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A UNIFIED THEORY OF BILATERAL DERIVATES

Abstract

We present here a unified theory of bilateral derivates, which we call here briefly “biderivates”. This unified theory is achieved with the help of two new fundamental theorems on biderivates, called the First and Second Biderivate Theorems. These two biderivate theorems are obtained in terms of bimonotonicity and bi-Lipschitz properties of a function on a set which also depend on the values of the function outside the set like the properties VB_* and AC_*. From these two biderivate theorems we further deduce the Third and Fourth Biderivate Theorems, which deal with the properties of a function on a portion of a given set and the Fifth Biderivate Theorem on the Baire class of biderivates.

Next, given $f : X \to \mathbb{R}$, where $X \subset \mathbb{R}$, let X' denote the set of limit points of X in X. Then we define the “median” Mf of f to be the multifunction $Mf(x) = [\underline{D}f(x), \overline{D}f(x)]$, $x \in X'$. We deduce here from the above biderivate theorems five basic median theorems which deal with the properties of the median.

The various known results on biderivates are deduced here from these biderivate and median theorems many of which are strengthened in this process. As the known results on biderivates were obtained earlier by ad hoc methods, the present theory provides a synthesis of these results and brings out their inter-relations leading thereby to a unified theory.

In particular, we deduce from one of the median theorems several monotonicity theorems in terms of biderivates including an extended version of the well known Goldowski-Tonelli theorem. Further, we deduce from another median theorem a mean-value theorem in terms of

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median and derivative, and the Darboux property of median and derivative. Also, the Denjoy property of derivatives is obtained from the Third Biderivate Theorem and results on the Baire class of derivatives and medians are obtained from the Fifth Biderivate Theorem. Also, a biderivate version of the classical Denjoy-Young-Saks theorem is obtained from the First Biderivate Theorem.

From the above mentioned biderivate and median theorems we also deduce some other known theorems of Denjoy, Young, Choquet, Zahorski, Kronrod, Fort and Marcus, along with the following two new results: (i) a theorem on biderivates similar to a fundamental theorem of Denjoy on unilateral derivates; and (ii) a theorem on median derivates similar to a theorem of Morse on unilateral derivates.

Contents

Chapter 1: Introduction, Definitions and Preliminaries

In this chapter we first present in §1.1 a more detailed introduction of the present work indicating the inter-relations between various results. In the next three sections of this chapter we present various definitions and notation that are used throughout the work.

In particular, we present in §1.2 some definitions, notation and other preliminaries that are needed for the present work. Then in §1.3 we define several weaker forms of continuity, and in §1.4 we define various notions of bimonotonicity and bi-Lipschitz properties of a function on a set. The first two biderivate theorems are obtained in terms of these properties.

§1.1. Introduction

We present here a unified theory of bilateral derivates, which we call here briefly "biderivates". This theory is similar to the unified theory of unilateral derivates presented in Part I of [G3].

The present unified theory is achieved with the help of two new fundamental theorems on biderivates which we call simply the First and Second Biderivate Theorems. In these two theorems we deal with bimonotonicity and bi-Lipschitz properties of a function on a set in terms of its biderivates at the points of the set. From these two theorems we deduce some other new biderivate theorems along with some basic median theorems which deal with the properties of the set-valued median related to biderivates.

The various known results on biderivates are deduced in turn from these biderivate and median theorems many of which are strengthened in this process. We refer to [G1] for several known results on biderivates. Apart from obtaining several new results in the present theory, most of the results of [G1] are strengthened by the results of this theory.

We have used here titles for theorems with common theme as in [G3], like the First and Second Biderivate Theorems, to eliminate confusion. This brings out more clarity and makes it easier to compare these theorems.

Next, we discuss the organization of the present work along with the nature of the results and their inter-relations in some detail. The work is divided into three chapters.

Apart from the introduction presented in this section, Chapter 1 is devoted to various preliminaries like definitions, notation and nomenclature that are used throughout the present work. In particular, we define in §1.3 several weaker forms of continuity, and in §1.4 we define various notions of bimonotonicity and bi-Lipschitz properties of a function on a set.

In Chapter 2 we deal with some fundamental theorems on biderivates. First, in §2.1, we obtain the First and Second Biderivate Theorems. Then, in §2.2, we obtain a derivability theorem and from the first two biderivate theorems deduce some of the known theorems on biderivates. In the derivability theorem we prove the differentiability of a function $f : X \to \mathbb{R}$ at almost all points of a set $E \subset X$ on which f is bimonotone. Then with the help of this theorem from the First Biderivate Theorem we deduce a version of the classical Denjoy-Young-Saks theorem in terms of biderivates. Further, from the Second Biderivate Theorem we deduce a theorem of Young on the set of points where a function has an infinite derivate, and an extended version of a theorem on the identity of a function with a Lipschitz function except for a set of arbitrarily small measure.

Next, in §2.3 from the first two biderivate theorems we deduce the Third and Fourth Biderivate Theorems which deal with properties of functions on some portion of a given set. Also in this section from the Third Biderivate Theorem we deduce the First Median Theorem which deals with properties of median at a residual set of points. Further, in §2.4, we define various lower and upper Baire classes of functions and multifunctions as in [G3], and from the First Biderivate Theorem deduce the Fifth Biderivate Theorem and Second Median Theorem which deal with the Baire class of biderivates and median, respectively.

Next, in Chapter 3, we deal with other results on biderivates, median and derivative. First, in §3.1 from the Third Biderivate Theorem we deduce some results on the properties of biderivates and median at residual sets of points which include the Third Median Theorem. Also, we obtain from the Third Biderivate Theorem a version of a fundamental theorem of Denjoy in terms of biderivates, and a version of a theorem of Choquet on the identity of biderivates with strong derivates at a residual set of points. Next, in §3.2 from the Third Biderivate Theorem we deduce the Fourth and Fifth Median Theorems which deal with the properties of the median of a function $f : I \to \mathbb{R}$ at its nonmonotonicity points; i.e., points in no neighborhood of which f is nondecreasing.

Next, in §3.3 from the Fourth Median Theorem we deduce several monotonicity theorems in terms of biderivates, including the well known Goldowski-Tonelli theorem on functions that are derivable nearly everywhere. Also from one of these monotonicity theorems we deduce a result on median derivates which is similar to a theorem of Morse on unilateral derivates.

Finally, in §3.4, we deal with the properties of derivative and median. First from the Fifth Biderivate Theorem we obtain a theorem of Zahorski on the property of derivability sets, and the Baire class of derivative. Then from the

First Median Theorem, which is deduced from the Third Biderivate Theorem, we deduce a mean-value theorem in terms of median and derivative. Further from the same median theorem we deduce the Darboux property of median and derivative, and we deduce from the Third Biderivate Theorem the Denjoy property of derivative. Also, we obtain here an extended version of a theorem of Marcus on stationary sets of derivatives.

§1.2. Definitions, Notation and Other Preliminaries

In this section we present some definitions, notations and other preliminaries that are needed for the present work.

We will employ $\mathbb{R}$ to denote the set of all real numbers, and $\overline{\mathbb{R}}$ to denote the set of extended real numbers, namely $[-\infty, +\infty]$. Further, we will use X to denote an arbitrary subset of $\mathbb{R}$, and I to denote any subinterval of $\mathbb{R}$. Further, unless stated otherwise, f will be assumed to be a real-valued function defined on I. However, X is also used frequently as the domain of f, and in that case we will state that "$f : X \to \mathbb{R}$", where X will be understood to be an arbitrary subset of $\mathbb{R}$.

Given $X \subset \mathbb{R}$, we will use X^d to denote the set of all limit points of X in $\mathbb{R}$, and X' to denote the set of limit points of X in X. Further, we will use X^-, X^+ and X^b to denote the sets of points in X which are left, right or bilateral limit points, respectively, of X.

Next, a set E is called *countable* if it is either finite or countably infinite. Now, suppose $E \subset X \subset \mathbb{R}$. Then E is said to be *meager* in X if it is of the first category in X, and when $X \setminus E$ is meager in X, E will be said to be *residual* in X. A nonempty subset of X is called a *portion* of X if it is of the form $X \cap I$, where I is an open subinterval of $\mathbb{R}$.

Further, when each portion of X includes a set of positive measure, X is called *metrically dense in itself.* Also, the set E will be said to be *c-dense* in X if for every portion X_0 of X the set $E \cap X_0$ has cardinality c.

Next, if any pointwise proposition P holds at a residual set of points in X, then P will be said to hold *residually everywhere* (or r.e.) in X and in case P holds at all but a countable set of points in X, then P will be said to hold *nearly everywhere* (or n.e.) in X. Also, we will use a.e. for *almost everywhere* in the measure-theoretic sense.

Next, given $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$, we will use f_{+c} to denote the function

$$f_{+c}(x) = f(x) + cx, \quad x \in X.$$

The notation f_{-c} will in turn be used to denote the function $f_{+(-c)}$.

Now, let $x \in X^d$. When x is a left limit point of X, we will use $\underline{f}(x-0)$ and $\overline{f}(x-0)$ to denote the lower and upper left limits of f at x, and when these two limits are equal, we will use $f(x-0)$ to denote their common value which is called the *left limit* of f at x. Note that the limit $f(x-0)$ may be finite or infinite. Further, when x is a right limit point of X, the right limits $\underline{f}(x+0)$, $\overline{f}(x+0)$ and $f(x+0)$ of f at x are defined analogously.

We include here a well known theorem of Young [Y2] on the symmetry of unilateral limits which will be used sometimes. Its proof given in [S, p. 261] for functions on an interval can be extended without difficulty to functions with an arbitrary domain $X \subset \mathbb{R}$.

Theorem (Young). *For every function $f : X \to \mathbb{R}$ at nearly all the points x in X*

$$\underline{f}(x-0) = \underline{f}(x+0) \leq f(x) \leq \overline{f}(x-0) = \overline{f}(x+0).$$

Further, given $f : X \to \mathbb{R}$, $x, y \in X$, $x \neq y$, we will use $Qf(x,y)$ to denote the following *difference quotient* of f on $[x,y]$ or $[y,x]$

$$Qf(x,y) = \{f(y) - f(x)\}/(y-x).$$

The lower and upper limits of $Qf(x,y)$ as $y \to x$ are called the *lower* and *upper biderivates* (or bilateral derivates) of f at x. We will use $\underline{D}f(x)$ and $\overline{D}f(x)$ to denote these two biderivates, respectively. When they are equal, then f is said to be *derivable* a x and their common value is called the *derivative* of f at x and denoted by $f'(x)$. Further, in the case when $f'(x)$ is finite, we will call f *differentiable* at x. We will use $\Delta(f)$ and $\Delta^*(f)$ to denote the sets of points where f is derivable or differentiable, respectively, and call both of them *derivability* sets of f.

Next, given $f : X \to \mathbb{R}$, a point $x \in X'$ is called [G1, p. 296] a *knot point* of f if $\underline{D}f(x) = -\infty$ and $\overline{D}f(x) = +\infty$. Also, when x is a knot point of f, f will be said to be *knotted* at x. The set of knot points of f is called on the other hand the *knot set* of f, and denoted by $K(f)$.

Clearly, a knot point x of f is a point where f lacks derivability in a sense to the extreme, viz. instead of the derivates $\underline{D}f(x)$ and $\overline{D}f(x)$ being equal, they are as far apart as possible.

Further, for each $x \in X'$ we define $Mf(x) = [\underline{D}f(x), \overline{D}f(x)]$. This set-valued function, introduced in [G3, p. 43], is called the (ordinary) *median* of f. Also, each element of $Mf(x)$ is called in turn a *median derivate* of f at x. It is interesting to note here that f is knotted at x iff $Mf(x) = \overline{\mathbb{R}}$.

Next, given $x \in X^d$, the lower and upper limits of $Qf(y,z)$ as y and z approach x through X are called the *lower* and *upper strong derivates* of f at x, and we will denote them by $\underline{D}_*f(x)$ and $\overline{D}_*f(x)$, respectively. Further, when these two derivates are equal, f is said to be *strongly derivable* at x and their common value is called the *strong derivative* of f at x, and will be denoted by $f'_*(x)$. Also, we define the *strong median* of f to be $M_*f(x) = [\underline{D}_*f(x), \overline{D}_*f(x)],\ x \in X^d$. The notions of strong derivates and strong derivative were introduced by Peano [P] without any nomenclature. (This nomenclature was introduced in [G3, p. 33]; see Chapter 14 of [G3] for a detailed investigation of these notions.)

Finally, given $f : I \to \mathbb{R}$, we will use $N(f)$ to denote the set of points x in I such that f is not nondecreasing in any neighborhood of x. The set $N(f)$ was first introduced in [G1], and has been called in [G3] a *nonmonotonicity set* of f. The points of $N(f)$ are called on the other hand the *nonmonotonicity points* of f.

It should be pointed out that in this work for each property, P, of functions defined at points, a function is said to have P at a set E if it has P at each point of E.

§1.3. Weaker Forms of Continuity

In this section we present several weaker forms of continuity which are used throughout this work. We will present here only local versions of these properties, but a function will be understood to have any of these properties globally when it has that property at each point of its domain.

We assume throughout this section that $f : X \to \mathbb{R}$. Also, unless stated otherwise, we assume that $x \in X$. The notions presented here in parts A and B have been defined earlier in [G3, pp. 12–14].

A. Lower and Upper Continuities

We will call f *lower continuous* (or LC) *from left* or *right* at x if

$$\overline{f}(x-0) \leq f(x) \text{ or } f(x) \leq \underline{f}(x+0), \text{ respectively,}$$

provided $x \in X^-$ or X^+, respectively. Similarly, f will be called *upper continuous* (or UC) *from left* or *right* at x if

$$\underline{f}(x-0) \geq f(x) \text{ or } f(x) \geq \overline{f}(x+0), \text{ respectively,}$$

provided $x \in X^-$ or X^+, respectively.

Further, when f is LC (or UC) from both sides at x, it will be called simply LC (or UC) at x.

It should be noted here that f is UC at x iff $-f$ is LC at x, and that f is continuous at x iff it is LC and UC at x. Further, the notions of LC and UC are not comparable in general with the ones of LSC and USC, respectively. Also, when f is nondecreasing or nonincreasing, it is clear that f is automatically LC or UC, respectively.

B. Weak Continuities

We next present some weaker versions of continuity and lower and upper continuities.

First, let $x \in X^d$. The function f is said to be *regulated from left* or *right* at x, or *left* or *right regulated* at x, if the limit $f(x-0)$ or $f(x+0)$, respectively, exists provided x is a left or right limit point, respectively, of X. Further, f is said to be *regulated* at x if it is so from both sides at x. Note that we do not assume here the limits $f(x-0)$ and $f(x+0)$ to be equal, and each of these limits may be finite or infinite.

Now, suppose $x \in X$. Then we will call f *weakly continuous* (or WC) *from left* or *right* at x if f is continuous from left or right, respectively, at x whenever it is regulated from that side at x. Further, when f is WC from both sides at x, it will be called simply WC at x.

Similarly, we will call f *weakly lower continuous* (or WLC) *from left* or *right* at x if f is LC from left or right, respectively, at x whenever it is regulated from that side at x. Further, when f is WLC from both sides at x, it will be called simply WLC at x. The notion of *weak upper continuity* (or WUC) is defined analogously.

It should be noted here that f is WUC at x iff $-f$ is WLC at x; and f is WC at x iff it is simultaneously WLC and WUC at x. Further, f is continuous at a point x iff it is regulated and WC at x. The notion of WC is however much weaker than that of continuity. For it is clear from the Young's Theorem 1.2.1 that every function f on X is WC n.e.

Next, we present three weaker versions of WC, WLC and WUC.

C. Subweak Continuities

Generalizing the above notions of weak continuities, f will be called *subweakly continuous*, *LC* or *UC* (or *SWC*, *SWLC* or *SWUC*, respectively) at x if it is continuous, LC or UC, respectively, at x whenever it is regulated from *both sides* at x.

Clearly, the properties SWC, $SWLC$ and $SWUC$ are weaker than WC, WLC and WUC, respectively.

D. Partial Weak Continuities

Given a subset E of X, we will call f *partially weakly lower continuous* (or $PWLC$) *with respect to* E *at* $x \in E$ if, provided x is a limit point of E from one and only one side, f is WLC from that side at x. The *partial weak upper continuity* (or $PWUC$) of f *with respect to* E *at* $x \in E$ is defined analogously;

and when f is simultaneously $PWLC$ and $PWUC$ with respect to E at x, then f will be said to be *partially weakly continuous* (or PWC) *with respect to* E *at* x.

Further, when f is $PWLC$, $PWUC$ or PWC with respect to E at each point of E, then f will be said to be $PWLC$, $PWUC$ or PWC, respectively, *on* E.

Note that the term "with respect to E" is used here in reference to the term "partially", and it should not be confused with the restriction of f to E.

E. Contiguous Weak Continuities

Any two points $x, y \in X$ are called [G1, p. 298] a *pair of contiguous points* of X if X does not include any point in between x and y. We will call f *contiguously* WLC (or $CWLC$) if it is WLC at at least one of each pair of contiguous points of X. The *contiguous* WUC (or $CWUC$) is defined analogously. Further, when f is simultaneously $CWLC$ and $CWUC$, it will be called *contiguously weakly continuous* (or CWC).

It should be noted that the following implications hold globally on X :

$$\begin{aligned} WC &\Longrightarrow SWC \Longrightarrow PWC \Longrightarrow CWC; \\ WLC &\Longrightarrow SWLC \Longrightarrow PWLC \Longrightarrow CWLC. \end{aligned}$$

Further, note that PWC and CWC are much weaker than WC or SWC. For, when E does not include any point which is a limit point of E from only one side, (e.g. if E is a dense subset of some open interval), then f is automatically PWC on E; and in this case we will say simply that f is *vacuously* PWC on E. Similarly, when X does not have any contiguous points, (e.g. when X is a dense subset of any interval), then f is automatically CWC; and in this case we will say simply that f is *vacuously* CWC.

Remark 1.3.1. It should be pointed out here that the notions of WC, WLC, WUC, SWC, $CWLC$ and $CWUC$ have been used earlier in [G1] under different terminology. The term "interned" was used there for SWC, and "bilaterally interned" for WC. Similarly, the term "lower interned" was used there for $SWLC$, and "bilaterally lower interned" for WLC. Similar terms were used for $SWUC$ and WUC. Further, the terms "contiguously lower and upper interned" were used there for $CWLC$ and $CWUC$, respectively, and "contiguously interned" for CWC.

§1.4. Bimonotonicity and Bi-Lipschitz Properties on a Set

In this section we define bimonotonicity and bi-Lipschitz properties of a function $f : X \to \mathbb{R}$ on a set $E \subset X$ which are used throughout the present work.

These properties were defined earlier in [G3, pp. 24, 25]. They depend on the values of f outside E as well like the properties VB_* and AC_* on E (see [S, pp. 228, 231]).

Suppose $f : X \to \mathbb{R}$ and $E \subset X$. We will call f *weakly biincreasing* (or WBI) on E if

$$f(a) \leq f(x) \leq f(b) \text{ whenever } a, b \in E,\ x \in X \text{ and } a \leq x \leq b. \qquad (1)$$

Further, if there is some real number $c > 0$ such that the function f_{-c} is WBI on E, then f will be called *strongly biincreasing* (or SBI) on E.

Next, when $-f$ is WBI on E, then f is called *weakly bidecreasing* (or WBD) on E. Similarly, when $-f$ is SBI on E, then f is called *strongly bidecreasing* (or SBD) on E.

Note here that when f is WBI on E, then it is weakly increasing (or WI) on E, or what is usually called nondecreasing on E.

Next, decomposing the Lipschitz property into two parts, we will call f *lower* or *upper Lipschitz* (or LL or UL, respectively) on E if there is a $c \in \mathbb{R}$ such that f_{+c} is nondecreasing or nonincreasing, respectively, on E. Clearly, f is Lipschitz on E iff it is LL and UL on E.

Further, we call f *lower* or *upper bi-Lipschitz* (or LBL or UBL) on E if there is a $c \in \mathbb{R}$ such that f_{+c} is WBI or WBD, respectively, on E. In case f is simultaneously LBL and UBL on E, then it will be called *bi-Lipschitz* (or BL) on E. Also, if there is a $c \geq 0$ such that f_{+c} is WBI and f_{-c} is WBD on E, then f is called *c-bi-Lipschitz* (or $BL(c)$) on E. Thus it is clear that f is BL on E iff it is $BL(c)$ on E for some $c \geq 0$.

Next, we call f *bimonotone* on E if it is either WBI or WBD on E. Similarly, f is called *strongly bimonotone* on E if it is either SBI or SBD on E.

Further, when there is a $c \in \mathbb{R}$ such that f_{+c} is monotone on E, then f is said to be of *monotonic type* (or MT) on E. If c can be chosen such that f_{+c} is bimonotone on E, then f is said to be of *bimonotonic type* (or BMT) on E.

It should be noted here that f is BMT on E iff it is either LBL or UBL on E. Also, the bimonotonicity of f on E is stronger than the ordinary monotonicity of f on E unless $E = X$.

When E coincides with X, it is clear from the above definitions that the bilateral version of each of the above properties becomes equivalent to its ordinary version, and so in that case this adjective will not be used in the nomenclature of that property, and likewise "B" will be dropped from its notation. Thus the function f is *strongly increasing* (on X) iff there is a $c > 0$ such that f_{-c} in nondecreasing, or, equivalently, increasing. Similarly, f is *strongly decreasing* iff there is a $c > 0$ such that f_{+c} is decreasing.

Finally, given any function-theoretic property P, and $f : X \to \mathbb{R}$ and $E \subset X$, we will call f *generalized* P (or GP) on E if E is the union of a sequence of sets $\{E_n\}$ on each of which f has P. Moreover, when the sets E_n can be chosen to be closed or G_δ-sets in E, then f will be called F*-generalized* P or G_δ*-generalized* P, respectively, (or F-GP or G_δ-GP, respectively), on E.

Further, f will be said to be *nowhere* P on E if it does not have P on any portion of E.

Remark 1.4.1. Note that the properties BMT and BL are stronger than VB_* and AC_*, respectively, (see [S, pp. 228, 231] for definitions of the latter properties). For, if $f : X \to \mathbb{R}$ is BMT on $E \subset X$, it is easy to see that it is VB_* on E; and in case f is BL on E, it can be seen easily that it is AC_* on E.

The significance of the properties strongly increasing and strongly decreasing (on X) in differentiation theory was first recognized in [G1]. However, they were called there "adequately increasing" and "adequately decreasing", respectively.

Chapter 2: Some Fundamental Theorems on Biderivates

In this chapter we deal with some fundamental theorems on biderivates. First, in §2.1, we obtain the First and Second Biderivate Theorems. Then, in §2.2, we obtain a derivability theorem, and then from the first two biderivate theorems deduce some of the known results on biderivates as described in §1.1.

Next, in §2.3 from the first two biderivate theorems we deduce the Third and Fourth Biderivate Theorems, and the First Median Theorem. Finally, in §2.4, we define various lower and upper Baire classes of functions and multifunctions, and deduce from the First Biderivate Theorem the Fifth Biderivate Theorem and Second Median Theorem which deal with the Baire class of biderivates and median, respectively.

§2.1. Two Fundamental Theorems on Biderivates

In this section we obtain the First and Second Biderivate Theorems which deal with generalized bimonotonicity and bi-Lipschitz properties of functions in terms of their biderivates. The various known results on biderivates will be deduced later from these two theorems. Here we include a few elementary consequences of the First Biderivate Theorem. Recall that we assume throughout that X is an arbitrary subset of $\mathbb{R}$. We begin with the following basic lemma on bimonotonicity.

Lemma 2.1.1. *Let $f : X \to \mathbb{R}$, $A \subset X$, and suppose f is WBI on A. If a is a left limit point of A, then f is left regulated at a, and if $a < x \leq b$ where $b \in A$ and $x \in X$, then $f(a-0) \leq f(x)$. Similarly, if a is a right limit point of A, then f is right regulated at a, and if $b \leq x < a$ where $b \in A$ and $x \in X$, then $f(x) \leq f(a+0)$.*

Proof. First, suppose a is a left limit point of A. Then there is an increasing sequence $\{a_n\}$ in A which converges to a. Now since f is WBI on A, the sequence $\{f(a_n)\}$ is clearly nondecreasing, and further, for each n, it is clear from (1) that $f(a_n) \leq f(t) \leq f(a_{n+1})$ for $t \in X \cap (a_n, a_{n+1})$. Hence it is clear that f is left regulated at a and $f(a-0) = \lim_n f(a_n)$. Next, suppose $a < x \leq b$, where $b \in A$ and $x \in X$. Then, for each n, since $a_n < a < x \leq b$, it follows from (1) that $f(a_n) \leq f(x)$, and hence $f(a-0) = \lim_n f(a_n) \leq f(x)$.

A similar argument holds in the case when a is a right limit point of A. □

We now obtain the following extension lemma on bimonotonicity which also plays a central role in the present chapter.

Lemma 2.1.2. (**First Extension**). *Let $f : X \to \mathbb{R}$, $E \subset X$, A be a dense subset of E, and suppose f is $PWUC$ with respect to E at each point of $E \setminus A$. If f is WBI or SBI on A, then it is so on $E \cap \operatorname{co}(A)$.*

Proof. Set $B = E \cap \mathrm{co}(A)$, and first suppose f is WBI on A. To prove that f has this property on B, suppose $a, b \in B$, $x \in X$ and $a \le x \le b$. We need to show that $f(a) \le f(x) \le f(b)$. We will give the proof for the first inequality; the second inequality is proved by a similar argument.

Suppose $a < x$, for there is nothing to prove otherwise. Clearly, there exist two points $a', b' \in A$ such that $a' \le a$ and $b \le b'$. If there is some point c in $A \cap [a, x]$, then since $a' \le a \le c \le x \le b'$ and f is WBI on A, it is clear from (1) that $f(a) \le f(c) \le f(x)$. Hence suppose $A \cap [a, x] = \emptyset$. Now, since A is dense in E, a is a limit point of A from the left, but not from the right. Hence f is by the hypothesis WUC from the left at a. Further, it follows from the above Lemma 2.1.1 that f is left regulated at a, and hence $f(a) \le f(a-0)$. Also, since $a < x \le b'$, where $b' \in A$, it follows from the same lemma that $f(a) \le f(a-0) \le f(x)$. This proves that f is WBI on B.

Next, if f is SBI on A, then there exists a $c > 0$ such that f_{-c} is WBI on A. Hence by the above f_{-c} is WBI on B, and so f is SBI on B. □

We include here another simple lemma which also will be used repeatedly in this section. Given any property P of functions on sets, we will call it *hereditary* if the following two conditions hold:

(C1) every function has P on each singleton set in its domain;

(C2) if a function has P on a set A, then it also has P on every subset of A.

Note that for the various properties considered here, condition (C1) holds vacuously.

Lemma 2.1.3. *Suppose P is a hereditary property of functions on sets. Then given $f : X \to \mathbb{R}$ and $E \subset X$, if there is a sequence of closed (or G_δ-) sets $\{E_n\}$ in X such that f has P on each E_n and $E \setminus \cup_n E_n$ is countable, then f is F-GP (or G_δ-GP) on E.*

Proof. We will give the proof in the case of closed sets; a similar argument holds in the other case.

Suppose there is a sequence of closed sets $\{E_n\}$ in X such that f has P on each E_n and the set $C = E \setminus \cup_n E_n$ is countable. Then C is the union of a sequence of singleton sets $\{S_n\}$ on each of which f has P again due to (C1). Now, rearrange the sets in $\{E_n\}$ and $\{S_n\}$ in the form of a single sequence $\{A_n\}$, and set for each n, $B_n = E \cap A_n$. Then it is clear that $E = \cup_n B_n$, where each B_n is closed in E. Further, it follows from (C2) that f has P on each B_n, and hence f is F-GP on E. □

Theorem 2.1.4. (**First Biderivate**). *Let $f : X \to \mathbb{R}$, $E \subset X$, and suppose $\underline{D}f > 0$ n.e. in E. Then f is G_δ-$GSBI$ on E and in case f is WUC on a* *set $F \supset E$, then f is F-GSBI on E. Moreover, there is in general a sequence of G_δ-sets $\{E_n\}$ in X such that $E \subset \cup_n E_n$ and f is SBI on each E_n and in case f is WUC on $F \supset E$, then the sets E_n can be chosen to be closed in F.*

PROOF. It is enough to prove the last two parts, for the first two parts follow directly from them with the help of Lemma 2.1.3 since the property SBI is clearly hereditary.

For each positive integer n let A_n denote the set of points x in E for which the following two relations hold:

(i) $f_{-1/n}(t) \le f_{-1/n}(x)$ for $t \in X \cap (x - 1/n, x)$,

(ii) $f_{-1/n}(t) \ge f_{-1/n}(x)$ for $t \in X \cap (x, x + 1/n)$.

Clearly, A_n is the union of a sequence of sets $\{A_{n,i} : i = 1, 2, \dots\}$ such that each $A_{n,i}$ has a diameter less than $1/n$. Now it is easy to see that

$$\{x \in E : \underline{D}f(x) > 0\} \subset \bigcup_n A_n = \bigcup_n \bigcup_i A_{n,i}.$$

Now let C denote the countable set of points in X where f is not WUC (see Theorem 1.2.1). For each pair of integers n, i, the function $f_{-1/n}$ is clearly WBI on $A_{n,i}$. Hence it follows from the Extension Lemma 2.1.2 that $f_{-1/n}$ is WBI on the set

$$B_{n,i} = X \cap A^-_{n,i} \cap \operatorname{co}(A_{n,i}) \setminus (C \setminus A_{n,i}). \tag{2}$$

Clearly, $B_{n,i}$ is a G_δ-set in X on which f is SBI. Further, since $\underline{D}f > 0$ n.e. in E, the set $E \setminus \cup_{n,i} B_{n,i}$ is countable, and hence it is the union of a sequence of singleton sets $\{S_n\}$ on each of which f is vacuously SBI. Now the desired sequence $\{E_n\}$ is obtained on rearranging the sets in $\{B_{n,i}\}$ and $\{S_n\}$ in the form of a sequence.

Next, to prove the second of the last two parts, suppose f is WUC on a set $F \supset E$. Then $F \cap C = \emptyset$. Hence for each pair of integers n, i, it is clear from (2) that the set $F \cap B_{n,i}$ is an F_σ-set in F, and so it is the union of a sequence $\{F_{n,i,j} : j = 1, 2, \dots\}$ of closed sets in F. The desired sequence $\{E_n\}$ is obtained in this case by rearranging the sets in $\{F_{n,i,j}\}$ and $\{S_n\}$ in the form of a sequence. □

Corollary 2.1.5. *Let $f : X \to \mathbb{R}$, $E \subset X$, and suppose $\underline{D}f > -\infty$ n.e. in E. Then f is G_δ-GLBL on E; and in case f is WUC on E, then it is F-GLBL on E.*

PROOF. Let $A = \{x \in E : \underline{D}f(x) > -\infty\}$, and for each positive integer n set $E_n = \{x \in E : \underline{D}f(x) > -n\} = \{x \in E : \underline{D}f_{+n}(x) > 0\}$. Then $A = \cup_n E_n$, and so $E \setminus \cup_n E_n$ is countable. Further, for each n, by the second part of the above theorem there is a sequence $\{E_{n,i} : i = 1, 2, \dots\}$ of G_δ-sets in E such that $E_n \subset \cup_i E_{n,i}$ and f_{+n} is SBI on each $E_{n,i}$. The function f is thus LBL on each $E_{n,i}$. Now, since $E \setminus \cup_n \cup_i E_{n,i}$ is countable, and the property LBL is clearly hereditary, it follows from Lemma 2.1.3 that f is G_δ-$GLBL$ on E. In the case when f is WUC on E, the sets $E_{n,i}$ can be chosen by the last part of the above theorem to be closed in E, and so it follows as before that f is F-$GLBL$ on E. □

Corollary 2.1.6. *Let $f : X \to \mathbb{R}$, $E \subset X$, and suppose f is knotted only at a countable set of points in E. Then f is G_δ-$GBMT$ on E and in case f is WC on E, then it is F-$GBMT$ on E.*

Proof. Let $A = \{x \in E : \underline{D}f(x) > -\infty\}$ and $B = E \setminus A$. Then according to the hypothesis, $\overline{D}f < \infty$ n.e. in B. Now it follows from the second part of Theorem 2.1.4 as in the proof of the previous corollary that there are two sequences $\{A_n\}$ and $\{B_n\}$ of G_δ-sets in E (or closed in E when f is WC on E) such that $A \subset \cup_n A_n$, $B \subset \cup_n B_n$ and such that f is LBL on each A_n and UBL on each B_n. Thus $E = (\cup_n A_n) \cup (\cup_n B_n)$, where f is BMT on each A_n and B_n for each n. Both the results are now clear. □

Next is another consequence of the First Biderivate Theorem 2.1.4 which deals with the existence of nonempty perfect sets on which the given function is monotone.

Corollary 2.1.7. *Let $f : X \to \mathbb{R}$, where X is a Borel set in $\mathbb{R}$, and let*

$$E = \{x \in X : \underline{D}f(x) > 0\}.$$

If E is uncountable, then there is a nonempty perfect set P in X on which f is SBI. Moreover, if $|E| > 0$, then P can be chosen to be metrically dense in itself.

Proof. Suppose E is uncountable. Then by the second part of Theorem 2.1.4 there is a sequence of G_δ-sets $\{E_n\}$ in X such that $E \subset \cup_n E_n$ and f is SBI on each E_n. Hence there is an integer n such that $E \cap E_n$ is uncountable. Then E_n is an uncountable Borel set in $\mathbb{R}$, and so it contains some nonempty perfect set P (see, e.g., [K, p. 44]) on which f is again SBI. Further, if $|E| > 0$, then there exists an integer n such that $|E_n| > 0$, and this E_n contains a metrically dense in itself perfect set (see, e.g., [H, p. 192]). □

In the next biderivate theorem we deal with the bi-Lipschitz property. In the case of a WC function this theorem can be deduced directly from Corollary 2.1.5, but to obtain it without any continuity hypothesis we need the following extension lemma on this property.

Lemma 2.1.8. (**Second Extension**). *Let* $f : X \to \mathbb{R}$, $E \subset X$ *and* $c \geq 0$. *If* f *is* $BL(c)$ *on a dense subset* A *of* E, *then it is so on* $E \cap \operatorname{co}(A)$.

Proof. Suppose the given hypothesis holds, and let $B = E \cap \operatorname{co}(A)$. To prove that f is $BL(c)$ on B, suppose $a, b \in B$, $x \in X$ and $a \leq x \leq b$. We will give the proof here for the inequality

$$|f(x) - f(a)| \leq c(x - a); \tag{3}$$

the inequality $|f(x) - f(b)| \leq c(b - x)$ is proved by a similar argument.

Suppose $a < x$, for there is nothing to prove otherwise. There clearly exist two points $a', b' \in A$ such that $a' \leq a$ and $b \leq b'$. If there is another point $c' \in A \cap [a, x]$, then since f is $BL(c)$ on A and $a' \leq a \leq c' \leq x \leq b'$, we clearly have

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f(c')| + |f(c') - f(a)| \\ &\leq c(x - c') + c(c' - a) = c(x - a). \end{aligned}$$

Next, suppose $A \cap [a, x] = \emptyset$. In case $c = 0$, we clearly have

$$|f(x) - f(a)| \leq |f(x) - f(a')| + |f(a) - f(a')| = 0 = c(x - a).$$

Hence suppose $c > 0$, and let $\varepsilon > 0$. Now since A is dense in E, a is a left limit point of A. Hence we can choose a point $d \in A$ such that $0 < a - d < \varepsilon/2c$. Since $d < a \leq x \leq b'$ and f is $BL(c)$ on A, we have

$$\begin{aligned} |f(x) - f(a)| &\leq |f(x) - f(d)| + |f(d) - f(a)| \\ &\leq c(x - d) + c(a - d) \\ &= c(x - a) + 2c(a - d) < c(x - a) + \varepsilon. \end{aligned}$$

As this holds for every $\varepsilon > 0$, this proves inequality (3). □

Theorem 2.1.9. (**Second Biderivate**). *Let* $f : X \to \mathbb{R}$ *and* $E \subset X$. *If the derivates* $\underline{D}f$ *and* $\overline{D}f$ *are finite* n.e. *in* E, *then* f *is* F-GBL *on* E. *Moreover, if there is a* $c > 0$ *such that* $\underline{D}f > -c$ *and* $\overline{D}f < c$ n.e. *in* E, *then* f *is* F-$GBL(c)$ *on* E.

Proof. We prove the second part first. Suppose there is a $c > 0$ such that $\underline{D}f > -c$ and $\overline{D}f < c$ n.e. in E. Then $\underline{D}f_{+c} > 0$ and $\overline{D}f_{-c} < 0$ n.e. in E. By the First Biderivate Theorem 2.1.4 there are two sequences of sets $\{A_n\}$ and $\{B_n\}$ with $E = \cup_n A_n = \cup_n B_n$ such that, for each n, the function f_{+c} is SBI on A_n and f_{-c} is SBD on B_n. Thus for each pair of integers n, i, the function f is $BL(c)$ on $A_n \cap B_i$, and hence by the above lemma it is so on $X \cap (A_n \cap B_i)^- \cap \operatorname{co}(A_n \cap B_i)$. Consequently, there is a sequence

$\{F_{n,i,j} : j = 1, 2, \dots\}$ of closed sets in X such that $A_n \cap B_i \subset \cup_j F_{n,i,j}$ and f is $BL(c)$ on $F_{n,i,j}$ for each j. Now on rearranging the sets $F_{n,i,j}$ in the form of a single sequence, we obtain a sequence of closed sets $\{F_n\}$ in X such that $E \subset \cup_n F_n$ and f is $BL(c)$ on each F_n. This proves that f is F-$GBL(c)$ on E.

To prove the first part, suppose $\underline{D}f$ and $\overline{D}f$ are finite n.e. in E. For each positive integer n set

$$E_n = \{x \in E : \underline{D}f(x) > -n, \overline{D}f(x) < n\}.$$

Then $E \setminus \cup_n E_n$ is countable. Now, for each n, it follows from above that there is a sequence $\{F_{n,i} : i = 1, 2, \dots\}$ of closed sets in X such that $E_n \subset \cup_i F_{n,i}$ and f is BL on $F_{n,i}$ for each i. Now since $E \setminus \cup_n \cup_i F_{n,i}$ is countable, it follows from Lemma 2.1.3 that f is F-GBL on E. □

Remark 2.1.10. By Remark 1.4.1, clearly Corollary 2.1.6 strengthens Theorem 10.1 of Denjoy in [S, p. 234] on biderivates. On the other hand the Second Biderivate Theorem 2.1.9 partly strengthens Theorem 10.5 of Denjoy in [S, p. 235] which was obtained by him in terms of unilateral derivates.

§2.2. Some Direct Applications of the First Two Biderivate Theorems

In this section we first obtain a derivability theorem and then from the First and Second Biderivate Theorems deduce some known results on biderivates. What we mean by direct applications is that they do not involve any of the other biderivate theorems obtained from these two theorems later in §§2.3 and 2.4.

First, in Theorem 2.2.1, we prove the differentiability of a function at almost all points of a set on which the function is bimonotone. Then with the help of this theorem from the First Biderivate Theorem we deduce a simpler version of the classical Denjoy-Young-Saks theorem [S, p. 271] in terms of biderivates (see Theorem 2.2.2). Next, in Theorem 2.2.4, from the Second Biderivate Theorem we deduce a theorem of Young on the set of points where a function has an infinite derivate. Also, from this theorem we deduce a theorem due to Kronrod. Finally, in Theorem 2.2.7 from the Second Biderivate Theorem we deduce an extended version of a theorem of [G1] which deals with the identity of a function with a Lipschitz function except for a set of arbitrarily small measure.

First with the help of Lemma 2.1.1 we obtain the following derivability theorem which deals with the differentiability of a function under bimonotonicity and BMT properties on a set. By Remark 1.4.1, this theorem is indeed a weaker version of a theorem of Lusin and Denjoy on VBG_* functions [S, p. 230].

Theorem 2.2.1. (**Derivability**). *Let $f : X \to \mathbb{R}$ and $E \subset X$. If f is bimonotone on E, then it is differentiable at almost all points of E. Consequently, the same holds if f is BMT on E.*

Proof. First, suppose f is WBI on E. There is clearly no loss of generality in assuming E to be bounded for in the general case E is a countable union of such sets. Set $F = \overline{E}$, $[a, b] = \mathrm{co}(F)$, $I = (a, b)$, and let $\{I_n = (a_n, b_n) : n = 1, 2, \dots\}$ be the sequence of contiguous intervals of F in I. Now, given n, if $a_n \notin X$, then a_n is clearly a left limit point of E, and so it follows from Lemma 2.1.1 that f is left regulated at a_n. Also, when $b_n \notin X$, it follows similarly from that lemma that f is right regulated at b_n. Hence we define $c_n = f(a_n)$ or $f(a_n - 0)$ according to whether a_n is in X or not, and, similarly, $d_n = f(b_n)$ or $f(b_n + 0)$ according to whether b_n is in X or not. Further, since f is WBI on E, by Lemma 2.1.1 for each n, $c_n \le f(x) \le d_n$ for $x \in X \cap (a_n, b_n)$.

We now define two functions g and h on $X \cap I$. Define $g(x) = h(x) = f(x)$ for $x \in X \cap I \cap F$, and $g(x) = c_n$ and $h(x) = d_n$ for $x \in X \cap (a_n, b_n)$, $n = 1, 2, \dots$. Then clearly g and h are two nondecreasing functions on $X \cap I$ such that $g(x) \le f(x) \le h(x)$ for each $x \in X \cap I$. Hence by Lebesgue's derivability theorem there is a subset A of E with no isolated points such that $|E \setminus A| = 0$ and g and h are both differentiable at the points of A. Now let $x \in A$. Since x is a limit point of E and $g = h$ on E,clearly $g'(x) = h'(x)$. Further, since $g \le f \le h$ on $X \cap I$, it is easy to see that f also is derivable at x and $f'(x) = g'(x)$. Hence f is differentiable a.e. in E.

Next, when f is WBD on E, the result follows on applying the above result to $-f$. Now suppose f is BMT on E. Then f is either LBL or UBL on E. Hence there is a real number c such that the function f_{+c} is either WBI or WBD on E. Hence from the above f_{+c} is differentiable a.e. in E, and consequently the same hold for f. □

Now, with the help of the above theorem from the First Biderivate Theorem we deduce the following version of Denjoy-Young-Saks theorem in terms of biderivates.

Theorem 2.2.2. (**Denjoy-Young-Saks**). *Every function $f : X \to \mathbb{R}$ is differentiable at almost all the points where it is not knotted.*

Proof. Let E denote the set of points in X where f is not knotted. Then according to Corollary 2.1.6 of the First Biderivate Theorem f is $GBMT$ on E. Hence there is a sequence $\{E_n\}$ such that $E = \cup_n E_n$ and f is BMT on each E_n. By Theorem 2.2.1 f is differentiable a.e. in each E_n, and so f is differentiable a.e. in E. □

From the above theorem we obtain directly the following extended version of a theorem of Banach [B].

Corollary 2.2.3. (**Banach**). *For every function $f : X \to \mathbb{R}$ the set where f has an infinite derivative is of measure zero.*

Next from the Second Biderivate Theorem we deduce the following result of Young [Y1] who obtained it for $X = I$. Note that a function has an infinite unilateral derivate at a point iff one of its biderivates is infinite there.

Theorem 2.2.4. (**Young**). *For every function $f : X \to \mathbb{R}$, the set where at least one derivate of f is infinite is a G_δ-set in X.*

Proof. Let E denote the set where at least one derivate of f is infinite, and set $A = X \setminus E$. Since both the biderivates of f are finite n.e. in A, by the Second Biderivate Theorem 2.1.9 there is a sequence of sets $\{A_n\}$ such that $A = \cup_n A_n$ and f is BL on each A_n. Further, for each n, by the Second Extension Lemma 2.1.8 f is BL on $B_n = X \cap \overline{A}_n \cap \text{co}(A_n)$. Then for each n, since $B_n \subset \text{co}(A_n)$ and f is BL on B_n, f does not have an infinite biderivate at any point of B_n. Hence $B_n \subset A$ for each n, and so $A = \cup_n B_n$. Now since each B_n is clearly an F_σ-set in X, so is A. Thus, E is a G_δ-set in X. □

From the above theorem we now deduce the following extended version of a theorem of Kronrod [Kr] with a simpler proof.

Theorem 2.2.5. (**Kronrod**). *Let $f : X \to \mathbb{R}$, and C be the set where f is continuous. If C is not an F_σ-set in X, then there is an uncountable set A in C where f is not differentiable. Further, if X is a Borel set in $\mathbb{R}$, then A has cardinality $\mathfrak{c}$.*

Proof. Let E denote the set where at least one derivate of f is infinite. Then by the above theorem E is a G_δ-set in X. Set $A = E \cap C$. Since $A \subset C \setminus \Delta^*(f)$, it suffices to prove the result for A. Further, since $X \setminus E \subset C$, we have $C = (C \cap E) \cup (X \setminus E) = A \cup (X \setminus E)$. Hence if A is countable, then it would follow that C is an F_σ-set in X, which is contrary to the hypothesis. Consequently, A is uncountable.

Next, assume X is a Borel set in $\mathbb{R}$. Since C is a G_δ-set in X, clearly A is a Borel set in $\mathbb{R}$, and hence by a theorem of Souslin [K, p. 479] A has cardinality $\mathfrak{c}$. □

To obtain the next theorem, we first prove the following lemma which seems to have some significance of its own.

Lemma 2.2.6. *Let $f : X \to \mathbb{R}$, and suppose E and F are two subsets of X which are closed in $\mathbb{R}$. If f is LBL on E and LL on F, then f is LL on $E \cup F$. Thus, if f is BL on E and Lipschitz on F, then it is Lipschitz on $E \cup F$.*

Proof. It is clearly enough to prove the first part, for the second part follows on applying the first part to f and $-f$. Let a and b denote the inf and sup of E, respectively. Similarly, let a' and b' denote the inf and sup of F, respectively. Further, if $a' < a$, let $c = \sup\{F \cap [a', a]\}$; otherwise let $c = a$. Similarly, if $b < b'$, let $d = \inf\{F \cap [b, b']\}$; otherwise let $d = b$. Then $c \le a \le b \le d$. Hence it is easy to see from the hypothesis that we can choose $k > 0$ such that the following four conditions hold for the function $g \equiv f_{+k}$: (i) g is biincreasing on E, (ii) g is increasing on F, (iii) $g(c) \le g(a)$, and (iv) $g(b) \le g(d)$. Now it is enough to show that g is nondecreasing on the set $A \equiv E \cup F$, for this will prove that f is LL on A. Hence, given $x, y \in A$, $x < y$, we need to show that

$$g(x) \le g(y). \tag{4}$$

When x and y are both in E, or both in F, (4) follows clearly from (i) or (ii), respectively. Hence, first suppose that $x \in E$ and $y \in F$. If $y \le b$, then since $b \in E$, (4) follows clearly from (i). Otherwise we have $b' \ge y > b$, and so $y \ge d$. Further, since F is closed in $\mathbb{R}$, clearly $d \in F$. Hence by (i), (iv) and (ii) $g(x) \le g(b) \le g(d) \le g(y)$, which proves (4). A similar argument holds when $x \in F$ and $y \in E$. □

Finally, with the help of the above lemma we deduce from the Second Biderivate Theorem the following extended version of Theorem 2.20 of [G1, p. 308] where X was assumed to be measurable. See also Rjazanov [R] for a partial result in this direction in the case of functions of bounded variation. It is interesting to note that the following proof is totally different from the one given in [G1, pp. 308-309].

Theorem 2.2.7. *Let $f : X \to \mathbb{R}$, where $|X| < \infty$. If f is knotted only at a set of points of measure zero, then for every $\varepsilon > 0$ there is a Lipschitz function g on $\mathbb{R}$ such that*

$$|\{x \in X : f(x) \ne g(x)\}| < \varepsilon. \tag{5}$$

Proof. Let A denote the set of points in X where f is differentiable. Then by the hypothesis and Theorem 2.2.2 $|X \setminus A| = 0$. Further by the Second Biderivate Theorem 2.1.9 there is a sequence $\{A_n\}$ such that $A = \cup_n A_n$ and f is BL on each A_n. Now, for each n, by the Second Extension Lemma 2.1.8 f is also BL on $\overline{A}_n \cap X \cap \mathrm{co}(A_n)$. But since $\mathrm{co}(A_n)$ is clearly a countable union of bounded closed sets in $\mathbb{R}$, we can thus assume the sets in the sequence $\{A_n\}$

to be bounded and closed in X. For each n let F_n denote the closure of A_n in $\mathbb{R}$, and set $B_n = X \cap F_n$. Then by Lemma 2.1.8, f is BL on each B_n.

Next, let m_X^* denote the outer measure obtained on X by restricting Lebesgue outer measure m^* (on $\mathbb{R}$) to the class of all subsets of X, and let m_X denote the measure induced by m_X^* in the usual manner on X. Now, given $\varepsilon > 0$, since the sets B_n are m_X^*-measurable (see [G3, p. 18, Theorem 1.4.3]), and $m_X(\cup_n B_n) = m_X(A) < \infty$, there exists an integer n such that $m_X(X \setminus \cup_{i \le n} B_i) < \varepsilon$. Set $B = \cup_{i \le n} B_i$. Then $|X \setminus B| < \varepsilon$. Now, let E_- and E_+ denote the sets of points x in $\mathbb{R} \setminus B$ which are left or right limit points, respectively, of B. Since f is BL on each B_i, clearly the limits $f(x-0)$ and $f(x+0)$ exist at the points of E_- and E_+, respectively, and are finite. Set $F = B \cup E_- \cup E_+$, and for each $x \in F$, define $g(x) = f(x)$, $f(x-0)$ or $f(x+0)$ according to whether $x \in B$, E_- or $E_+ \setminus E_-$, respectively. Next, for each $i \le n$, since f is BL on B_i, clearly g is also BL on B_i, and hence it is BL on F_i by Lemma 2.1.8. Now by Lemma 2.2.6 using induction g is Lipschitz on the compact set $K = \cup_{i \le n} F_i$.

Next, let g be defined linearly on the bounded closed contiguous intervals of K, and let g be constant on the two unbounded closed contiguous intervals of K. Then since g is Lipschitz on K, obviously g is also Lipschitz on $\mathbb{R}$. Now since $g = f$ on B and $|X \setminus B| < \varepsilon$, clearly (5) holds for g. □

§2.3. Properties on Some Portion in Terms of Biderivates

In this section from the first two biderivate theorems of §2.1 we deduce the Third and Fourth Biderivate Theorems, which deal with the bimonotonicity and bi-Lipschitz properties of a function on a portion of a given set. These two theorems are also found to play a basic role in the development of the present unified theory. We need here the following lemma.

Lemma 2.3.1. *Suppose $f : X \to \mathbb{R}$ is $CWUC$. If f is WBI on a dense subset A of X, then it is nondecreasing on $X \cap \text{co}(A)$.*

PROOF. Suppose the hypothesis holds, and let $B = X \cap \text{co}(A)$. Then, given $a, b \in B$, $a < b$, we show that $f(a) \le f(b)$. Clearly, there exist two points $a', b' \in A$ such that $a' \le a$ and $b \le b'$. If there is a $c \in A \cap [a, b]$, then since $a' \le a \le c \le b \le b'$ and f is WBI on A, it is clear that $f(a) \le f(c) \le f(b)$. Hence, suppose $A \cap [a, b] = \emptyset$. Then a and b are two contiguous points of X, and so by the hypothesis f is WUC at a or b. First, suppose f is WUC at a. Since A is dense in X, clearly a is a left limit point of A, and so by Lemma 2.1.1 f is left regulated at a. Now since f is WUC at a, $f(a) \le f(a-0)$. Also, since $a < b \le b'$, where $b' \in A$, by the same lemma $f(a) \le f(a-0) \le f(b)$. A similar argument holds when f is WUC at b. □

Theorem 2.3.2. (**Third Biderivate**). *Let $f : X \to \mathbb{R}$ and $E \subset X$. If f is PWUC on E and $\underline{D}f > 0$ at a nonmeager set of points in E, then f is SBI on some portion of E. Moreover, when $E = X$, the same holds provided f is CWUC.*

Proof. First, suppose f is *PWUC* on E and $A = \{x \in E : \underline{D}f(x) > 0\}$ is nonmeager in E. Then by the First Biderivate Theorem 2.1.4 there is a sequence $\{A_n\}$ such that $A = \cup_n A_n$ and f is *SBI* on each A_n. Hence clearly there is an n such that A_n is dense in some portion E_0 of E. Now by the First Extension Lemma 2.1.2 f is *SBI* on $E_0 \cap \mathrm{co}(A_n)$, which in turn contains some portion of E. When $E = X$ and f is *CWUC*, the result is obtained by a similar argument with the help of the above lemma. □

We include here some consequences of the above theorem dealing with the properties *LBL* and *BMT*.

Corollary 2.3.3. *Let $f : X \to \mathbb{R}$ and $E \subset X$. If f is PWUC on E and $\underline{D}f > -\infty$ at a nonmeager set of points in E, then f is LBL on some portion of E. Moreover, when $E = X$, the same holds provided f is CWUC.*

Proof. Clearly from the hypothesis there is a positive integer n such that $\underline{D}f > -n$ at a nonmeager set A in E. Then $\underline{D}f_{+n} > 0$ at each point of A, and so in each of the two cases by the above theorem the function f_{+n} is *SBI* on some portion E_0 of E. Consequently, f is *LBL* on E_0. □

Corollary 2.3.4. *Let $f : X \to \mathbb{R}$ and $E \subset X$. If f is PWC on E and there is a nonmeager set of points in E where f is not knotted, then f is BMT on some portion of E. Moreover, when $E = X$, the same holds provided f is CWC.*

Proof. Clearly from the hypothesis one of the inequalities (i) $\underline{D}f > -\infty$ and (ii) $\overline{D}f < \infty$ holds at a nonmeager set in E. Hence the result follows on applying Corollary 2.3.3 to one of the functions f or $-f$. □

Next we obtain two interesting results on the median of a function which follow directly from the Third Biderivate Theorem 2.3.2 and its last corollary. Note that a function f is knotted at x iff $Mf(x) = \overline{\mathbb{R}}$.

Theorem 2.3.5. (**First Median**). *Let $f : X \to \mathbb{R}$, $E \subset X$, and suppose f is PWC on E (or CWC when $E = X$).*

(a) *If f is nowhere strongly bimonotone on E, then $0 \in MF$* r.e. *in E.*

(b) *If f is nowhere BMT on E, then it is knotted* r.e. *in E.*

Next from the Second Biderivate Theorem we deduce a result on the bi-Lipschitz property on a portion.

Theorem 2.3.6. (**Fourth Biderivate**). *Let $f : X \to \mathbb{R}$ and $E \subset X$. If there is a nonmeager set in E where $\underline{D}f$ and $\overline{D}f$ are finite, then f is BL on a portion of E. Moreover, given $c > 0$, if there is a nonmeager set in E where $\underline{D}f > -c$ and $\overline{D}f < c$, then f is $BL(c)$ on a portion of E.*

Proof. It is enough to prove the second part, for if $\underline{D}f$ and $\overline{D}f$ are finite at a nonmeager set of points in E, then there clearly exists a positive integer c for which the hypothesis of the second part holds. Hence suppose there is some $c > 0$ such that the set

$$A = \{x \in E : \underline{D}f(x) > -c, \overline{D}f(x) < c\}$$

is nonmeager in E. Then by the Second Biderivate Theorem 2.1.9 there is a sequence $\{A_n\}$ such that $A = \cup_n A_n$ and f is $BL(c)$ on each A_n. Now clearly there is an n such that A_n is dense in some portion E_0 of E. By Lemma 2.1.8 f is $BL(c)$ on $E_0 \cap \mathrm{co}(A_n)$, which in turn contains a portion of E. □

It was proved by Fort [Fo] that if $f : I \to \mathbb{R}$ is discontinuous at a dense set, then it is differentiable only at a meager set. Since the bi-Lipschitz property clearly implies continuity (except at the extreme points), the following consequence of the above theorem provides a considerable strengthening of the Fort's result and also of Corollary 2.14 of [G1] where this result was obtained for $E = X$.

Corollary 2.3.7. *Let $f : X \to \mathbb{R}$ and $E \subset X$. If f is nowhere BL on E, then there is a residual set in E where at least one derivate of f is infinite.*

§2.4. Baire Class of Biderivates and Median

In this section we deal with the Baire class of biderivates and median. First, we define various lower and upper Baire classes of functions and multifunctions (i.e., set-valued functions). Then from the First Biderivate Theorem we obtain the Baire class of biderivates and medians. The following lower and upper Baire classes of functions and multifunctions were defined in [G3, §§1.4, 1.5] in a more general setting. We present them here in a setting that is needed in the present work.

We first deal with functions. Given $f : X \to \overline{\mathbb{R}}$, where $X \subset \mathbb{R}$, and a countable ordinal α, f is said to be of *lower* or *upper Baire class* α if for every $c \in \mathbb{R}$,$\{x \in X : f(x) > c\}$ or $\{x \in X : f(x) < c\}$, is respectively of additive class α in X.

We will use B_α, LB_α and UB_α to denote the classes of all extended real valued functions of Baire class α, lower Baire class α and upper Baire class α, respectively, relative to their domain. Then obviously for each $\alpha < \Omega$,

$$B_\alpha = LB_\alpha \cap UB_\alpha \quad \text{and} \quad LB_\alpha \cup UB_\alpha \subset B_{\alpha+1}. \tag{6}$$

Further, B_0, LB_0 and UB_0 are indeed the classes of continuous, LSC and USC functions, respectively.

Next, we deal with multifunctions. We will use $2^{\overline{\mathbb{R}}}$ to denote the space of all "nonempty" closed subsets of $\overline{\mathbb{R}}$ with the *Vietoris* (or exponential) *topology* [K, p. 160]; namely, the coarsest topology on $2^{\overline{\mathbb{R}}}$ for which the sets

$$\{F \in 2^{\overline{\mathbb{R}}} : F \cap U \neq \emptyset\} \text{ and } \{F \in 2^{\overline{\mathbb{R}}} : F \subset U\}$$

are open for every open set U in $\overline{\mathbb{R}}$. Now, let $\varphi : X \to 2^{\overline{\mathbb{R}}}$ be any multifunction. The continuity and the Baire class of φ are defined as usual in terms of the Vietoris topology of $2^{\overline{\mathbb{R}}}$. Further, given any countable ordinal α, φ will be said to be of *lower* or *upper Baire class* α if for every open set U in $\overline{\mathbb{R}}$

$$\{x \in X : \varphi(x) \cap U \neq \emptyset\} \text{ or } \{x \in X : \varphi(x) \subset U\},$$

respectively, is of additive class α in X.

We will use the notation B_α, LB_α and UB_α to denote the Baire class α and lower and upper Baire classes α of multifunctions as well. Also, when φ is of LB_0 or UB_0, φ is called as before LSC or USC, respectively. These two semicontinuities of multifunctions are also defined locally; see [K, p. 173] for their definitions. Then the relations in (6) hold also in the case of multifunctions. Note that when $\varphi(x)$ is singleton for each $x \in X$, then for each $\alpha < \Omega$, φ is of B_α, LB_α or UB_α as a multifunction iff it is of B_α as a function.

We now deal with the Baire class of biderivates and median. We begin with the following theorem on biderivates, which we call the Fifth Biderivate Theorem due to its basic nature.

Theorem 2.4.1. (**Fifth Biderivate**). *Let $f : X \to \mathbb{R}$. Then for each $c \in \overline{\mathbb{R}}$ $\{x \in X : \underline{D}f(x) > c\}$ is of the form $F_\sigma \setminus C$ in X, where C is a countable set where f is not WUC. Consequently, $\underline{D}f$ is always of B_2; and in case f is WUC, then $\underline{D}f$ is of LB_1, and so then $\underline{D}f$ is LSC* r.e. *in X'.*

Proof. First, to prove the first part, let C denote the countable set where f is not WUC (see Theorem 1.2.1), and for each $c \in \overline{\mathbb{R}}$, set

$$H_c = \{x \in X : \underline{D}f(x) > c\}.$$

First consider $c = 0$. Then by the First Biderivate Theorem 2.1.4 there is a sequence $\{A_n\}$ such that $H_0 = \cup_n A_n$ and f is SBI on each A_n. Further, for each n, by the First Extension Lemma 2.1.2 there is a set $C_n \subset C \setminus A_n$ such that f is SBI on

$$B_n = X \cap \mathrm{co}(A_n) \cap (\overline{A}_n \setminus C_n).$$

Now set $E_n = B_n \cap X'$. Then obviously $A_n \subset E_n \subset H_0$ and there is an F_σ-set P_n in X such that $E_n = P_n \setminus C_n$. Consequently, $H_0 = \cup_n E_n = \cup_n P_n \setminus C_0$, where $\cup_n P_n$ is an F_σ-set in X and $C_0 \subset C$. This proves the result for $c = 0$.

Now, for any other $c \in \mathbb{R}$, the result follows on applying the above result to f_{-c}. This in turn yields the result for $c = -\infty$ as well, for $H_{-\infty} = \cup_n H_{-n}$. If $c = \infty$, the result is on the other hand obvious, for $H_\infty = \emptyset$. This completes the proof of the first part.

The second part is obtained directly from the first, whereas the LSC of $\underline{D}f$ is obtained with the help of Theorem 1.4.1 of [G3, p. 17]. □

The part of the above theorem dealing with B_2 was obtained earlier by Hájek [Ha] when X is an interval.

On applying the above theorem to $-f$, a similar result is obtained on the Baire class of $\overline{D}f$. Further from the above theorem we obtain the following result on the knot set and on any single median derivate of a function.

Corollary 2.4.2. *Let $f : X \to \mathbb{R}$. Then each of the sets $K(f)$ and $\{x \in X : c \in Mf(x)\}$, $c \in \overline{\mathbb{R}}$, is of the form $G_\delta \cup C$ in X, where C is a countable set where f is not WC. Consequently, if f is WC, then each of these sets is a G_δ-set in X.*

Proof. Since $K(f) = \{x \in X : \underline{D}f(x) = -\infty\} \cap \{x \in X : \overline{D}f(x) = +\infty\}$, the result on $K(f)$ follows clearly by applying the above theorem to f and $-f$; and a similar argument holds in the other case since for each $c \in \overline{\mathbb{R}}$,

$$\{x \in X : c \in Mf(x)\} = \{x \in X : \underline{D}f(x) \leq c\} \cap \{x \in X : \overline{D}f(x) \geq c\}.$$

The second part is of course a direct consequence of the first. □

Further, with the help of Theorem 1.5.1 and Corollary 1.5.2 of [G3, pp. 20, 21], we obtain from the above theorem the following median theorem on the Baire class of median.

Theorem 2.4.3. (**Second Median**). *Let $f : X \to \mathbb{R}$. The median Mf is always of B_2. Further, if f is WC, then Mf is of UB_1, and so then Mf is USC* r.e. *in X'.*

Chapter 3: Other Results on Biderivates, Median and Derivative

In this chapter we deal with various other results on biderivates, medians and derivatives. First, in §3.1 from the Third Biderivate Theorem we deduce some results on the properties of biderivates and medians at residual sets, including the Third Median Theorem. Also, we obtain biderivate versions of two theorems of Denjoy and Choquet. Next, in §3.2 from the Third Biderivate Theorem we deduce the Fourth and Fifth Median Theorems which deal with the properties of medians at nonmonotonicity points. Next, in §3.3 from the Fourth Median Theorem we deduce several monotonicity theorems in terms of biderivates, including the well known Goldowski-Tonelli theorem. Also from one of these theorems we deduce a result on median derivates similar to a theorem of Morse. Finally, in §3.4, we deal with various properties of derivatives, medians and derivability sets. First from the Fifth Biderivate Theorem we obtain a theorem of Zahorski on a property of derivability sets and the Baire class of derivatives. Then from the First Median Theorem we deduce a mean-value theorem in terms of medians and derivatives, and the Darboux property of medians and derivatives. Further from the Third Biderivate Theorem we obtain the Denjoy property of derivatives and an extended version of a theorem of Marcus on stationary sets of derivatives.

§3.1. Properties of Biderivates and Median at Residual Sets of Points

In this section from the Third Biderivate Theorem we deduce some theorems on the properties of biderivates and medians at residual sets. One of these theorems is analogous to a fundamental theorem of Denjoy on unilateral derivates and another theorem deals with a version of a theorem of Choquet on the identity of biderivates with strong derivates at a residual set.

Theorem 3.1.1. *Let $f : X \to \mathbb{R}$, $E \subset X$ and $c \in \overline{\mathbb{R}}$. If $\underline{D}f \leq c$ at a dense set in E where f is PWUC with respect to E, then $\underline{D}f \leq c$* r.e. *in E. Moreover, if f is CWUC and $\underline{D}f \leq c$ at a dense set in X, then the same holds* r.e. *in X.*

Proof. First, suppose there is a dense set A in E where $\underline{D}f \leq c$ and f is $PWUC$ with respect to E. Note first that $E \subset X'$; for since $\underline{D}f$ is defined at the points of A, $A \subset X'$, and since A is dense in E, it follows that $E \subset X'$. Further, it is enough to prove the result for $c = 0$. For, on applying this result to the function f_{-c}, the result is obtained for each $c \in \mathbb{R}$; and in the case when $c = -\infty$, then for each positive integer n the hypothesis holds also for $c = -n$, so that $\underline{D}f \leq -n$ r.e. in E, and this implies that $\underline{D}f = -\infty$ r.e. in E; and when $c = +\infty$, the result holds trivially since $E \subset X'$.

Hence let $c = 0$, and set $B = \{x \in E : \underline{D}f(x) \leq 0\}$. Suppose, if possible, that $E \setminus B$ is nonmeager in E. Then there is a portion E_0 of E such that $E_0 \setminus B$ is nonmeager in each portion of E_0. Now since B includes A, B is dense in E_0, and hence E_0 is dense in itself. Let H denote the set in E_0 where f is $PWUC$ with respect to E. Then the countable set $E_0 \setminus H$ is meager in E_0, and so $H \setminus B$ is nonmeager in H. Further, since A is dense in E and $A \subset H$, the function f is $PWUC$ on H. Consequently, it follows from the Third Biderivate Theorem 2.3.2 that f is SBI on some portion H_0 of H. Now since H is clearly dense in itself, and A is dense in H, there exist two points, $a, b \in H_0$ and a point $c \in A$ such that $a < c < b$. But since f is SBI on H_0, this implies that $\underline{D}f(c) > 0$, which contradicts the fact that $c \in A$.

The other part is obtained by a similar argument from the second part of Theorem 2.3.2 with $E = X$. □

On taking into account the definitions of $PWUC$ and $CWUC$, it is interesting to note here that the first part of the above theorem holds without any continuity hypothesis provided the dense set in E is included in E^b, and the same holds for its second part provided X does not have any contiguous points. The second part has been obtained by Filipczak [F] in the case when X is an interval.

Now, on combining the above theorem with the result obtained on applying it to $-f$, we obtain the following theorem on median.

Theorem 3.1.2. (**Third Median**). *Let $f : X \to \mathbb{R}$, $E \subset X$ and $c \in \overline{\mathbb{R}}$. If $c \in Mf$ (or f is knotted) at a dense set in E where f is PWC with respect to E, then $c \in Mf$ (or f is knotted)* r.e. *in E. Moreover, if f is CWC and either $c \in Mf$ or f is knotted at a dense set in X, then the same holds* r.e. *in X.*

Further, since f is clearly UC (or LC) at each point where $\overline{D}f < \infty$ (or $\underline{D}f > -\infty$), we obtain the following from the above two theorems.

Corollary 3.1.3. *Let $f : X \to \mathbb{R}$, $E \subset X$ and $c \in \mathbb{R}$. If there is a dense set in E where $\overline{D}f \leq c$ (or $f' = c$), then $\underline{D}f \leq c$ (or $c \in Mf$)* r.e. *in E.*

Next from the Third Biderivate Theorem we deduce the following result on biderivates which is similar to a fundamental theorem of Denjoy on unilateral derivates of continuous functions. (See [D1, p. 149] and [D3, p. 187], or see Theorem 2.5.1 of [G3, p. 32] for a strengthened version of Denjoy's theorem.) Recall that we use $Qf(a,b)$ to denote $\{f(b) - f(a)\}/(b - a)$ when $b \neq a$.

Theorem 3.1.4. *Let $f : X \to \mathbb{R}$, $c \in \overline{\mathbb{R}}$, and suppose $E \subset X$ is dense in itself. Suppose $\{a_n\}$ is dense in E, and $\{b_n\}$ is another sequence in X such that $b_n \neq a_n$ for each n, $b_n - a_n \to 0$ as $n \to \infty$, and that f is $PWUC$ with respect to E at each a_n (or f is $CWUC$ when $E = X$). If $\limsup_{n\to\infty} Qf(a_n, b_n) \leq c$, then $\underline{D}f \leq c$* r.e. *in E.*

Proof. We give the proof when f is $PWUC$ with respect to E at each a_n. A similar and more direct argument holds in the other case. Further, as in the case of Theorem 3.1.1, it suffices to prove the result for $c = 0$. Hence suppose the hypothesis holds for $c = 0$, but $A = \{x \in E : \underline{D}f(x) > 0\}$ is nonmeager in E. Then there is a portion E_0 of E such that $A \cap E_0$ is nonmeager in each portion of E_0. Now let H be the set in E_0 where f is $PWUC$ with respect to E. Since the countable set $E_0 \setminus H$ is clearly meager in E_0, the set $A \cap H$ is nonmeager in H. Now by the Third Biderivate Theorem 2.3.2 f is SBI on a portion H_0 of H. Hence there is a $d > 0$ such that f_{-d} is WBI on H_0. Since H_0 is clearly dense in itself, there exists an open interval $I \equiv (a, b)$ such that $a, b \in H_0$ and $I \cap H_0 \neq \emptyset$. Now it is clear from the hypothesis that there is an n such that $a_n \in I \cap H_0$, $b_n \in I$ and $Qf(a_n, b_n) < d$. But then $Qf_{-d}(a_n, b_n) < 0$, which contradicts the fact that f_{-d} is WBI on H_0. □

Finally from the Third Biderivate Theorem we deduce the following theorem on the identity of the medians Mf and M_*f at a residual set. When X is an interval, the following theorem holds clearly without any continuity hypothesis. In this particular case the identity of Mf and M_*f at a residual set was obtained by Choquet [C1; C2] in 1947 in terms of the identity of contingent and paratingent of the function at a residual set.

Theorem 3.1.5. *Let $f : X \to \mathbb{R}$, where X is dense in itself. If f is $CWUC$, then there is a residual set in X where $\underline{D}f = \underline{D}_*f$, and where $\underline{D}f$ is LSC. Consequently, if f is CWC, then there is a residual set in X where $Mf = M_*f$ (or $\underline{D}f = \underline{D}_*f$ and $\overline{D}f = \overline{D}_*f$), and where Mf is USC.*

Proof. First, suppose f is $CWUC$ and that $\{x \in X : \underline{D}f(x) > \underline{D}_*f(x)\}$ is nonmeager in X. Then clearly there is a rational number r such that

$$A = \{x \in X : \underline{D}f(x) > r > \underline{D}_*(f)\}$$

is nonmeager in X. Hence there is a portion X_0 of X such that A is nonmeager in each portion of X_0. Now by the Third Biderivate Theorem 2.3.2 f_{-r} is increasing on some portion X_1 of X_0. Choose $a, b \in X_1$ such that $X \cap (a, b) \neq \emptyset$. Then at each point of $X \cap (a, b)$ we have $\underline{D}_*f_{-r} \geq 0$, or $\underline{D}_*f \geq r$, which contradicts the fact that A is nonmeager in $X \cap (a, b)$. Consequently, there is a residual set B in X where $\underline{D}f = \underline{D}_*f$. Now since $\underline{D}_*f$ is LSC (see Theorem

14.1.1 of [G3, p. 317]), and $\underline{D}f \geq \underline{D}_* f$ everywhere, clearly $\underline{D}f$ is LSC at the points of B.

The second part follows on the other hand from the first on combining it with the result obtained by applying it to $-f$, with the help of Theorem 1.5.1 of [G3, p. 20]. □

§3.2. Properties of Median at Nonmonotonicity Points

In this section from the Third Biderivate Theorem we further deduce some median theorems which deal with the properties of the median of a function at its nonmonotonicity points. We include here some direct consequences of these median theorems, but in the next section we deduce several monotonicity theorems from one of these median theorems.

Following our convention, it will be understood throughout this section that $f : I \to \mathbb{R}$, where I is some subinterval of $\mathbb{R}$. Recall that we use $N(f)$ to denote the set of points x in I such that f is not nondecreasing in any neighborhood of x. The set $N(f)$ is called the "nonmonotonicity set" of f, and its points are called the "nonmonotonicity points" of f. These notions were introduced in [G1] and [G3, p. 191]. We further need some nomenclature that was introduced in [G1].

We call f *lower* or *upper absolutely continuous* (or LAC or UAC, respectively) if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each finite family $\{[a_i, b_i] : i = 1, \dots, n\}$ of nonoverlapping intervals with endpoints in I, if $\sum_{i \leq n}(b_i - a_i) < \delta$, then $\sum_{i=1}^{n}\{f(b_i) - f(a_i)\} > -\varepsilon$ or $< \varepsilon$, respectively. Clearly, f is AC iff it is LAC and UAC. Further, it is easy to see that if f is lower Lipschitz, then it is LAC.

Further, we call f *lower* or *upper singular* if its derivative is ≥ 0 or ≤ 0, respectively, at almost all points where it exists. Also, f is called *singular* if its derivative is zero at almost all points where it exists.

Note that every nonderivable function is vacuously singular. Further, it is easy to see from the Denjoy-Young-Saks Theorem 2.2.2 that f is lower singular (or singular) iff $\overline{D}f \geq 0$ (or $0 \in Mf$) a.e.

We now state the following monotonicity theorem in terms of the above notions which was obtained in [G1, p. 310] with the help of the Vitali covering theorem. This theorem seems to be fundamental since it is used repeatedly in the present section.

Theorem 3.2.1. (**First Monotonicity**). *A function f is nondecreasing iff it is LAC and lower singular.*

The following consequence of this theorem is a well known theorem of Lebesgue.

Corollary 3.2.2. (**Lebesgue**). *If a function f is AC and singular, then it is constant.*

We will need the following two lemmas.

Lemma 3.2.3. *Suppose f is $SWLC$, and let $N = N(f)$. Then*

(a) *N is perfect in I;*

(b) *if f is bimonotone on N^b, then it is LC at the points of $N \setminus N^b$; and*

(c) *if f is WBI on N^b, then it is nondecreasing on I.*

PROOF. suppose $N \neq \emptyset$, for there is nothing to prove otherwise. To prove (a), note first that N is closed in I. Suppose, if possible, that N has some isolated point x. Then x is a common endpoint of two contiguous intervals (a, x) and (x, b) of N, on each of which f is nondecreasing. Thus f is regulated from both sides at x, and since it is $SWLC$, it follows that f is LC at x. Consequently, f is nondecreasing on (a, b); i.e., $x \notin N$, which is a contradiction.

Next, to prove (b), first suppose f is WBI on N^b, and let $x \in N \setminus N^b$. Since N is perfect in I, x is a limit point of N^b from one side, say from the left. Then there is an increasing sequence $\{a_n\}$ in N^b which converges to x. Then $\{f(a_n)\}$ is clearly nondecreasing, and for each n, $f(a_n) \leq f(t) \leq f(a_{n+1})$ for $a_n < t < a_{n+1}$. Hence f is left regulated at x. Further, since x is not a right limit point of N, either x is the right endpoint of I, or it is the left endpoint of some open subinterval J of I which is disjoint from N, so that f is nondecreasing on J. Hence in any case f is also right regulated at x. Now since f is $SWLC$, f is LC at x. A similar argument holds when x is a right limit point of N^b. Also, when f is WBD on N^b, the result is obtained by a similar argument.

Next, to prove (c), suppose f is WBI on N^b. Then, by (b), f is LC at the points of $N \setminus N^b$. So f is nondecreasing on the closure of each contiguous interval of N in I. Now clearly f is nondecreasing on N, and hence the same holds on I. ☐

Lemma 3.2.4. *Let f be lower singular. Then for each $c > 0$, $N(f_{+c}) = N(f)$.*

PROOF. Set $N = N(f)$, and let $c > 0$. Since f_{+c} is clearly nondecreasing on each interval on which f is, $N(f_{+c}) \subset N$. To prove the reverse inclusion, suppose there is an $x \in N \setminus N(f_{+c})$. Then there is an open interval J containing x such that f_{+c} is nondecreasing on J. Thus f is LL on J. So by Theorem 3.2.1 f is nondecreasing on J; i.e., $x \notin N$, which is a contradiction. ☐

With the help of the above two lemmas, we now deduce from the Third Biderivate Theorem the following median theorem.

Theorem 3.2.5. (**Fourth Median**). *Suppose f is $SWLC$. Then*

(a) $\underline{D}f \leq 0$ r.e. *in* $N(f)$;

(b) *if f is nowhere strongly decreasing, then* $0 \in Mf$ r.e. *in* $N(f)$; *and*

(c) *if f is lower singular, then* $[-\infty, 0] \subset Mf$ r.e. *in* $N(f)$.

PROOF. Suppose as before that $N \equiv N(f)$ is nonempty, for there is nothing to prove otherwise. To prove (a), suppose $A = \{x \in N : \underline{D}f(x) > 0\}$ is nonmeager in N. Set $E = N^b$. Now since, by Lemma 3.2.3(a), N is dense in itself, the countable set $N \setminus E$ is clearly meager in N. Hence $A \cap E$ is nonmeager in N. Further, since f is vacuously $PWUC$ on E, by the Third Biderivate Theorem 2.3.2 f is biincreasing on some portion of E. Hence there is an open subinterval J of I such that $E \cap J \neq \emptyset$ and f is biincreasing on $E \cap J$. Now by part (c) of Lemma 3.2.3 f is nondecreasing on J; i.e., $N \cap J = \emptyset$. This contradiction proves that $\underline{D}f \leq 0$ r.e. in N.

Next, to prove (b), suppose f is nowhere strongly decreasing. Now by (a) it suffices to prove that $B = \{x \in N : \overline{D}f(x) < 0\}$ is meager in N. Suppose B is nonmeager in N. Then there is an open subinterval J of I such that $B \cap J$ is nonmeager in each portion of $N \cap J$. We claim that N is nowhere dense in J. For, if N includes some open subinterval J_0 of J, then since $B \cap J_0$ is nonmeager in J_0, and f is vacuously $PWLC$ on J_0, by Theorem 2.3.2 f is strongly decreasing on a subinterval J_1 of J_0, which contradicts the hypothesis.

Further, we claim that f is $PWLC$ on N. For, since f is obviously left (or right) regulated at the points of N which are not left (or right) limit points of N, the claim follows from the fact that f is $SWLC$ on I. Hence it follows from Theorem 2.3.2 that f is SBD on some portion N_0 of $N \cap J$. Let (a, b) be any contiguous interval of N_0 whose endpoints are in N_0. Then since f is by Lemma 3.2.3(b), LC at a and b, f is indeed nondecreasing on $[a, b]$. Hence $f(a) \leq f(b)$, which contradicts the fact that f is SBD on N_0.

Next, to prove (c), suppose f is lower singular. If there is a real number $c > 0$ such that f_{+c} is decreasing on some open subinterval J of I, then f_{+c} is derivable a.e. in J, and so f has a derivative $\leq -c < 0$ a.e. in J, which contradicts the lower singularity of f. Consequently, f is nowhere strongly decreasing, and hence according to (b), $0 \in Mf$ r.e. in N. Further, for each positive integer n it follows from Lemma 3.2.4 and part (a) that the set $E_n = \{x \in N : \underline{D}f_{+n}(x) \leq 0\} = \{x \in N : \underline{D}f(x) \leq -n\}$ is residual in N. Now since the set $E = \cap_n E_n$ is again residual in N, it follows that $-\infty \in Mf$ r.e. in N. Consequently, $[-\infty, 0] \subset Mf$ r.e. in N. □

Next we obtain a result in the direction of part (a) of the above median theorem.

Theorem 3.2.6. *If f is SWLC, then $\underline{D}f < 0$ at a $\mathfrak{c}$-dense set in $N(f)$.*

Proof. Set $N = N(f)$ and $E = \{x \in N : \underline{D}f(x) < 0\}$, and suppose there is an open subinterval J of I such that $N \cap J \neq \emptyset$ but card $(E \cap J) < c$. Then f is clearly lower singular on J, and since by Lemma 3.2.3(a), N is perfect in I, it follows from part (c) of the above theorem that $N \cap J = \emptyset$. Thus f is nondecreasing on J, which is a contradiction. □

We next deduce from the Third Biderivate Theorem the following median theorem on lower singular functions without any continuity hypothesis.

Theorem 3.2.7. (**Fifth Median**). *Let f be lower singular. If f is nowhere nondecreasing, then $[-\infty, 0] \subset Mf$ r.e. in I.*

Proof. We can assume without loss of generality that I is open. Then f is vacuously PWC on I. Suppose f is nowhere nondecreasing, and set $A = \{x \in I : \overline{D}f(x) < 0\}$ and $B = \{x \in I : \underline{D}f(x) > -\infty\}$. Then it is enough to prove that A and B are meager in I. First, suppose A is nonmeager in I. Then by the Third Biderivate Theorem 2.3.2, f is SD on some open subinterval J of I. Hence there is a $c > 0$ such that f_{+c} is decreasing on J. Consequently, f is derivable a.e. in J and its derivative is $\leq -c < 0$ a.e. in J, which contradicts the lower singularity of f.

Next, suppose B is nonmeager in I. Then by Corollary 2.3.3, f is LL on some open subinterval J of I. But according to Theorem 3.2.1, f is then nondecreasing on J, which contradicts the hypothesis. □

Next are two interesting direct consequences of the above median theorem. What is particularly interesting about these results is that they hold without any continuity hypothesis.

Corollary 3.2.8. *If f is singular and nowhere monotone, then it is knotted* r.e. *in I, and thus f is nowhere MT.*

Corollary 3.2.9. *Suppose f is derivable at a nonmeager set in I. If f is lower singular (or singular), then it is nondecreasing (or constant) on some subinterval of I.*

In conclusion, it should be pointed out that earlier in §9.4 of [G3], median theorems have been obtained in terms of unilateral derivates which are similar to the fourth and fifth median theorems.

§3.3. Monotonicity in Terms of Biderivates and Goldowski-Tonelli Theorem

In this section from the Fourth Median Theorem of the last section we deduce some monotonicity theorems in terms of biderivates including the well known

Goldowski-Tonelli theorem on monotonicity. Also from one of these monotonicity theorems we deduce a result on median derivates which is similar to a theorem of Morse on unilateral derivates.

Following our convention, it will be understood again in this section that $f : I \to \mathbb{R}$, where I is a subinterval of $\mathbb{R}$. We begin with the following monotonicity lemma which seems to be fundamental. The first two monotonicity theorems of this section and the Goldowski-Tonelli theorem are direct consequences of this lemma.

Lemma 3.3.1. (**Monotonicity**). *A function f is nondecreasing iff it is $SWLC$ and lower singular, and the set where $[-\infty, 0] \subset Mf$ has cardinality less than $\mathfrak{c}$.*

Proof. The necessity is obvious. To prove its sufficiency, suppose f is $SWLC$ and lower singular, and that the set where $[-\infty, 0] \subset Mf$ has cardinality less than $\mathfrak{c}$. Then by Lemma 3.2.3(a), $N(f)$ is perfect in I, and so the result follows directly from part (c) of the Fourth Median Theorem 3.2.5. □

The following two monotonicity theorems are direct consequences of the above monotonicity lemma. Among these theorems, the Second Monotonicity Theorem 3.3.2 is well known (see, e.g., Natanson [N, p. 266]), and the Third Monotonicity Theorem 3.3.3 provides a more general version of the Second.

Theorem 3.3.2. (**Second Monotonicity**). *A function f is nondecreasing iff $\underline{D}f \geq 0$ everywhere.*

Theorem 3.3.3. (**Third Monotonicity**). *Suppose f is SWLC. If $\overline{D}f \geq 0$* a.e. *and $\underline{D}f > -\infty$* n.e.*, then f is nondecreasing.*

The following version of the Goldowski-Tonelli theorem also follows directly from the Monotonicity Lemma 3.3.1. This theorem was obtained initially by Goldowski [Go] and Tonelli [T] in the case of continuous functions, and was extended later by Zahorski [Z3] to Darboux functions which are indeed WC.

Theorem 3.3.4. (**Goldowski-Tonelli**). *If f is $SWLC$ and lower singular, and it is derivable* n.e.*, then f is nondecreasing.*

We add the following monotonicity theorem which in turn follows directly from part (b) of the Fourth Median Theorem 3.2.5.

Theorem 3.3.5. (**Fourth Monotonicity**). *Suppose f is $SWLC$. If f is nowhere strongly decreasing and the set where $0 \in Mf$ has cardinality less than $\mathfrak{c}$, then f is increasing.*

The second condition used in this theorem is clearly not necessary for a function to be increasing. However, from this theorem we obtain the following necessary and sufficient condition for a function to be nondecreasing. Note that when $\overline{D}f \geq 0$ at a dense set, it follows easily from the Second Monotonicity Theorem 3.3.2 that f is nowhere strongly decreasing.

Corollary 3.3.6. *Let f be SWLC. Then f is nondecreasing iff* (i) $\overline{D}f \geq 0$ *at a dense set and* (ii) *for every real number $r < 0$ there is a number $s \in [r, 0]$ such that the set where $s \in Mf$ has cardinality less than $\mathfrak{c}$.*

Finally from the Fourth Monotonicity Theorem 3.3.5 we deduce the following result on median derivates which is similar to a well known theorem of Morse [M, Theorem 1] on unilateral derivates.

Theorem 3.3.7. *Suppose f is SWLC, and let $r \in \mathbb{R}$. If f has a median derivate $\geq r$ at a dense set in I and a median derivate $< r$ at some point, then the set where r is a median derivate of f has cardinality $\mathfrak{c}$.*

Proof. Assume the hypothesis and let $g = f_{-r}$. Clearly $\overline{D}g \geq 0$ at a dense set in I, and $\underline{D}g < 0$ at some point. Hence g is nowhere strongly decreasing, and since g is not increasing, by Theorem 3.3.5 $E = \{x \in I : 0 \in Mg(x)\}$ has cardinality $\mathfrak{c}$. Further, it is clear that $r \in M(f)$ at each point of E. □

§3.4. Properties of Derivative, Median and Derivability Sets

In this final section we obtain various properties of derivative, median and derivability sets of a function with the help of the results of the previous sections. We begin with the property of derivability sets. Recall that for $f : X \to \mathbb{R}$, we use $\Delta(f)$ and $\Delta^*(f)$ to denote the sets in X where f is derivable or differentiable, respectively. Both of these sets are called *derivability sets* of f.

The following theorem was obtained originally by Zahorski [Z1, Z2] for continuous functions on an interval, and was extended later by Brudno [Br] to arbitrary functions on an interval. In its following form it was obtained on the other hand in [G1, p. 315]. We obtain this theorem here from the results of §§2.2 and 2.4 which in turn were deduced from the First and Second Biderivate Theorems of §2.1.

Theorem 3.4.1. (**Zahorski-Brudno**). *For every function $f : X \to \mathbb{R}$, the complement of each of the derivability sets $\Delta(f)$ and $\Delta^*(f)$ is a set of the form $G_\delta \cup G_{\delta\sigma}$ in X, where the $G_{\delta\sigma}$-set is of measure zero.*

Proof. Set $N = X \setminus \Delta(f)$, $N^* = X \setminus \Delta^*(f)$, $K = K(f)$, and let $\mathbb{Q}$ denote the set of rational numbers. Then clearly

$$N = (X \setminus X') \cup \bigcup_{r \in \mathbb{Q}} \big[\{x : \underline{D}f(x) < r\} \cap \{x : \overline{D}f(x) > r\}\big].$$

Hence, by the Fifth Biderivate Theorem 2.4.1 N is a $G_{\delta\sigma}$-set in X. Further, by Corollary 2.4.2, $K = G_\delta \cup C$, where C is countable. Hence, $N = K \cup (N \setminus K) = G_\delta \cup H$, where $H = C \cup (N \setminus K)$ is a $G_{\delta\sigma}$-set in X. Also, according to Theorem 2.2.2, $|N \setminus K| = 0$, and so $|H| = 0$. This proves the result for N.

Next, to obtain the result for N^*, let E denote the set where at least one derivate of f is infinite. Then $N^* = N \cup E$, where according to Theorem 2.2.4, E is a G_δ-set in X. Hence the result on N^* follows clearly from the above result on N. □

Next from the Fifth Biderivate Theorem we deduce the following theorem on the Baire class of derivative. This theorem seems to have been obtained first by Zahorski [Z3, p. 15] for everywhere derivable functions on $\mathbb{R}$, and in the following form it was obtained by Preiss [Pr] for connected X. See [G1, p. 322] for its following version.

Theorem 3.4.2. *For every function $f : X \to \mathbb{R}$, its derivative f' is of B_1 relative to the set where it exists.*

Proof. Given $c \in \mathbb{R}$, set $\Delta = \Delta(f)$ and $E = \{x \in \Delta : f'(x) > c\}$. Then by the Fifth Biderivate Theorem 2.4.1

$$E = \Delta \cap \{x : \underline{D}f(x) > c\} = \Delta \cap (F_\sigma \setminus C) = \Delta \cap F_\sigma \setminus \Delta \cap C,$$

where F_σ is an F_σ-set in X and f is not WUC at any point of C. Now, at each $x \in \Delta \cap C$ we have $f'(x) = +\infty$, so that $x \in E$. Hence $E = \Delta \cap F_\sigma$, which proves that f' is of LB_1 on Δ. Now applying this result to $-f$ it follows that f' is also of UB_1 on Δ, and hence f' is of B_1 on Δ. □

Now, with the help of Theorem 3.4.1, we obtain the following from the above theorem.

Corollary 3.4.3. *Let $f : X \to \mathbb{R}$ and $c \in \mathbb{R}$. Then $\{x \in \Delta(f) : f'(x) = c\}$ is a G_δ-set in $\Delta(f)$ and an $F_{\sigma\delta}$-set in X.*

We refer here to the Second Median Theorem 2.4.3 for the Baire class of median which is set-valued in general.

From now on it will be understood that $f : I \to \mathbb{R}$, where I is a subinterval of $\mathbb{R}$. From the Third Biderivate Theorem we first deduce a mean-value theorem in terms of median and derivative.

The function f is said to have *symmetrical derivates* if $D_+f = D_-f$ and $D^+f = D^-f$ at each point of I^0. We will further need the following lemma which was obtained in [G1, p. 323].

Lemma 3.4.4. *Suppose f is WC. If f is not monotone and it does not have a relative extremum at any point, then $N = N(f) \cap N(-f)$ is nonempty and perfect in I and f is nowhere monotone on N.*

Theorem 3.4.5. (**Mean-Value**). *Suppose f is WC. Then for each $x, y \in I$, with $x < y$, there is a point z between x and y such that $Qf(x,y) \in Mf(z)$. Further, in case f has symmetrical derivates and it is derivable n.e., then z can be so chosen that f is derivable at z and $Qf(x,y) = f'(z)$.*

Proof. Let $c = Qf(x,y)$. It is enough to prove the result here in the case when $c = 0$; i.e., when $f(x) = f(y)$, for in the general case the result follows on applying this result to f_{-c}. Hence, suppose $f(x) = f(y)$. Then we need to find a point z in (x,y) such that $0 \in Mf(z)$. As this holds trivially in the case when f is constant on $[x,y]$, suppose it is not so. Then f is not monotone on this interval, and since f is WC, it is not monotone on (x,y) as well.

Now set $J = (x,y)$, and let g denote the restriction of f to J. We need to show that there is a point z in J such that $0 \in Mg(z)$. Suppose there is no such point z. Then g does not have any point of relative extremum, for it is easy to see that $0 \in Mg$ at such a point. Hence by Lemma 3.4.4 $N \equiv N(g) \cap N(-g)$ is nonempty and perfect in J, and g is nowhere monotone on N. Thus g is also nowhere bimonotone on N, and it is clearly PWC on N. Hence by the First Median Theorem 2.3.5(a), deduced from the Third Biderivate Theorem, $0 \in Mg$ at a residual set in N, which is a contradiction.

Next, to prove the second part, suppose f has symmetrical derivates and it is derivable n.e. Then the same holds for g, and it is easy to see in this case that g has a zero derivative at each point where it has a relative extremum. Hence it is clear from the above proof that in this case z can be so chosen in J that g is derivable at z and $f'(z) = 0$. □

Next from the Third Biderivate Theorem we deduce the Darboux property of median and derivative. Since the median is a multifunction, we will use the following notion of Darboux property of a multifunction as introduced in [G3, p. 229].

Given a set $E \subset \mathbb{R}$, a multifunction $\varphi : E \to 2^{\overline{\mathbb{R}}}$ is said to have the *Darboux property* if for every connected set C in $\mathbb{R}$, the set $\cup\{\varphi(x) : x \in E \cap C\}$ is connected in $\overline{\mathbb{R}}$.

In the case when E is connected and φ is single-valued, it is clear that the above property coincides with the usual Darboux property which is weaker than continuity. Also, when E is connected and φ is a connected-valued multifunction, it is easy to see that φ has Darboux property whenever it is LSC or USC.

Theorem 3.4.6. *Suppose f is WC. Then Mf possesses Darboux property. Further, in case f has symmetrical derivates and it is derivable n.e., then f' possesses Darboux property relative to the set where it exists.*

Proof. Given an arbitrary subinterval J of I, let $H = \cup\{Mf(x) : x \in J\}$. To show that H is connected, let $c \in \text{co}(H)$. Then to obtain the Darboux property of Mf, it is enough to show that $c \in H$. Suppose $c \notin H$. Then there exist $a, b \in H$ such that $a < c < b$. Hence there exist two points $x, y \in J$ such that $a \in Mf(x)$ and $b \in Mf(y)$. In case $x = y$, then since $Mf(x)$ is connected, clearly $c \in Mf(x) \subset H$. Hence $x \neq y$. We may assume without loss of generality that $x < y$.

Further, since $Mf(x)$ and $Mf(y)$ are connected and they don't include c, it is clear that $\overline{D}f(x) < c < \underline{D}f(y)$. Hence $D^+f(x) < c < D_-f(y)$; so that $D^+f_{-c}(x) < 0 < D_-f_{-c}(y)$. The function f_{-c} is thus not monotone on $[x, y]$, and since f_{-c} is also WC, it cannot be monotone on (x, y) as well. Now set $J_0 = (x, y)$, and let g denote the restriction of f_{-c} to J_0. It is now enough to show that there is a point z in J_0 such that $0 \in Mg(z)$; for then $c \in Mf(z)$. Now since g is not monotone, the existence of the required point z follows from Lemma 3.4.4 and the First Median Theorem 2.3.5 by the arguments used in the proof of Theorem 3.4.5. The result when f has symmetrical derivates and it is derivable n.e. also follows from the proof of that theorem. $\square$

Next, we deal with the Denjoy property of derivatives. Given a function f, its derivative f' will be said to have the *Denjoy property* if for every $a, b \in \mathbb{R}$, $a < b$, the set $\{x \in \Delta(f) : a < f'(x) < b\}$ either is empty or has positive measure. Further, f is said to be *nonangular* [G3, p. 29] if $D_-f \leq D^+f$ and $D_+f \leq D^-f$ at each point of I^0.

The Denjoy property for derivatives was obtained originally by Denjoy [D2] for a continuous function which is derivable everywhere. In the following theorem we deduce a more general result from the Third Biderivate Theorem. There was a slight error in the proof of this theorem given earlier in [G1, p. 326], which has been corrected here.

Theorem 3.4.7. *Suppose f is WC and nonangular, and it is derivable n.e. Given $a, b \in \overline{\mathbb{R}}$, $a < b$, if the set where f has a derivative between a and b is of measure zero, then either $\overline{D}f \leq a$ everywhere or $\underline{D}f \geq b$ everywhere. Consequently, the derivative f' possesses Denjoy property.*

Proof. First, suppose $a = -\infty$. Then since the set where $f' = -\infty$ is by Corollary 2.2.3 of measure zero, we have $f' \geq b$ a.e., or $(f_{-b})' \geq 0$ a.e. Hence it follows from the Goldowski-Tonelli Theorem 3.3.4 that f_{-b} is nondecreasing, and so $\underline{D}f \geq b$ everywhere. In the case when $b = +\infty$, it is proved by a similar augment that $\overline{D}f \leq a$ everywhere. Hence, suppose $a, b \in \mathbb{R}$.

Now set $N = N(-f_{-a}) \cap N(f_{-b})$. This set is clearly closed in I. We claim that N is dense in itself. Let $\mathcal{I}_-$ and $\mathcal{I}_+$ denote the families of contiguous intervals of $N(-f_{-a})$ and $N(f_{-b})$, respectively, in I. Then $\overline{D}f \leq a$ on each interval of $\mathcal{I}_-$, and $\underline{D}f \geq b$ on each interval of $\mathcal{I}_+$. The intervals of the family $\mathcal{I} = \mathcal{I}_- \cup \mathcal{I}_+$ are thus mutually disjoint, and they constitute the contiguous intervals of N in I. Now since f is WC, clearly f_{-a} is nonincreasing on the closure of each interval of $\mathcal{I}_-$, and f_{-b} is nondecreasing on the closure of each interval of $\mathcal{I}_+$. Thus the intervals of $\mathcal{I}_-$ are nonabutting, and so are the intervals of $\mathcal{I}_+$. If, on the other hand, $(x, y) \in \mathcal{I}_-$ and $(y, z) \in \mathcal{I}_+$, then f_{-a} is nonincreasing on $[x, y]$ and f_{-b} is nondecreasing on $[y, z]$. Hence $D^- f(y) \leq a < b \leq D_+ f(y)$, which contradicts the nonangularity of f. Similarly, it is not possible that $(x, y) \in \mathcal{I}_+$ and $(y, z) \in \mathcal{I}_-$. Hence the intervals of $\mathcal{I}$ are mutually nonabutting, and the set N is consequently dense in itself.

Now it is enough to show that $N = \emptyset$, for then $I \in \mathcal{I}_-$ or $\mathcal{I}_+$, which will establish the result. Hence suppose $N \neq \emptyset$. We first claim that f_{-a} is nowhere SBI on N. Suppose to the contrary that f_{-a} is SBI on some portion P of N. Then there is clearly a $c \in (a, b)$ such that f_{-c} is WBI on P. Let x, y be two bilateral limit points of P with $x < y$, and set $J = (x, y)$. Then $J \cap P \neq \emptyset$. Now, since f_{-c} is WBI on P, as before by the WC of f, f_{-c} is nondecreasing on the closure of each contiguous interval of P in J. Hence clearly f_{-c} is nondecreasing on J. Thus $\underline{D}f \geq c > a$ on J, and hence by the hypothesis $f' \geq b$ a.e. in J. Now by the Goldowski-Tonelli Theorem 3.3.4 f_{-b} is nondecreasing on J. Hence $J \cap N(f_{-b}) = \emptyset$, which contradicts the fact that $J \cap P \neq \emptyset$. This proves the claim.

Now since f_{-a} is nowhere SBI on N, by the Third Biderivate Theorem 2.3.2 $\underline{D}f_{-a} \leq 0$, or $\underline{D}f \leq a$, r.e. in N. Using a similar argument it is proved that f_{-b} is nowhere SBD on N, and hence by the same theorem $\overline{D}f \geq b$ r.e. in N. Thus $\underline{D}f \leq a < b \leq \overline{D}f$ r.e. in N, which contradicts the hypothesis that f is derivable n.e. □

Finally, in the following theorem we obtain an extended version of a theorem of Marcus [Ma] on the stationary sets of derivatives of continuous functions to the ones of derivatives of SWC functions. A subset E of $\mathbb{R}$ is said to be a *stationary set* of a class $\mathcal{C}$ of functions on $\mathbb{R}$ if every $f \in \mathcal{C}$ that is constant on E is constant on $\mathbb{R}$.

Theorem 3.4.8. *A set $E \subset \mathbb{R}$ is a stationary set of derivatives of SWC functions iff the complement of E has zero inner measure.*

Proof. Let E be a subset of $\mathbb{R}$ whose complement has zero inner measure and let f be the derivative of some SWC function g on $\mathbb{R}$ such that f is constant on E, say c. Now, since $\{x : f(x) = c\}$ is measurable by Corollary 3.4.3, $f(x) = c$ for a.e. x. Hence the function g_{-c} is singular, and so by Theorem 3.3.4 g_{-c} is constant on $\mathbb{R}$. Consequently, $f = g' = c$ on $\mathbb{R}$. Further, when the complement of E has an inner measure > 0, it is known [Z3] that there exists a finite derivative which is constant on E but not on $\mathbb{R}$. Hence the theorem. □

Remark 3.4.9. We conclude with a few comments on the results of this section.

(a) The hypothesis of symmetrical derivates is essential in the second parts of Theorems 3.4.5 and 3.4.6. For, let $f(x) = 0$ or $x - 1$ according to whether x is in $[0, 1]$ or $(1, 2]$, respectively. Then f is derivable n.e., but since f' has only values 0 and 1, it is clear that the second part of neither of these two theorems holds for f.

(b) In the case of a function $f : I \to \mathbb{R}$ with symmetrical derivates, it is interesting to point out that under a stronger continuity hypothesis it has been proved in Corollaries 10.4.2 and 8.1.3 of [G3, pp. 165, 230] that Theorems 3.4.5 and 3.4.6 hold without any derivability hypothesis in terms of derivative of f relative to the set where it exists. What is surprising about the two theorems obtained here is that they hold for every WC function.

(c) Similar to Theorem 8.3.1 of [G3, p. 171], it is possible to deduce from the Mean-Value Theorem 3.4.5 a characterization of strong derivative of f in terms of U-limit of Mf at the point in question.

(d) In connection with Theorem 3.4.7, it should be pointed out that under stronger continuity hypothesis the Denjoy property of (ordinary) derivatives continues to hold under some weaker forms of derivability (see Theorems 10.4.6 and 10.4.8 of [G3, p. 232]).

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