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ON HENSTOCK'S INNER VARIATION AND STRONG DERIVATIVES

Abstract

The Lebesgue and Bochner integrals are characterized by strong derivatives, inner variation and Lusin condition in this note.

In his book [6, p.148], Henstock surmises that by using the concept of inner variation, eventually a theory of differentiation not based on Vitali's covering theorem will emerge. Along this direction, the differentiation of Henstock integrals in n-dimensional Euclidean space has been discussed in [2, 7]. In this note, we shall discuss the differentiation of McShane integrals, which provides another example, based on inner variation. We remark that even in the one-dimensional case, we need to use inner variation, since Vitali's covering theorem cannot be applied. For McShane integrals, we should use strong derivatives [1, 4, 12, 16], since they correspond to McShane interval-point pairs, which are used in the definition of McShane integrals. The family of those interval-point pairs, for which derivation property does not hold may not be a Vitali's cover. Some interesting properties of strong derivatives are mentioned in [1, 4, 16].

1 Derivatives of Lebesgue Integrals

In this section, we shall characterize Lebesgue integrals using strong derivatives. Let $\mathbb{R}$ be the real line; and $[a, b]$ be a compact interval in $\mathbb{R}$.

Definition 1.1. Let $F : [a, b] \to \mathbb{R}$. F is said to be McShane differentiable at $x \in [a, b]$ with the McShane derivative $D_M F(x)$ if for every $\epsilon > 0$, there exists

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a positive number $\delta(x)$ such that whenever $([u,v],x)$ is McShane δ-fine, i.e., $[u,v] \subset (x-\delta(x),\, x+\delta(x))$, we have

$$|F(v) - F(u) - D_M F(x)(v-u)| < \epsilon |v-u|.$$

We remark that the McShane derivative is called the strong derivative in [1, 4, 16] or the sharp derivative in [12, p.199] or full derivative in [9, p.136]. The McShane derivative is named after its corresponding McShane interval-point pairs. It is clear that the McShane derivative is stronger than the ordinary derivative, which, in fact, is induced by Henstock interval-point pairs, where $x \in [u,v]$ in the above definition. It is well-known that the primitive (the indefinite integral) of a Lebesgue integrable function has the ordinary derivative almost everywhere. However it may not have the McShane derivative on a set of positive Lebesgue measure. An example can be constructed by using a Cantor set of positive Lebesgue measure, see [16]. In fact, Henstock has already proved that although it is of positive measure, it has inner variation zero, [6, p.14], or see Theorem 1.1 in this note.

In the following, we shall consider the converse of Henstock's result above, using the idea in [2, 7]. First we introduce inner covers and inner variation zero. The formulations are slightly different from those given in [6, 2, 7]. However their concepts are not different.

For each positive function δ on $[a,b]$ and each $\eta > 0$, let $\Gamma(\delta,\eta)$ be a family of McShane δ-fine interval -point pairs. Assume that for fixed δ, $\Gamma(\delta,\eta_1) \subset \Gamma(\delta,\eta_2)$ if $\eta_2 \le \eta_1$ and for fixed η, $\Gamma(\delta_1,\eta) \subset \Gamma(\delta_2,\eta)$ if $\delta_1(\xi) \le \delta_2(\xi)$ on $[a,b]$. A family $\Gamma(\delta,\eta)$ is called an inner cover of $X \subset [a,b]$ if for each $x \in X$, there is at least one $(I,x) \in \Gamma(\delta,\eta)$. An inner cover is also called a fine cover in [12]. Assume that for each δ, $\Gamma(\delta,\eta)$ is an inner cover of X if η is small enough. Let G be a real-valued function defined on the family of all interval-point pairs (I,x) with $I \subset [a,b]$, $x \in [a,b]$. The set X has inner G-variation zero with respect to the given collection $\{\Gamma(\delta,\eta)\}$ as given above if for each $\epsilon > 0$, there exists a positive function δ such that for any McShane δ-fine partial division $D = \{(I,x)\}$ of $[a,b]$ with $x \in X$ and $D \subset \bigcup_\eta \Gamma(\delta,\eta)$, we have

$$(D)\sum |G(I,x)| < \epsilon.$$

Note that if $D \subset \bigcup_\eta \Gamma(\delta,\eta)$ then $D \subset \Gamma(\delta,\eta)$ for some η, if $G(I,x)$ represents the length of I, then inner G-variation is simply called inner variation. We shall use the following notations: Let

$$IV(G,X,\Gamma(\delta,\eta)) = \sup_D (D)\sum |G(I,x)|$$

where $\sup_{D}$ is the supremum over all McShane δ-fine partial division $D = \{(I,x)\}$ of $[a,b]$ with $D \subset \Gamma(\delta,\eta)$ and $x \in X$.

Let

$$IV(G,X) = \inf_{\delta} \sup_{\eta} IV(G,X,\Gamma(\delta,\eta)).$$

Note that $IV(G,X) = 0$ if and only if X has inner G-variation zero. If $G(I,x)$ represents the length of I, $IV(G,X,\Gamma(\delta,\eta))$ and $IV(G,X)$ are simply denoted by $IV(X,\Gamma(\delta,\eta))$ and $IV(X)$ respectively.

We need to specify $\Gamma(\delta,\eta)$ when we discuss derivatives.

Let f, F be real-valued functions on $[a,b]$. For each $\delta(\xi) > 0$ and each $\eta > 0$, define

$$\Gamma(f,F,\delta,\eta) = \{(I,x) : |F(I) - f(x)|I|| \geq \eta|I|\},$$

where $F(I) = F(v) - F(u)$ and $|I| = v - u$ if $I = [u,v]$.

$$X(f,F,\delta,\eta) = \{x \in [a,b] : \text{there exists I such that } (I,x) \in \Gamma(f,F,\delta,\eta)\},$$
$$X(f,F) = \bigcup_{\eta}\bigcap_{\delta} X(f,F,\delta,\eta).$$

Note that $X(f,F)$ consists of points x where $D_M F(x) \neq f(x)$. However when $x \in X(f,F)$, some (certainly not all) interval-point pairs (I,x) may still satisfy derivation inequality

$$|F(I) - f(x)|I|| < \epsilon|I|.$$

$X(f,F)$ may not be a Vitali's cover. Hence inner variation will be used to discuss the McShane derivatives. $\{\Gamma(f,F,\delta,\eta)\}$ satisfies all the conditions imposed on $\Gamma(\delta,\eta)$ mentioned above. From now on, we shall use $\Gamma(\delta,\eta)$ instead of $\Gamma(f,F,\delta,\eta)$ if it is obvious that we are discussing f and F; and inner variation is with respect to this specific family $\{\Gamma(f,F,\delta,\eta)\}$, when we are discussing differentiation.

Theorem 1.2. *[6, p. 143] If f is Lebesgue integrable on $[a,b]$ with primitive F, then $D_M F(x) = f(x)$ except at points of a set X with inner variation zero.*

Definition 1.3. Let $F : [a,b] \to \mathbb{R}$. Then F is said to have strong Lusin condition if $IV(Y) = 0$ then $IV(F,Y) = 0$.

We remark that if f is Lebesgue integrable on $[a,b]$ with primitive F, then F has strong Lusin condition, in view of Henstock's Lemma [6, pp.86-87] and $IV(Y) = 0$. Now we shall prove the converse of Theorem 1.2.

Theorem 1.4. *Let f and F be real-valued functions defined on $[a, b]$. Suppose (i) $D_M F(x) = f(x)$ except at points of a set X with inner variation zero; and (ii) F has strong Lusin condition. Then f is Lebesgue integrable on $[a, b]$ with primitive F.*

PROOF. It can be proved by similar idea used for Henstock integrals [2, 7]. It is sufficient to assume that f is bounded on $[a, b]$, say $|f(x)| \leq \alpha$ for all x, since we may consider $[a, b] = \bigcup_{k=1}^{\infty} \{x : k - 1 \leq |f(x)| < k\}$. Let $\epsilon > 0$. There exists a positive function $\delta(x)$ on $[a, b]\backslash X$ such that

$$|F(I) - f(x)|I|| < \epsilon|I|$$

whenever (I, x) is McShane δ-fine. On the other hand, there exists a positive function δ on X such that

$$\sup_{\eta} IV(X, \Gamma(\delta, \eta)) < \epsilon.$$

Hence $IV(X, \Gamma(\delta, \eta)) < \epsilon$ for all $\eta > 0$. Then $(D) \sum |I| < \epsilon$ whenever $D = \{(I, x)\}$ is a McShane δ-fine partial division with $D \subset \Gamma(\delta, \eta)$ for some $\eta > 0$ and $x \in X$. Recall that

$$|F(I) - f(x)|I|| \geq \eta|I|$$

for all $(I, x) \in D \subset \Gamma(\delta, \eta)$. Suppose (I, x) is McShane δ-fine with $x \in X$ and $(I, x) \notin \Gamma(\delta, \eta)$. Then

$$|F(I) - f(x)|I|| < \eta|I|.$$

By given $IV(F, X) = 0$, so we may assume that with the same δ, we have

$$\sup_{\eta} IV(F, X, \Gamma(\delta, \eta)) < \epsilon.$$

Hence $(D) \sum |F(I)| < \epsilon$, when $D = \{(I, x)\}$ is a McShane δ-fine partial division with $D \subset \Gamma(\delta, \eta)$ for some $\eta > 0$ and $x \in X$. Now let $D = \{(I, x)\}$ be a McShane δ-fine division of $[a, b]$ with $x \in [a, b]$.
Then

$$\begin{aligned}(D) \sum_{x \notin X} |F(I) - f(x)|I|| &< \epsilon (D) \sum_{x \notin X} |I| \\ &\leq \epsilon|b - a|.\end{aligned}$$

On the other hand,

$$\begin{aligned}D' &= \{(I, x) \in D : x \in X, (I, x) \notin \Gamma(\delta, \epsilon)\}, \\ D'' &= \{(I, x) \in D : x \in X, (I, x) \in \Gamma(\delta, \epsilon)\}\end{aligned}$$

Then

$$(D')\sum |F(I)-f(x)|I|| < \epsilon|b-a,|$$
$$(D'')\sum |I| < \epsilon,$$
$$(D'')\sum |F(I) < \epsilon.$$

Hence $(D)\sum |F(I)-f(x)|I|| < 2\epsilon|b-a|+\alpha\epsilon+\epsilon$. Thus f is Lebesgue integrable on $[a,b]$ with primitive F. □

Remark 1.5. In general, we cannot change the values of f on X, where X is of inner variation zero. We can only change the values of f on $Y \subset X$, when Y is of variation zero. Recall that variation $V(Y)$ is defined by replacing $\Gamma(\delta,\eta)$ by all McShane δ-fine interval-point pairs, [6, p.76].

2 Derivatives of Bochner Integrals

Now we shall consider the Bochner integral which is equivalent to the strong McShane integral, see [5, 14, 11]. Let $(E, \|\ \|)$ be a Banach space.

Definition 2.1. Let $f:[a,b]\to E$. f is said to be Bochner (or strongly McShane) integrable on $[a,b]$ with primitive F if for every $\epsilon>0$, there exists a positive function δ on $[a,b]$ such that whenever $D=\{(I,x)\}$ is a McShane δ-fine partial division of $[a,b]$, we have

$$(D)\sum \|F(I)-f(x)|I|\| < \epsilon.$$

In [10], some examples are given for the strong Henstock integral.

Definition 2.2. Let $F:[a,b]\to E$. F is said to be McShane differentiable at $x\in[a,b]$ with the McShane derivative $D_MF(x)$ if for every $\epsilon>0$, there exists a positive number $\delta(x)$ such that whenever (I,x) is McShane δ-fine, we have

$$\|F(I)-D_MF(x)|I|\| < \epsilon|I|.$$

Note that $D_MF(x):[a,b]\to E$.

Using the idea in section 1 with $|\cdot|$ replaced by $\|\ \|$ we have

Theorem 2.3. *Let f and F be E-valued functions defined on $[a,b]$. Then f is Bochner integrable on $[a,b]$ with primitive F if and only if (i) $D_MF(x)=f(x)$ except at points of a set X with inner variation zero, and (ii) F has strong Lusin condition.*

Example 2.4. In the classical stochastic analysis, we need to consider $L_1(\Omega \times [a,b])$, see [13, 15]. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a measure space, where Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω and P is a measure on $\mathcal{F}$, with $P(\Omega) = 1$. Let $L_1(\Omega \times [a,b])$ be the space of all real-valued functions $f(w,t)$ which are classical integrable on the product measure space $\Omega \times [a,b]$.

It is reasonable to guess that the integral above can be defined by the McShane approach and the positive function $\delta(w,t)$ in the approach depends on $w \in \Omega$ and $t \in [a,b]$. In the following, we shall point out that if we consider the above integral as a Bochner integral for $L_1(\Omega)$-valued functions, the δ depends only on $t \in [a,b]$.

Let $(L_1(\Omega), \|\|)$ be the L_1-space with $\|g\| = \int_\Omega |g| dP$. Let $\int_a^b f(w,t)\, dt$ belong to $L_1(\Omega)$ for almost all $w \in \Omega$ and $\int_\Omega f(w,t) dP \in L_1[a,b]$ for almost all $t \in [a,b]$. However in order that the Bochner integral in Definition 2.1 is well-defined, we have to assume that $\int_\Omega f(w,t) dP \in L_1[a,b]$ for all $t \in [a,b]$. It can be assumed, since we can change the values of $\int_\Omega f(w,t) dP$ on points t of a set of Lebesgue measure zero.

Theorem 2.5. *Let $f \in L_1(\Omega \times [a,b])$, and $g(t)(w) = f(w,t)$. Then $g : [a,b] \to (L_1(\Omega), \|\|)$ is Bochner integrable on $[a,b]$.*

PROOF. It is clear that $g : [a,b] \to (L_1(\Omega), \|\|)$, as we assume that for each $t \in [a,b]$, $f(w,t) \in L_1(\Omega)$. Observe that

$$\int_a^b \int_\Omega |f(w,t)| dP \ dt \text{ exists}$$

and hence $\int_a^b \|f(\cdot,t)\|\, dt = \int_a^b \|g(t)\|\, dt$ exists.

Thus $\|g(t)\|$ is Lebesgue integrable on $[a,b]$. Therefore g is Bochner integrable on $[a,b]$ with primitive $G(t)(w) = \int_a^t g(w,s)\, ds$, see [3, p 45]. □

The converse of the above theorem is also true.

Theorem 2.6. *Let $f(w,t) : \Omega \times [a,b] \to \mathbb{R}$ and $g(t)(w) = f(w,t)$. Suppose $g : [a,b] \to (L_1(\Omega), \|\|)$ and g is Bochner integrable on $[a,b]$. Then $f \in L_1(\Omega \times [0,1]$.*

PROOF. Suppose g is Bochner integrable on $[a,b]$, then $\|g(t)\|$ is Lebesgue integrable [3, p 45]. Hence

$$\int_a^b \int_\Omega |g(t)(w)| dP \ dt = \int_a^b \int_\Omega |f(w,t)| dP \ dt \text{ exists}$$

Hence $f \in L_1(\Omega \times [a,b])$. □

Consequently, by Theorem 2.1, we have

Theorem 2.7. *Let* $f \in L_1(\Omega \times [a,b])$ *and* $F(t)(w) = \int_a^t f(w,s)\,ds$. *Then (i)* $D_M F(t)(w) = f(w,t)$ *except at points of a set with inner variation zero, and (ii)* F *has strong Lusin condition.*

Remark 2.8. If we use ordinary derivatives D_H i.e. derivation with respect to Henstock interval-point pairs, then instead of (i) and (ii) in Theorem 2.7, we have (i)$'$ $D_H F(t)(w) = f(w,t)$ except at points of a set with Lebesque measure zero and (ii)$'$ F has strong Lusion condition (with respect to Henstock interval-point pairs). Certainly we can replace (ii)$'$ by (ii)* F is absolutely continuous on $[a,b]$ with respect to $\| \ \|$.

Finally we remark that a set of inner variation (with respect to Henstock interval-point pairs) zero if and only if it is a set of Lebesgue measure zero, since we can apply Vitali's covering theorem to the corresponding $\bigcup_{\delta,\eta} \Gamma(\delta,\eta)$ used in Section 1. Hence (i)$'$ is true in Remark 2.1 or see [3, p.49].

Example 2.9. In the classical stochastic analysis, we also consider the belated Bochner integral (or the belated strong McShane integral), see [13, 15].

Definition 2.10. Let f and B defined on $[a,b]$ with values in $(E, \|\|)$. f is said to be belated Bochner (belated strongly McShane) integrable with respect to B on $[a,b]$ with primitive F if for every $\epsilon > 0$, there exists a positive function δ on $[a,b]$ such that whenever $D = \{(I,x)\}$ is a belated McShane δ-fine partial division of $[a,b]$, we have

$$(D)\sum \|F(I) - f(x)(I)\| < \epsilon.$$

Recall that an interval-point pair (I,x) is said to be belated δ-fine if $I \subset (x, x+\delta(x))$. Note that the point x is always on the left-hand side of I. We may not have full belated McShane δ-fine division of $[a,b]$.
In the classical stochastic analysis, we always assume that F is absolutely continuous with respect to $\|\|$. Hence the primitive is unique. Note that if $(E, \|\|)$ is $(\mathbb{R}, |\ |)$ then the belated Bochner integral is equivalent to the Bochner (Lebesgue) integral, see [8]. In general they are not equivalent.

Definition 2.11. Let $F : [a,b] \to E$. F is said to be belated McShane differentiable at $x \in [a,b]$ with respect to B, where $B : [a,b] \to E$, with the belated McShane derivative $D_{bM}F(x)$ if for every $\epsilon > 0$, there exists a positive number $\delta(x)$ such that whenever (I,x) is belated McShane δ-fine, we have

$$\|F(I) - D_{bM}F(x)B(I)\| < \epsilon\|B(I)\|.$$

In the following, we assume that the variation $V(B,[a,b])$ of B over $[a,b]$ is finite. Recall $V(B,[a,b]) = \inf_{\delta}\sup_{D}(D)\sum\|B(I)\|$ where $D = \{(I,x)\}$ is belated McShane δ-fine partial division of $[a,b]$.

Similar to Theorem 2.1 with $\sum|I| \leq (b-a)$ replaced by $\sum\|B(I)\| \leq V(B,[a,b])$ in the proof of Theorem 1.4, we have

Theorem 2.12. *Let f and F be E-valued functions defined on $[a,b]$. Then f is belated Bochner integrable with respect to B (with finite variation) on $[a,b]$ with primitive F if and only if*

(i) $D_{bM}F(x) = f(x)$ except at points of a set X with inner variation zero, and (ii) F has strong Lusin condition.

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