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MOST C$^\infty$ FUNCTIONS ARE NOWHERE GEVREY DIFFERENTIABLE OF ANY ORDER

Abstract

We define a complete metric on C^∞, and find that most functions in C^∞ are nowhere Gevrey differentiable of any order. For any $s > 1$ we prove there exists an everywhere Gevrey differentiable function of order s that is nowhere Gevrey differentiable of any order less than s.

In this paper C^∞ denotes the family of continuously differentiable functions of all orders on the compact interval I. Recently Gevrey differentiable functions have interested analysts studying partial differentiable equations [C]. For any real number $s > 0$ a function f in C^∞ is said to be Gevrey differentiable of order s at a point a if a has a compact neighborhood K for which $\sup_{x\in K}\left|f^{(p)}(x)\right| \le Ch^p(p!)^s$ for all $p \ge 0$ and for some positive constants C and h. Thus f is analytic at a if $s = 1$.

Let GD stand for Gevrey differentiable. We say that f is everywhere (nowhere) GD if f is GD at each point (at no point) of I. For each $s > 1$, an everywhere GD function of order s that is nowhere analytic, has been constructed [C]. In this note we prove the existence of an everywhere GD function of order s that is nowhere GD of any order less than s. We prove the existence of a C^∞ function that is nowhere GD of any order.

As in [D], we define

$$d(f,g) = \sum_{n=0}^{\infty} \min\Big(\frac{1}{2^n},\ \sup\left|f^{(n)} - g^{(n)}\right|\Big) \quad \text{for} \quad f, g \in C^\infty.$$

It follows that d is a complete metric on C^∞. By a category argument much like the one in [D], we prove the following.

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Theorem 1. *Most functions in C^∞ are nowhere GD of any order.*

Proof. Fix a point a in I. For numbers $s \geq 1$, $c \geq 1$, $h \geq 1$, let $T(a,s,c,h)$ denote the family of all f in C^∞ for which $\left|f^{(p)}(a)\right| \leq ch^p(p!)^s$ for $p \geq 0$. It follows that $T(a,s,c,h)$ is a closed subset of C^∞. Our next task is to prove that $T(a,s,c,h)$ is nowhere dense in C^∞.

Choose any $v > 0$. Choose an integer n so large that $\sum_{i=n}^\infty 2^{-i} < v$ and then select $b > 2$ so large that $vb^n > 3ch^{2n}\big((2n)!\big)^s$. Let $f \in T(a,s,c,h)$. Put

$$w(x) = f(x) + v\frac{1}{b^n}\cos\big(b(x-a)\big).$$

Now

$$d(f,w) \leq \sum_{i=0}^{n-1} vb^{i-n} + \sum_{i=n}^{\infty}\frac{1}{2^i} \leq v + v = 2v.$$

But $\left|f^{(2n)}(a) - w^{(2n)}(a)\right| = vb^n > 3ch^{2n}\big((2n)!\big)^s$, and $\left|f^{(2n)}(a)\right| \leq ch^{2n}\big((2n)!\big)^s$. It follows that $\left|w^{(2n)}(a)\right| > ch^{2n}\big((2n)!\big)^s$ and $w \notin T(a,s,c,h)$.

So $T(a,s,c,h)$ is nowhere dense. We let s, c and h run over the rational numbers ≥ 1 and let a run over the rational numbers in I, and we see that the set of all the functions in C^∞ that are GD of some order at some point in I is a first category subset of C^∞. This completes the proof. $\square$

For any $s > 1$ let F_s denote the function on $\mathbb{R}$ defined by

$$F_s(t) = \begin{cases} \exp\left[-t^{-\frac{1}{s-1}}\right] & \text{for } t > 0 \\ 0 & \text{for } t \leq 0 \end{cases}$$

In [C] it was proved that F_s is not GD at 0 of any order less than s. It was also proved that F_s is everywhere GD of order s on $\mathbb{R}$. In fact, there are constants C and k such that

$$\sup_{t\in\mathbb{R}}\left|F_s^{(p)}(t)\right| \leq Ck^p(p!)^s \quad (p \geq 0).$$

We use C and k to find a constant H_s such that

$$\sup_{t\in\mathbb{R}}\left|F_s^{(p)}(t)\right| \leq H_s^p(p!)^s \quad (p > 0).$$

Theorem 2. *For any $s > 1$, there exists an everywhere GD function of order s that is nowhere GD of any order less than s.*

PROOF. Let Y denote the family of all functions in C^∞ such that

$$\sup_{t \in I} \left| f^{(p)}(t) \right| \leq H_s^p (p!)^s \quad (p > 0),$$

where H_s is defined as before. Then F_s lies in Y and Y is nonvoid. Clearly Y is a closed subset of C^∞. Thus Y is a complete metric space under d.

Let $s_0 \in [1, s)$, $c \geq 1$, $h \geq 1$, $a \in$ interior of I be rational numbers. Let Y_0 denote the set of all $f \in C^\infty$ for which $\left| f^{(p)}(a) \right| \leq ch^p (p!)^{s_0}$ $(p \geq 0)$. Clearly Y_0 is a closed subset of C^∞, so $Y \cap Y_0$ is a closed subset of the complete metric space Y.

Now let $g \in Y \cap Y_0$, and let n be a positive integer. Put

$$w_n(t) = \Big(1 - \frac{1}{n}\Big) g(t) + \frac{1}{n} F_s(t - a).$$

Then

$$\sup_{t \in I} \left| w_n^{(p)}(t) \right| \leq \Big(1 - \frac{1}{n}\Big) H_s^p (p!)^s + \frac{1}{n} H_s^p (p!)^s = H_s^p (p!)^s$$

for $p > 0$, and hence $w_n \in Y$. Also

$$d(w_n, g) \leq d\Big(\frac{1}{n} g,\, 0\Big) + d\Big(\frac{1}{n} F_s,\, 0\Big)$$

and consequently $d(w_n, g) \to 0$. But g is GD of order s_0 at a and $F_s(t - a)$ is not. So w_n is not GD of order s_0, and hence $w_n \notin Y \cap Y_0$. Thus $Y \cap Y_0$ is nowhere dense in the space Y.

We let s_0 run over all the rational numbers in the interval $[1, s)$, let a run over all the rational numbers in interior of I, and let c and h run over all the rational numbers in $[1, \infty)$, to find that the set of all f in Y that are GD at any point of order less than s is a first category subset of the space Y. Finally, there are many functions in Y that are everywhere GD of order s, and nowhere GD of any order less than s.

In particular, there are many everywhere GD functions of order s in Y that are nowhere analytic. For a concrete example of one such function, consult [C]. □

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