = \{0\}$ is connected. So assume that $P \cap A \cap (a_0, b_0) \neq \emptyset$. Then $\mathrm{Int}(P) \cap A_\alpha \neq \emptyset$ for each $\alpha < \mathbf{c}$. Hence $f(P) \supset \xi(\{A_\alpha : \alpha < \mathfrak{c}\}) = \mathbb{R}$ is a connected set.

The function $f \restriction_{\mathbb{R}\setminus A} \equiv 0$ is quasi-continuous, but f is not quasi-continuous at any point of the interval (a_0, b_0). □

If $A \subset X$ (where X is a topological space) let C_A denote a set of all functions $f : X \to \mathbb{R}$ which are continuous at some point of the set A.

Theorem 3. *Let $\mathcal{F}$ be a space of functions $f : \mathcal{A}^\star \to \mathbb{R}$ such that $f \restriction_{\hat{\mathcal{D}}} \in \mathcal{D}_{\mathcal{P}}$ (with the metric of uniform convergence). Then the set $\mathcal{D}_{\mathcal{P}} \subset \mathcal{F}$ is porous at each point $\eta \in C_{\hat{\mathcal{D}}} \cap \mathcal{F}$.*

PROOF. Let $\eta \in C_{\hat{\mathcal{D}}} \cap \mathcal{F}$. Then there exists a point $g_0 \in \hat{\mathcal{D}}$ of continuity of the function η. Let $R > 0$. Then there exists $\delta > 0$ such that

$$\eta(\overline{B}(g_0, \delta)) \subset (\eta(g_0) - \frac{R}{8}, \eta(g_0) + \frac{R}{8}).$$

By Lemma 3 there exists an arc $L_0 = L(g_0, t_0)$ such that $L_0 \setminus \{g_0\} \subset \mathcal{A}^\star \setminus \hat{\mathcal{D}}$. We may assume that $L_0 \subset B(g_0, \frac{\delta}{2})$. For each $\alpha \geq 0$ let $T_\alpha = \{t \in \mathcal{A}^\star : \rho_{L_0}(t) = \alpha\}$. Let $\alpha_0 > 0$ be such that $T_{\alpha_0} \subset B(g_0, \frac{\delta}{2})$. Define $h : \mathcal{A}^\star \to \mathbb{R}$ by

$$h(t) = \begin{cases} \eta(t) & \text{if } t \in \mathcal{A}^\star \setminus B(g_0, \delta) \\ \frac{R}{8} \sin \frac{1}{\alpha} + \eta(g_0) & \text{if } t \in T_\alpha, 0 < \alpha \leq \alpha_0 \\ \frac{R}{8} \sin \frac{1}{\alpha_0} + \eta(g_0) & \text{if } t \in \bigcup\limits_{\alpha > \alpha_0} T_\alpha \cap B(g_0, \dfrac{\delta}{2}) \\ \frac{R}{8} \sin \frac{1}{\delta - r} + \eta(g_0) & \text{if } t \in S(g_0, r), r \in [\frac{3}{4}\delta, \delta) \\ \frac{R(\sin \frac{4}{\delta} - \sin \frac{1}{\alpha_0})}{2\delta} r + \eta(g_0) + \frac{R(3 \sin \frac{1}{\alpha_0} - 2 \sin \frac{4}{\delta})}{8} & \text{if } t \in S(g_0, r), r \in [\frac{\delta}{2}, \frac{3}{4}\delta] \\ \eta(g_0) - \frac{R}{8} & \text{if } t = g_0 \\ \eta(g_0) + \frac{R}{8} & \text{if } t \in L_0 \setminus \{g_0\}. \end{cases}$$

First we shall show that

$$h \in \mathcal{F}. \tag{4}$$

Let $L \subset \hat{\mathcal{D}}$ be an arc. Then there are 6 possible cases with some subcases.
1^0 $L \subset \mathcal{A}^\star \setminus B(g_0, \delta)$.

Then $h(L) = \eta(L)$ is a connected set, because $\eta \in \mathcal{F}$.
2^0 $L \subset B(g_0, \frac{\delta}{2})$.

If $L \subset T_\alpha$ for some $\alpha \in [0, +\infty)$ or $L \subset \bigcup_{\alpha > \alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})$, then $h(L)$ is a singleton; thus a connected set. Hence assume that these cases fail. Let $\alpha_1 = \inf\{\alpha \in [0, +\infty) : L \cap T_\alpha \neq \emptyset\}$ and $\alpha_2 = \sup\{\alpha \in [0, +\infty) : L \cap T_\alpha \neq \emptyset\}$. Then $0 \leq \alpha_1 < \alpha_2 < +\infty$. Moreover, since an arc L is connected, we can conclude that $L \cap T_\alpha \neq \emptyset$ for each $\alpha \in (\alpha_1, \alpha_2)$. Hence

$$h(L) = h(L \cap \bigcup_{\alpha \in \langle \alpha_1, \alpha_2 \rangle} T_\alpha) = \xi(\langle \alpha_1, \alpha_2 \rangle),$$

where $\xi : [0, +\infty) \to \mathbb{R}$ is defined by

$$\xi(x) = \begin{cases} \eta(g_0) - \frac{R}{8} & \text{for } x = 0 \\ \frac{R}{8} \sin \frac{1}{x} + \eta(g_0) & \text{for } x \in (0, \alpha_0] \\ \frac{R}{8} \sin \frac{1}{\alpha_0} + \eta(g_0) & \text{for } x \in [\alpha_0, +\infty) \end{cases}$$

and $\langle \alpha_1, \alpha_2 \rangle$ is an interval with endpoints α_1 and α_2 such that $\alpha_1 \in \langle \alpha_1, \alpha_2 \rangle \iff L \cap T_{\alpha_1} \neq \emptyset$ and $\alpha_2 \in \langle \alpha_1, \alpha_2 \rangle \iff L \cap T_{\alpha_2} \neq \emptyset$. Hence $h(L)$ is a connected set, because ξ is a Darboux function.
3^0 $L \subset B(g_0, \delta) \setminus B(g_0, \frac{\delta}{2})$.

If $L \subset S(g_0, r)$ for some $r \in [\frac{\delta}{2}, \delta)$, then $h(L)$ is a singleton; hence a connected set. So assume that this case fails. Let $r_1 = \inf\{r \in [\frac{\delta}{2}, \delta) : L \cap S(g_0, r) \neq \emptyset\}$ and $r_2 = \sup\{r \in [\frac{\delta}{2}, \delta) : L \cap S(g_0, r) \neq \emptyset\}$. Then it is not difficult to see that $\frac{\delta}{2} \leq r_1 < r_2 \leq \delta$. Let $\langle r_1, r_2 \rangle$ be an interval with endpoints r_1 and r_2 such that $r_1 \in \langle r_1, r_2 \rangle \iff L \cap S(g_0, r_1) \neq \emptyset$ and $r_2 \in \langle r_1, r_2 \rangle \iff L \cap S(g_0, r_2) \neq \emptyset$. Then $\langle r_1, r_2 \rangle \subset [\frac{\delta}{2}, \delta)$ and $L \cap S(g_0, r) \neq \emptyset$ for each $r \in (r_1, r_2)$. Hence

$$h(L) = h(L \cap \bigcup_{r \in \langle r_1, r_2 \rangle} S(g_0, r)) = \tau(\langle r_1, r_2 \rangle),$$

where $\tau : [\frac{\delta}{2}, \delta) \to \mathbb{R}$ is defined by

$$\tau(x) = \begin{cases} \dfrac{R(\sin \frac{4}{\delta} - \sin \frac{1}{\alpha_0})}{2\delta} x + \eta(g_0) + \dfrac{R(3 \sin \frac{1}{\alpha_0} - 2 \sin \frac{4}{\delta})}{8} & \text{for } x \in [\frac{\delta}{2}, \frac{3\delta}{4}] \\ \frac{R}{8} \sin \frac{1}{\delta - x} + \eta(g_0) & \text{for } x \in [\frac{3\delta}{4}, \delta). \end{cases}$$

Hence $h(L)$ is a connected set, because τ is the continuous function.

4^0 $L \subset B(g_0, \delta)$ and $L \cap B(g_0, \delta) \setminus B(g_0, \frac{\delta}{2}) \neq \emptyset$ and $L \cap B(g_0, \frac{\delta}{2}) \neq \emptyset$.

Let $r_2 = \sup\{r \in [\frac{\delta}{2}, \delta) : L \cap S(g_0, \delta) \neq \emptyset\}$. Note that $\frac{\delta}{2} \leq r_2 < \delta$. Then

$$L \cap S(g_0, r) \neq \emptyset \text{ for each } r \in [\frac{\delta}{2}, r_2). \tag{5}$$

Consider the following two subcases:

4a) $L \cap \bigcup_{0\leq\alpha\leq\alpha_0} T_\alpha \neq \emptyset$.

Then let $\alpha_1 = \inf\{\alpha \in [0, \alpha_0] : L \cap T_\alpha \neq \emptyset\}$. Hence $L \cap T_\alpha \neq \emptyset$ for each $\alpha \in (\alpha_1, \alpha_0]$. By (5)

$$\begin{aligned} h(L) =& h(L \cap \bigcup_{0\leq\alpha\leq\alpha_0} T_\alpha) \cup h(L \cap \bigcup_{\alpha>\alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \\ & \cup h(L \cap B(g_0, \delta) \setminus B(g_0, \frac{\delta}{2})) \\ =& h(L \cap \bigcup_{\alpha\in\langle\alpha_1,\alpha_0]} T_\alpha) \cup h(L \cap \bigcup_{\alpha>\alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \cup h(L \cap \bigcup_{r\in[\frac{\delta}{2},r_2\rangle} S(g_0, r)) \\ =& \xi(\langle\alpha_1, \alpha_0]) \cup h(L \cap \bigcup_{\alpha>\alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \cup \tau([\frac{\delta}{2}, r_2\rangle), \end{aligned}$$

where ξ is defined as in the case 2^0, τ is defined as in the case 3^0 and $[\frac{\delta}{2}, r_2\rangle$ is an interval with endpoints $\frac{\delta}{2}$ and r_2 such that $r_2 \in [\frac{\delta}{2}, r_2\rangle \iff L \cap S(g_0, r_2) \neq \emptyset$ and $\langle\alpha_1, \alpha_0]$ is an interval with endpoints α_1 and α_2 such that $\alpha_1 \in \langle\alpha_1, \alpha_0] \iff L \cap T_{\alpha_1} \neq \emptyset$. So, from the fact that

$$h(L \cap \bigcup_{\alpha>\alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \subset \{\frac{R}{8} \sin\frac{1}{\alpha_0} + \eta(g_0)\} \subset \xi(\langle\alpha_1, \alpha_0]),$$

we get that $h(L) = \xi(\langle\alpha_1, \alpha_0]) \cup \tau([\frac{\delta}{2}, r_2\rangle)$, where sets $\xi(\langle\alpha_1, \alpha_0])$, $\tau([\frac{\delta}{2}, r_2\rangle)$ are connected and are not disjoint (because $\xi(\alpha_0) = \frac{R}{8}\sin\frac{1}{\alpha_0} + \eta(g_0) = \tau(\frac{\delta}{2})$). Therefore $h(L)$ is a connected set.

4b) $L \cap \bigcup_{0\leq\alpha\leq\alpha_0} T_\alpha = \emptyset$.

Then

$$\begin{aligned} h(L) =& h(L \cap \bigcup_{\alpha>\alpha_0} T_\alpha(L_0) \cap B(g_0, \frac{\delta}{2})) \cup h(L \cap B(g_0, \delta) \setminus B(g_0, \frac{\delta}{2})) \\ =& \{\frac{R}{8} \sin\frac{1}{\alpha_0} + \eta(g_0)\} \cup \tau([\frac{\delta}{2}, r_2\rangle) = \tau([\frac{\delta}{2}, r_2\rangle), \end{aligned}$$

where τ and $[\frac{\delta}{2}, r_2\rangle$ are defined as in the case 4a). So $h(L)$ is a connected set.

5^0 $L \subset \mathcal{A}^\star \setminus B(g_0, \frac{\delta}{2})$ and $L \setminus B(g_0,\delta) \neq \emptyset$ and $L \cap B(g_0,\delta) \neq \emptyset$.

Let $r_1 = \inf\{r \in [\frac{\delta}{2}, \delta) : L \cap S(g_0, r) \neq \emptyset\}$. Then $L \cap S(g_0, r) \neq \emptyset$ for each $r \in (r_1, \delta)$. Let $\{S_i\}_{i\in I}$ be the family of all nonempty components of $L \setminus B(g_0, \delta)$. Then

$$S_i \cap S(g_0,\delta) \neq \emptyset \text{ for each } i \in I. \tag{6}$$

Indeed, suppose that there exists $i_0 \in I$ such that $S_{i_0} \cap S(g_0, \delta) = \emptyset$. Let $H : [0,1] \to L$ be a homeomorphism. Then for each $i \in I$, $H^{-1}(S_i) = [a_i, b_i]$ for some a_i, b_i such that $0 \le a_i \le b_i \le 1$. It is easy to see that $a_{i_0} \neq 0$ or $b_{i_0} \neq 1$ (if not $S_{i_0} = L$, which is impossible). Let, for instance, $a_{i_0} \neq 0$. Then

$$\begin{aligned}&\text{there exists } \{a_n\}_{n\in\mathbb{N}} \subset [0,1] \text{ such that } a_n \uparrow a_{i_0}\\ &\text{and } a_n \notin H^{-1}(\bigcup_{i\neq i_0} S_i) \text{ for each } n \in \mathbb{N}.\end{aligned} \tag{7}$$

Indeed, suppose that there is $0 \le a^\star < a_{i_0}$ such that $(a^\star, a_{i_0}) \subset H^{-1}(\bigcup_{i\neq i_0} S_i)$. Let

$$I_0 = \{i \neq i_0 : H^{-1}(S_i) \cap (a^\star, a_{i_0}) \neq \emptyset\}.$$

Then $(a^\star, a_{i_0}) \subset H^{-1}(\bigcup_{i\in I_0} S_i)$. Hence

$$H^{-1}(\bigcup_{i\in I_0} S_i) = H^{-1}(\bigcup_{i\in I_0} S_i) \cup (a^\star, a_{i_0})$$

is a connected set, because for each $i \in I_0$ we have that $H^{-1}(S_i) \cap (a^\star, a_{i_0}) \neq \emptyset$. So $\bigcup_{i\in I_0} S_i$ is also a connected set (in L). Since S_i, $i \in I_0$, are components of some set, $I_0 = \{j_0\}$ for some $j_0 \in I_0$. Then $j_0 \neq i_0$. Hence $(a^\star, a_{i_0}) \subset H^{-1}(S_{j_0})$. Hence

$$H^{-1}(S_{i_0}) \cap \mathrm{cl}(H^{-1}(S_{j_0})) \supset [a_{i_0}, b_{i_0}] \cap [a^\star, a_{i_0}] \neq \emptyset.$$

Thus $H^{-1}(S_{i_0}) \cup H^{-1}(S_{j_0})$ is a connected set as a sum of non-separated connected sets. So $S_{i_0} \cup S_{j_0}$ is a connected set in L, which is impossible completing the proof of (6).

Let $\{a_n\}_{n\in\mathbb{N}} \subset [0,1]$ be a sequence such that $a_n \uparrow a_{i_0}$ and $a_n \notin H^{-1}(\bigcup_{i\neq i_0} S_i)$ for each $n \in \mathbb{N}$. Hence $H(a_n) \in L \cap B(g_0, \delta)$ for each $n \in \mathbb{N}$. Then $H(a_{i_0}) \in L \cap \overline{B}(g_0, \delta)$. On the other hand $H(a_{i_0}) \in S_{i_0} \subset L \setminus B(g_0,\delta) \setminus S(g_0,\delta) = L \setminus \overline{B}(g_0,\delta)$. This contradiction finishes the proof of (7).

So we have

$$\begin{aligned} h(L) &= h(L\cap B(g_0,\delta))\cup h(L\setminus B(g_0,\delta)) \\ &= h(L\cap \bigcup_{r\in\langle r_1,\delta)} S(g_0,\delta))\cup \eta(L\setminus B(g_0,\delta)) = \tau(\langle r_1,\delta))\cup\eta(\bigcup_{i\in I} S_i) \\ &= [\eta(g_0)-\frac{R}{8},\eta(g_0)+\frac{R}{8}]\cup\bigcup_{i\in I}\eta(S_i), \end{aligned} \tag{8}$$

where τ is defined as in the case 3^0 and $\langle r_1,\delta)$ is an interval with endpoints r_1, δ such that $r_1\in\langle r_1,\delta)\Longleftrightarrow L\cap S(g_0,r_1)\neq\emptyset$. Moreover note that for each $i\in I$

$$\eta(S_i\cap S(g_0,\delta))\subset\eta(\overline{B}(g_0,\delta))\subset(\eta(g_0)-\frac{R}{8},\eta(g_0)+\frac{R}{8});$$

so $\eta(S_i)\cap[\eta(g_0)-\frac{R}{8},\eta(g_0)+\frac{R}{8}]\neq\emptyset$ for each $i\in I$. From the fact that for each $i\in I$, $\eta(S_i)$ is a connected set and from the equality (8) we can infer that $h(L)$ is a connected set.

6^0 $L\cap B(g_0,\frac{\delta}{2})\neq\emptyset$, $\quad L\cap B(g_0,\delta)\setminus B(g_0,\frac{\delta}{2})\neq\emptyset$ and $L\setminus B(g_0,\delta)\neq\emptyset$.

Then obviously

$$L\cap S(g_0,r)\neq\emptyset \text{ for each } r\in[\frac{\delta}{2},\delta). \tag{9}$$

Let $\{S_i\}_{i\in I}$ be the family of all nonempty components of $L\setminus B(g_0,\delta)$. Then, as in the case 5^0 we can prove that $S_i\cap S(g_0,\delta)\neq\emptyset$ for each $i\in I$. Let us consider the following two cases

6a) $L\cap\bigcup_{0\leq\alpha\leq\alpha_0}T_\alpha\neq\emptyset$.

Let $\alpha_1=\inf\{\alpha\in[0,\alpha_0]:L\cap T_\alpha\neq\emptyset\}$. Then $L\cap T_\alpha\neq\emptyset$ for each

$\alpha \in (\alpha_1, \alpha_0]$. Hence by (9)

$$\begin{aligned} h(L) =& h(L \cap B(g_0, \frac{\delta}{2})) \cup h(L \cap B(g_0, \delta) \setminus B(g_0, \frac{\delta}{2})) \cup h(L \setminus B(g_0, \delta)) \\ =& h(L \cap \bigcup_{0 \le \alpha \le \alpha_0} T_\alpha) \cup h(L \cap \bigcup_{\alpha > \alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \\ & \cup h(L \cap \bigcup_{r \in [\frac{\delta}{2}, \delta)} S(g_0, r)) \cup \eta(L \setminus B(g_0, \delta)) \\ =& h(L \cap \bigcup_{\alpha \in \langle \alpha_1, \alpha_0]} T_\alpha) \cup h(L \cap \bigcup_{\alpha > \alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \\ & \cup \tau([\frac{\delta}{2}, \delta)) \cup \eta(\bigcup_{i \in I} S_i) \\ =& \xi(\langle \alpha_1, \alpha_0]) \cup h(L \cap \bigcup_{\alpha > \alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \\ & \cup [\eta(g_0) - \frac{R}{8}, \eta(g_0) + \frac{R}{8}] \cup \bigcup_{i \in I} \eta(S_i) \\ =& \xi(\langle \alpha_1, \alpha_0]) \cup [\eta(g_0) - \frac{R}{8}, \eta(g_0) + \frac{R}{8}] \cup \bigcup_{i \in I} \eta(S_i), \end{aligned} \tag{10}$$

where ξ and τ are defined as in previous cases and $\langle \alpha_1, \alpha_0]$ denotes an interval with endpoints α_1, α_0 such that $\alpha_1 \in \langle \alpha_1, \alpha_0] \Longleftrightarrow L \cap T_{\alpha_1} \neq \emptyset$. As in case 5^0 we can see that for each $i \in I$, $\eta(S_i) \cap [\eta(g_0) - \frac{R}{8}, \eta(g_0) + \frac{R}{8}] \neq \emptyset$ and

$$\xi(\langle \alpha_1, \alpha_0]) \cap [\eta(g_0) - \frac{R}{8}, \eta(g_0) + \frac{R}{8}] \supset \{\xi(\alpha_0)\} \neq \emptyset.$$

Moreover $\eta(S_1)$ is connected for each $i \in I$ and $\xi(\langle \alpha_1, \alpha_0])$ is connected. Hence from the equality (10) we can infer that $h(L)$ is a connected set.

6b) $L \cap \bigcup_{0 \le \alpha \le \alpha_0} T_\alpha = \emptyset$.

Then

$$\begin{aligned} h(L) =& h(L \cap \bigcup_{\alpha>\alpha_0} T_\alpha \cap B(g_0, \frac{\delta}{2})) \cup h(L \cap B(g_0,\delta) \setminus B(g_0, \frac{\delta}{2})) \cup h(L \setminus B(g_0,\delta)) \\ =& \{\frac{R}{8}\sin\frac{1}{\alpha_0} + \eta(g_0)\} \cup h(L \cap \bigcup_{r\in[\frac{\delta}{2},\delta)} S(g_0,r)) \cup \eta(\bigcup_{i\in I} S_i) \\ =& \{\frac{R}{8}\sin\frac{1}{\alpha_0} + \eta(g_0)\} \cup \tau([\frac{\delta}{2},\delta)) \cup \bigcup_{i\in I}\eta(S_i) \\ =& [\eta(g_0) - \frac{R}{8}, \eta(g_0) + \frac{R}{8}] \cup \bigcup_{i\in I}\eta(S_i) \end{aligned}$$

and in analogous way as in the case 6a) we can prove that for each $i \in I$

$$[\eta(g_0) - \frac{R}{8}, \eta(g_0) + \frac{R}{8}] \cap \eta(S_i) \neq \emptyset;$$

thus $h(L)$ is a connected set. This completes the proof of (4).

It is easy to prove (using (3)) that $\rho(h,\eta) < \frac{R}{2}$. From this inequality it is easy to conclude that

$$B(h, \frac{R}{8}) \subset B(\eta, R). \tag{11}$$

Now we show that

$$B(h, \frac{R}{8}) \cap \mathcal{D}_{\mathcal{P}} = \emptyset. \tag{12}$$

Indeed, let $\mu \in B(h, \frac{R}{8})$. Then

$$\mu(g_0) \in (h(g_0) - \frac{R}{8}, h(g_0) + \frac{R}{8}) = (\eta(g_0) - \frac{R}{4}, \eta(g_0))$$

and for $t \in L_0 \setminus \{g_0\}$, $\mu(t) \in (h(t) - \frac{R}{8}, h(t) + \frac{R}{8}) = (\eta(g_0), \eta(g_0) + \frac{R}{4}))$. Hence $\mu(L_0)$ is not a connected set; so $\mu \notin \mathcal{D}_{\mathcal{P}}$ and the proof of (12) is finished.

Hence from (11) and (12) we conclude that $\gamma(\eta, R, \mathcal{D}_{\mathcal{P}}) \geq \frac{\mathcal{R}}{\forall}$ and consequently $\limsup_{R\to 0^+} \dfrac{\gamma(\eta, R, \mathcal{D}_{\mathcal{P}})}{R} \geq \dfrac{1}{8}$; so the set $\mathcal{D}_{\mathcal{P}}$ is porous at the point η. The proof of theorem is finished. □

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